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ON GEODESIC MAPPINGS WITH CERTAIN INITIAL CONDITIONS

JOSEF MIKEŠ AND HANA CHUDÁ

Abstract. In this paper we studied geodesic mappings f between (pseudo-) Riemannian spaces V_n and $\bar{V}_n$ with the following initial condition $\bar{g}(f(x_o)) = k \cdot g(x_o)$, where g and $\bar{g}$ are the metrics of V_n and $\bar{V}_n$.

We proved that if the Weyl tensor of the projective curvature is not vanishing at the point $x_o \in V_n$ then f is homothetic.

1. Introduction

Many monographs and papers are devoted to the theory of geodesic mappings, see [4]-[8]. Geodesic mappings were studied under additional conditions based on the proportionality of metrics on a certain subset of the points [1, 2, 4], see [7]. It turn out that even under this weaker conditions, the mapping is homothetic.

We prove, that the under certain circumstances (very simple condition) a condition for proportional metrics holds only for a single pair of points $x_o \mapsto f(x_o)$.

We suppose, that metrics on studied Riemannian spaces V_n have generally signature, i.e. a Riemannian space in our sense, is classical Riemannian or (pseudo-) Riemannian. We talk about classical Riemannian spaces or (pseudo-) Riemannian spaces.

2. Main properties of Geodesic mappings

It is known, that the diffeomorphism f between (pseudo-) Riemannian spaces V_n and $\bar{V}_n$ is called a *geodesic mapping*, if f maps any geodesic on V_n onto a geodesic on $\bar{V}_n$.

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Using properties of a diffeomorphism f, we can suppose that $\bar{M} = M$, where M is a 'common' manifold on which the metrics g and $\bar{g}$ of V_n and $\bar{V}_n$ are defined.

The mapping of V_n onto $\bar{V}_n$ is geodesic if the Levi-Civita equations hold

$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z), \tag{1}$$

where ∇ is the Levi-Civita connection on V_n, ψ is a linear form and X, Y, Z are tangent vectors. If $\psi = 0$ then a geodesic mapping is called *trivial* or *affine.* The equations (1) we re-write in local coordinates:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{2}$$

where $\bar{g}_{ij}(x)$ and $\psi_k(x)$ are components of $\bar{g}$ and ψ, "$,$" is a covariant derivative on V_n, $x = (x^1, x^2, \ldots, x^n)$ is a point of a coordinate neighbourhood $U \subset M$. Equations (1) and (2) hold when $V_n, \bar{V}_n \in C^1$, i.e. $g_{ij}(x), \bar{g}_{ij}(x) \in C^1$ in any coordinate neighbourhood U.

The Weyl tensor of projective curvature of the Riemannian space has an invariant form:

$$W(X,Y,Z) = R(X,Y)Z - \frac{1}{(1-n)}(\mathrm{Ric(Z,X)\cdot Y - Ric(Z,Y)\cdot X}), \tag{3}$$

or locally

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{(1-n)}(\delta^h_k R_{ij} - \delta^h_j R_{ik}), \tag{4}$$

where R^h_{ijk} are components of the Riemannian tensor R and $R_{ij} = R^\alpha_{ij\alpha}$ are components of the Ricci tensor.

The Weyl tensor of the projective curvature is an invariant under geodesic mappings $V_n \to \bar{V}_n$, i.e. $W = \bar{W}$. For dimension $n = 2$, the Weyl tensor W is always vanishing.

3. Fundamental equations of geodesic mapping

In [6], [7], we analyzed formula (2). In the case, if V_n and $\bar{V}_n \in C^3$, then:

$$\begin{aligned} &(a) \quad \bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik};\\ &(b) \quad n\psi_{i,j} = n\psi_i\psi_j + \mu \bar{g}_{ij} - R_{ij} + \bar{g}_{i\alpha}\bar{g}^{\beta\gamma}R^\alpha_{\beta\gamma j};\\ &(c) \ (n-1)\mu_{,i} = 2(n-2)\psi_\alpha \bar{g}^{\alpha\beta}R_{\beta i} + \bar{g}^{\alpha\beta}(2R_{\alpha i,\beta} - R_{\alpha\beta,i}) \end{aligned} \tag{5}$$

where μ is a function on V_n, $\bar{g}^{ij}(x)$ are components of the inverse matrix to $(\bar{g}_{ij}(x))$, R^h_{ijk} and $R_{ij} = R^\alpha_{ij\alpha}$ are components of Riemannian, respectively Ricci, tensors on the manifold V_n.

The system (5) has no more than one solution at the point x_o for initial conditions:

$$\bar{g}_{ij}(x_o) = \bar{g}^o_{ij},\ \psi_i(x_o) = \psi^o_i, \mu(x_o) = \mu^o,\ x_o \in U.$$

Evidently, if

$$\bar{g}_{ij}(x_o) = k \cdot g_{ij}(x_o),\ k = \mathrm{const} \neq 0,\ \psi_i(x_o) = 0,\ \mu(x_o) = 0, \tag{6}$$

then the initial conditions correspond to a trivial solution

$$\bar{g} = k \cdot g. \tag{7}$$

Elementary, this solution is uniquely on whole manifold. It means, that V_n and $\bar{V}_n$ are homothetic.

4. Geodesic mapping with the condition $\bar{g}(x_0) = k \cdot g(x_0)$

It was proved [6, 7], that in a neighbourhood U, where the Weyl tensor of projective curvature W is not vanishing, a covector $\psi_i(x)$ will be express as a function of components $\bar{g}_{ij}(x)$. The functions will be derived from a metric tensor $\bar{g}_{ij}(x)$ of $U \subset V_n$.

It is known, that, for $n = 2$, is the tensor of projective curvature W is vanishing. Therefore, the next results for $n \geq 3$ hold. It means, that equations (5) are reduced in this neighbourhood by the following way:

$$\bar{g}_{ij,k} = 2F_k(x, \bar{g})\, \bar{g}_{ij} + F_i(x, \bar{g})\, \bar{g}_{jk} + F_j(x, \bar{g})\, \bar{g}_{ik}, \tag{8}$$

where F_i are the functions which depend on geometric objects of V_n and also on components of an unknown metric tensor $\bar{g}$ of $\bar{V}_n$. It means, that equations (8) are the set of partial differential equations with unknown functions $\bar{g}_{ij}(x)$.

We prove following:

Lemma 4.1. *Let $V_n = (M, g)$ and $\bar{V}_n = (M, \bar{g})$ be (pseudo-) Riemannian spaces of the class C^3, $n \geq 3$, and let the Weyl tensor of projective curvature $W(x)$ be not vanishing for all $x \in U$ in a coordinate neighbourhood $U \subset M$.*

If the condition $\bar{g} = k \cdot g$ is satisfied at the point $x_o \in U$ and spaces V_n and $\bar{V}_n$ have the same geodesics, then metrics g and $\bar{g}$ in U are homothetics, i.e. