Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 349–358
www.emis.de/journals
ISSN 1786-0091

ON THE PARALLEL DISPLACEMENT AND PARALLEL VECTOR FIELDS IN FINSLER GEOMETRY

TETSUYA NAGANO

This paper is dedicated to Professor Masao Hashiguchi on the occasion of his 80-th birthday

Abstract. In Finsler geometry, the notion of parallel for Finsler tensor fields is defined, already. In this paper, however, for vector fields on a base manifold, the author studies the notion of parallel. In the last section, the obtained results are shown as compare corresponding properties to them of Riemannian geometry.

Introduction

The author gave the definition (1.1) of the parallel displacement along a curve in [7] and explained the geometrical meaning by HTM(the horizontal subbundle of TTM) in it. Further, the author proved that the inverse vector field was parallel along the inverse curve under some conditions, in addition, that the inner product of two parallel vector fields was preserved along a path. Next,the notion of the autoparallel curves was stated and, last, the author formulated the notion of the geodesics by using the Cartan Finsler connection and the notions of the path and autoparallel curves.

In this paper, the author studies the notion of parallel displacement for vector fields on a base manifold. The notion of parallel for Finsler tensor fields is already defined([3],[5],[6]) as same as done the notion of parallel displacement along curves([2],[1]). However, the study in detail of parallel vector field on the base manifold are not done, the author thinks so. As the first step of the study in detail, the author investigates the geometrical meaning of parallel in HTM (Proposition 3.1) and from the obtained results the author have the conditions for Finsler connections (Theorem 3.1). Lastly, in Riemannian geometry, parallel

2000 *Mathematics Subject Classification.* 53B40.
Key words and phrases. Finsler geometry, vector field, parallel displacement, path, autoparallel, geodesic.

vector fields have the special properties, for examples, “be parallel along any curve” and “its norm is constant on the base manifold”. So the author checks the above two properties for the ”parallel vector field” in Finsler geometry and investigates being parallel along curves under our definition (1.1) of the parallel displacement in particular.

The terminology and notations are referred to the books [4] and [5]. The author is given very useful suggestions by Prof. T. Aikou and Prof. M. Hashiguchi frequently, and greatly appreciates their kindness.

1. The definition of the parallel displacement along a curve

Firstly, we put the terminology and notations used in this paper. Let M be an n-dimensional differentiable manifold and TM its tangent bundle, and let $(N^i_j, F^i_{jr}, C^i_{jr})$ be a Finsler connection(or the coefficients of a Finsler connection). Further let $F(x,y)$ be a Finsler fundamental function and $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the Finsler metric.

Now, for a vector field along a curve c, we put the definition of the parallel displacement along c. It is as follows

Definition 1.1. For a curve $c = (c^i(t))$ $(a \le t \le b)$ and a vector field $v = (v^i(t))$ along c, if the equations

$$\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0 \quad (\dot{c}^r = \frac{dc^r}{dt}) \tag{1.1}$$

are satisfied, then v is said to be parallel along c, and we call the map $v(a) \to v(b)$ the parallel displacement along c.

Let set the state as the curve $c = (c^i(t))$ $(a \le t \le b)$ passes through two points $p = c(a)$, $q = c(b)$ on M, and we assume that the vector field $v = (v^i(t))$ is parallel along c and $A = (A^i) = v(a)$, $B = (B^i) = v(b)$. Then we have another curve c^{-1} and vector field v^{-1} as follows

$$c^{-1}(\tau) = (c^{-1\,i}(\tau)), \text{ where } c^{-1\,i}(\tau) = c^i(-\tau + a + b), \tag{1.2}$$

$$v^{-1}(\tau) = (v^{-1\,i}(\tau)), \text{ where } v^{-1\,i}(\tau) = v^i(-\tau + a + b) \tag{1.3}$$

and $t = -\tau + a + b,\ a \le \tau \le b$. Then $c^{-1}(a) = c(b) = q,\ c^{-1}(b) = c(a) = p$ and $v^{-1}(a) = v(b) = B,\ v^{-1}(b) = v(a) = A$.

We have the following theorem, for the inverse curve c^{-1} and vector field v^{-1},

Theorem 1.1. *For any differentiable curve $c(t)$. Let $v(t)$ be parallel vector field along the curve $c(t)$. Then the vector field v^{-1} is parallel along the curve c^{-1} if and only if*

$$F^i_{0j}(x,y) + F^i_{0j}(x,-y) = 0 \tag{1.4}$$

is satisfied, where $(N^i_j, F^i_{jr}, C^i_{jr})$ is a Finsler connection satisfying $F^i_{jr}(x,y) = F^i_{rj}(x,y)$.

Now, let HTM be the subbundle of the bundle TTM. HTM is the collection of horizontal vectors at every point (x, y) on the tangent bundle TM, namely

$$HTM = \bigcup_{(x,y)\in TM} \left\{ z^i \delta_i \in T_{(x,y)}TM \;\middle|\; z^i \in \mathbb{R} \right\}, \tag{1.5}$$

where $\delta_i = \frac{\partial}{\partial x^i} - N_i^r \frac{\partial}{\partial y^r}$ is the horizontal basis of $T_{(x,y)}TM$. We put a local coordinate system (x^i, y^i, z^i) of HTM. This system have the coordinate transformation $(x^i, y^i, z^i) \longrightarrow (\bar{x}^a, \bar{y}^a, \bar{z}^a)$ attended with the coordinate transformation $(x^i) \longrightarrow (\bar{x}^a)$ of the base manifold M, where

$$\begin{cases} \bar{x}^a = \bar{x}^a(x) \\ \bar{y}^a = y^j \dfrac{\partial \bar{x}^a}{\partial x^j} \\ \bar{z}^a = z^j \dfrac{\partial \bar{x}^a}{\partial x^j}. \end{cases} \tag{1.6}$$

Then we can take the derivative operator with respect to x^i

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^r \frac{\partial}{\partial y^r} - F_i^r \frac{\partial}{\partial z^r}, \tag{1.7}$$

where $F_j^i = z^r F_{rj}^i(x, y)$.

Next, let $c = (c^i(t))$ be a curve on M and $v = (v^i(t))$ a vector field along c. We take the lift $\tilde{c} = (c^i, \dot{c}^i, v^i)$ to HTM and the tangent vector $\dfrac{d\tilde{c}}{dt}$ of $\tilde{c}$ is written in

$$\begin{aligned} \frac{d\tilde{c}}{dt} &= \frac{dc^i}{dt}\frac{\partial}{\partial x^i} + \frac{d\dot{c}^i}{dt}\frac{\partial}{\partial y^i} + \frac{dv^i}{dt}\frac{\partial}{\partial z^i} \\ &= \dot{c}^i \frac{\delta}{\delta x^i} + \left(\frac{d\dot{c}^i}{dt} + N_r^i(c, \dot{c})\dot{c}^r \right) \frac{\partial}{\partial y^i} + \left(\frac{dv^i}{dt} + v^j F_{jr}^i(c, \dot{c})\dot{c}^r \right) \frac{\partial}{\partial z^i}. \end{aligned} \tag{1.8}$$

Therefore Definition 1.1 means that the lift $\tilde{c}$ is a horizontal curve in HTM. So we have

Theorem 1.2. *If a vector field $v = (v^i(t))$ along the curve $c = (c^i(t))$ is parallel along c, then the lift $\tilde{c} = (c, \dot{c}, v)$ to HTM is a horizontal curve in HTM. The inverse property is also true.*

2. Paths and Autoparallel curves

First, we treat paths on M ([4]). It is the curve $c = (c^i(t))$ satisfying

$$\frac{d\dot{c}^i}{dt} + N_r^i(c, \dot{c})\dot{c}^r = 0. \tag{2.1}$$

In other words, the canonical lift $(c, \dot{c})$ to TM is a horizontal curve in TM.

Then we have the following theorem.

Theorem 2.1. *The inverse curve c^{-1} of any path c is also a path if and only if $N^i_j(x,-y) = -N^i_j(x,y)$ satisfies.*

Next, we consider a Finsler space $(M, F(x,y))$ with a Finsler connection $(N^i_j, F^i_{jr}, C^i_{jr})$. We assume the curve $c = (c^i(t))$ is a path, namely, satisfies (2.1) and two vector fields $v = (v^i(t))$, $u = (u^i(t))$ are parallel along c. Then the "*inner product*" $g_{ij}(c,\dot{c})v^i u^j$ satisfies

$$\begin{aligned}\frac{d}{dt}(g_{ij}(c,\dot{c})v^i u^j) &= \frac{\partial g_{ij}}{\partial x^r}\dot{c}^r v^i u^j + \frac{\partial g_{ij}}{\partial y^r}\frac{d\dot{c}^r}{dt}v^i u^j + g_{ij}\frac{dv^i}{dt}u^j + g_{ij}v^i\frac{du^j}{dt} \\ &= \left(\frac{\delta g_{ij}}{\delta x^r} - g_{kj}F^k_{ir} - g_{ik}F^k_{jr}\right)\dot{c}^r v^i u^j = g_{ij|r}(c,\dot{c})v^i u^j \dot{c}^r.\end{aligned} \tag{2.2}$$

Hence if $g_{ij|r} = 0$, then $g_{ij}(c,\dot{c})v^i u^j$ is constant on c.

Inversely, we assume that the inner product $g_{ij}(c,\dot{c})v^i u^j$ is preserved on the curve $c = (c^i(t))$. Then we have the following calculations

$$\frac{d}{dt}(g_{ij}(c,\dot{c})v^i u^j) = \frac{\partial g_{ij}}{\partial y^r}v^i u^j\left(\frac{d\dot{c}^r}{dt} + N^r_k(c,\dot{c})\dot{c}^k\right) \equiv 0. \tag{2.3}$$

The vector fields $v,\ u$ are arbitrarily. So we have

$$\frac{\partial g_{ij}(c,\dot{c})}{\partial y^r}\left(\frac{d\dot{c}^r}{dt} + N^r_k(c,\dot{c})\dot{c}^k\right) = 0. \tag{2.4}$$

Obviously, on the curve c satisfying (2.4), the inner product of the parallel vector fields v and u are preserved. So we have

Theorem 2.2. *Let $(M, F(x,y))$ be a Finsler space with a Finsler connection $(N^i_j, F^i_{jr}, C^i_{jr})$ satisfying h-metrical $g_{ij|r} = 0$ and $v = (v^i(t))$, $u = (u^i(t))$ vector fields parallel along the curve $c = (c^i(t))$. Then the inner product $g_{ij}(c,\dot{c})v^i u^j$ is preserved on c if and only if the equation* (2.4) *is satisfied.*

Furthermore if $N^i_j(x,y) + N^i_j(x,-y) = 0$, $T^i_{jk}(x,y) = 0$ and $F^i_{0j}(x,y) + F^i_{0j}(x,-y) = 0$ are satisfied, then according to Theorem 1.1 we have

Theorem 2.3. *For a Finsler space $(M, F(x,y))$ with a Finsler connection $(N^i_j, F^i_{jr}, C^i_{jr})$ satisfying h-metrical $g_{ij|r} = 0$, if the curve $c = (c^i(t))$ is a path and vector fields $v = (v^i(t))$, $u = (u^i(t))$ are parallel along c, then the inner products $g_{ij}(c,\dot{c})v^i u^j$ and $g_{ij}(c^{-1},\dot{c}^{-1})v^{-1i}u^{-1j}$ are preserved on c and c^{-1}, respectively.*

Last, let's define the autoparallel curve. It is as follows

Definition 2.1. If the canonical lift $(c,\dot{c},\dot{c})$ to HTM is horizontal of HTM, then we call c an autoparallel curve.

By the above definition, autoparallel curves c satisfy

$$\frac{d\dot{c}^i}{dt} + F^i_{jr}(c,\dot{c})\dot{c}^j\dot{c}^r = 0. \tag{2.5}$$

In addition we consider the geodesics from the viewpoint of the parallel displacement. It is well known that for the Cartan connection $(\overset{c}{N}{}^i_j, \overset{c}{F}{}^i_{jr}, \overset{c}{C}{}^i_{jr})$ the equations of the geodesic $c(t)$ are written

$$\frac{d^2c^i}{dt^2} + \overset{c}{N}{}^i_j(c, \dot{c})\frac{dc^j}{dt} = 0 \text{ (or } \frac{d^2c^i}{dt^2} + \overset{c}{F}{}^i_{rj}(c, \dot{c})\frac{dc^r}{dt}\frac{dc^j}{dt} = 0). \tag{2.6}$$

This means that the geodesic is path and autoparallel curve, from $\overset{c}{N}{}^i_j = \overset{c}{F}{}^i_{0j}$.

3. Parallel vector fields

In Riemannian geometry, when the equations

$$\frac{\partial v^i(x)}{\partial x^j} + \Gamma^i_{rj}(x)v^r(x) = 0 \tag{3.1}$$

are satisfied, we call $v(x)$ the parallel vector field, where $\Gamma^i_{jk}(x)$ are the coefficients of the Riemannian connection. We want the notion of parallel vector fields in Finsler geometry.

According to Section 1 and 2, we treat the lift $\tilde{v} = (x, v, v)$ to HTM and calculate the differential with respect to x^i as follows

$$\begin{aligned}\frac{\partial \tilde{v}}{\partial x^i} &= \delta^r_i \frac{\partial}{\partial x^r} + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial y^r} + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial z^r} \\ &= \frac{\partial}{\partial x^i} + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial y^r} + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial z^r} \\ &= \left(\frac{\delta}{\delta x^i} + N^r_i\frac{\partial}{\partial y^r} + F^r_i\frac{\partial}{\partial z^r}\right) + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial y^r} + \frac{\partial v^r}{\partial x^i}\frac{\partial}{\partial z^r} \\ &= \frac{\delta}{\delta x^i} + \left(\frac{\partial v^r}{\partial x^i} + N^r_i\right)\frac{\partial}{\partial y^r} + \left(\frac{\partial v^r}{\partial x^i} + F^r_i\right)\frac{\partial}{\partial z^r} \\ &= \frac{\delta}{\delta x^i} + \left(\frac{\partial v^r}{\partial x^i} + N^r_i(x, v)\right)\frac{\partial}{\partial y^r} + \left(\frac{\partial v^r}{\partial x^i} + F^r_{ji}(x, v)v^j\right)\frac{\partial}{\partial z^r}.\end{aligned} \tag{3.2}$$

So we consider the tangential case of $\frac{\partial \tilde{v}}{\partial x^i}$, namely, we assume that $v(x)$ satisfies

$$\frac{\partial v^r}{\partial x^i} + N^r_i(x, v) = 0 \tag{3.3}$$

$$\frac{\partial v^r}{\partial x^i} + F^r_{ji}(x, v)v^j = 0. \tag{3.4}$$

From (3.2) if the vector field $v(x)$ on M satisfies (3.3) and (3.4), then the lift $\tilde{v}$ on HTM is tangential to the n-dimensional subspace spanned by $\frac{\delta}{\delta x^i}$. So we have

Proposition 3.1. *Let M be an n-dimensional differential manifold and $(N^i_j, F^i_{jr}, C^i_{jr})$ a Finsler connection. For a vector field $v(x)$ on M, the lift $\tilde{v} = (x, v, v)$ on HTM is tangential to the n-dimensional subspace $\left\{\sum_i \lambda_i \frac{\delta}{\delta x^i} \middle| \lambda_i \in \mathbf{R},\ i = 1, 2, \cdots, n\right\}$ at every point (x, v, v) if and only if v satisfies* (3.3) *and* (3.4).

Next we study the conditions in order for the vector field satisfying (3.3) and (3.4) to exist in locally at every point $(x, y) \in TM$.

First of all, we must have the necessary condition as follows

$$D^i_j(x, v) = 0 \tag{3.5}$$

because of $\frac{\partial v^r}{\partial x^i} = -N^r_i(x, v) = -F^r_{ji}(x, v)v^j$ and $D^i_j(x, v) = F^i_{0j}(x, v) - N^i_j(x, v)$.

Next, by the integrability conditions $\dfrac{\partial^2 v^i}{\partial x^j \partial x^k} = \dfrac{\partial^2 v^i}{\partial x^k \partial x^j}$ of (3.3), the equation

$$R^i_{jk}(x, v) = 0 \tag{3.6}$$

are satisfied because that

$$\begin{aligned} \frac{\partial^2 v^i}{\partial x^j \partial x^k} &= -\frac{\partial N^i_j}{\partial x^k} - \frac{\partial N^i_j}{\partial y^r}\frac{\partial v^r}{\partial x^k} = -\frac{\partial N^i_j}{\partial x^k} + \frac{\partial N^i_j}{\partial y^r} N^r_k \\ &= -\left(\frac{\partial N^i_j}{\partial x^k} - N^r_k \frac{\partial N^i_j}{\partial y^r}\right) = -\frac{\delta N^i_j}{\delta x^k} \end{aligned} \tag{3.7}$$

and

$$\frac{\partial^2 v^i}{\partial x^k \partial x^j} = -\frac{\delta N^i_k}{\delta x^j}, \tag{3.8}$$

so we have

$$R^i_{jk}(x, v) = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} = 0. \tag{3.9}$$

Furthermore by the integrability conditions $\dfrac{\partial^2 v^i}{\partial x^j \partial x^k} = \dfrac{\partial^2 v^i}{\partial x^k \partial x^j}$ of (3.4), the equation

$$R^i_{0jk}(x, v) = 0 \tag{3.10}$$

are satisfied because that

$$
\begin{aligned}
\frac{\partial^2 v^i}{\partial x^j \partial x^k} &= -\frac{\partial (F^i_{rj} v^r)}{\partial x^k} = -\frac{\partial F^i_{rj}}{\partial x^k} v^r - F^i_{rj} \frac{\partial v^r}{\partial x^k} \\
&= -\left(\frac{\partial F^i_{rj}}{\partial x^k} + \frac{\partial F^i_{rj}}{\partial y^m} \frac{\partial v^m}{\partial x^k} \right) v^r + F^i_{rj} F^r_{mk} v^m \\
&= -\left(\frac{\partial F^i_{rj}}{\partial x^k} - \frac{\partial F^i_{rj}}{\partial y^m} N^m_k \right) v^r + F^i_{rj} F^r_{mk} v^m \\
&= -\frac{\delta F^i_{rj}}{\delta x^k} v^r + F^i_{rj} F^r_{mk} v^m = -\left(\frac{\delta F^i_{rj}}{\delta x^k} - F^i_{mj} F^m_{rk} \right) v^r
\end{aligned} \tag{3.11}
$$

and

$$
\frac{\partial^2 v^i}{\partial x^k \partial x^j} = -\left(\frac{\delta F^i_{rk}}{\delta x^j} - F^i_{mk} F^m_{rj} \right) v^r, \tag{3.12}
$$

so we have

$$
\left(\frac{\delta F^i_{rj}}{\delta x^k} - \frac{\delta F^i_{rk}}{\delta x^j} + F^i_{mk} F^m_{rj} - F^i_{mj} F^m_{rk} \right) v^r = 0. \tag{3.13}
$$

From (3.6), (3.13) and $R^i_{rjk} = \dfrac{\delta F^i_{rj}}{\delta x^k} - \dfrac{\delta F^i_{rk}}{\delta x^j} + F^i_{mk} F^m_{rj} - F^i_{mj} F^m_{rk} + C^i_{rm} R^m_{jk}$, we have (3.10).

At last we have the conditions that there exist a vector field $v(x)$ with initial value $v(x_0) = y_0$ satisfying (3.3) and (3.4). Of course, the solutions of the differential equations (3.3) or (3.4) are locally. So we have

Theorem 3.1. *For any point $x_0 \in M$ and any direction y_0 at x_0, there exists an neighborhood U of x_0 and on U the vector field $v(x)$ with the initial value $v(x_0) = y_0$ satisfies* (3.3) *and* (3.4) *if and only if the Finsler connection $(N^i_j(x,y), F^i_{jr}(x,y), C^i_{jr}(x,y))$ satisfies*

$$D^i_j(x,y) = 0, \ \textit{(for the deflection tensor)} \tag{3.14}$$

$$R^i_{jk}(x,y) = 0, \ \textit{(for the } R^1 \textit{ torsion tensor)} \tag{3.15}$$

$$R^i_{0jk}(x,y) = 0 \ \textit{(for the } R^2 \textit{ curvature tensor)}. \tag{3.16}$$

Now, in Riemannian geometry, if a vector field $v(x)$ on M satisfies (3.1), namely, parallel vector field, then $v(x)$ have the following properties

(1) v is parallel along any curve c,
(2) the norm $\|v\|$ is constant on M.

So, for the vector field $v(x)$ stated in Theorem 3.1, we investigate the above two properties.

First of all, we consider the curve $c(t) = (c^i(t))$ as the solution of the following differential equation

$$\frac{dc^i}{dt} = v^i(c(t)). \tag{3.17}$$

Then the restriction $v|_c = v(c(t))$ satisfies $\dfrac{dv^i|_c}{dt} = \dfrac{\partial v^i}{\partial x^r}\dfrac{dc^r}{dt} = -F^i_{jr}(c, v|_c)v^j|_c\dot{c}^r$ from (3.4). Further, from $\dot{c} = v|_c$, we have

$$\frac{dv^i|_c}{dt} + v^j|_c F^i_{jr}(c, \dot{c})\dot{c}^r = 0. \tag{3.18}$$

Thus, according to Definition 1.1, we may call v *parallel along the curve* c.

Next the solution $c(t)$ of (3.17) satisfies $\dfrac{d\dot{c}^i}{dt} = \dfrac{\partial v^i}{\partial x^r}\dfrac{dc^r}{dt} = -N^i_r(c, v|_c)\dot{c}^r$ from (3.3). Thus we have

$$\frac{d\dot{c}^i}{dt} + N^i_r(c, \dot{c})\dot{c}^r = 0. \tag{3.19}$$

By the definition (2.1), the flow c of $v(x)$ is a *path*.

Therefore let (M, F) be a Finsler space with a Finsler connection satisfying (3.14), (3.15), (3.16) and $g_{ij|k}(x, y) = 0$. According to Theorem 2.2, the inner product $\|v\|^2 = g_{ij}(c, \dot{c})v^iv^j$ is constant on the path c that is the solution of (3.17). This means the norm $\|v\|$ is constant on c. Thus we have

Theorem 3.2. *Let (M, F) be a Finsler space with a Finsler connection satisfying (3.14), (3.15), (3.16) and $g_{ij|k} = 0$. Then, at every point $x(\in M)$, there are an neighborhood U of x, a local vector field v and a certain path on U, and the following properties are satisfied on U*

(1) *v is parallel along c,*
(2) *the norm $\|v\|$ is constant on c.*

Remark 3.1. We notice the interesting similarity when we compare the notions of geodesics and parallel vector fields with the ones of Riemannian geometry, see Table 1.

For the metrical properties (inner product, geodesic, norm), we need not only the horizontal property of HTM, but also the one of TM. We may define the notion of parallel vector fields on M in Finsler geometry. It is Proposition 3.1, namely, if the lift $\tilde{v} = (x, v, v)$ to HTM is tangential to the n-dimensional subspace spanned by $\frac{\delta}{\delta x^i}$, then we call v the parallel vector field on M.

Riemannian space (M, g)	Finsler space (M, F)
(I) v is parallel along c	
• $\frac{dv^i}{dt} + v^j \Gamma^i_{jr}(c)\dot{c}^r = 0$ (c, v) is horizontal of TM	• $\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0$ $(c, \dot{c}, v)$ is horizontal of HTM
(II) the inner product	
• it is constant on c	• $(c, \dot{c})$ is horizontal of TM if c is a path, it is constant on c
(III) c is a geodesic	
• $(c, \dot{c})$ is horizontal of TM c is an autoparallel curve	• $(c, \dot{c})$ is horizontal of TM c is a path • $(c, \dot{c}, \dot{c})$ is horizontal of HTM c is an autoparallel curve
(IV) v is a parallel vector field	
• $\frac{\partial v^i(x)}{\partial x^j} + \Gamma^i_{rj}(x)v^r(x) = 0$ (x, v) is horizontal of TM • v is parallel along any c • the norm $\lVert v \rVert$ is constant on M	• $\frac{\partial v^r}{\partial x^i} + F^r_{ji}(x, v)v^j = 0$ (x, v, v) is horizontal of HTM • v is parallel along the curve c that is the solution of (3.17) • $\frac{\partial v^r}{\partial x^i} + N^r_i(x, v) = 0$ if (x, v) is horizontal of TM, the norm $\lVert v \rVert$ is constant on c

TABLE 1

REFERENCES

[1] T. Aikou and L. Kozma. Global aspects of Finsler geometry. In *Handbook of global analysis*, pages 1–39, 1211. Elsevier Sci. B. V., Amsterdam, 2008.

[2] M. Hashiguchi. On parallel displacements in Finsler spaces. *J. Math. Soc. Japan*, 10:365–379, 1958.

[3] M. Matsumoto. *The theory of Finsler connections.* Publications of the Study Group of Geometry, Vol. 5. Department of Mathematics, Okayama University, Okayama, 1970.

[4] M. Matsumoto. *Foundations of Finsler geometry and special Finsler spaces.* Kaiseisha Press, Shigaken, 1986.

[5] M. Matsumoto. Finsler geometry in the 20th-century. In *Handbook of Finsler geometry. Vol. 1, 2*, pages 557–966. Kluwer Acad. Publ., Dordrecht, 2003.

[6] R. Miron. Introduction to the theory of finsler spaces. In *Proc. Nat. Sem. Finsler spaces*, pages 131–183. Braşov, 1980.

[7] T. Nagano. Notes on the notion of the parallel displacement in Finsler geometry. *Tensor (N.S.)*, 70(3):302–310, 2008.

Department of Information and Media Studies,
University of Nagasaki,
Nagasaki, Japan
E-mail address: hnagano@sun.ac.jp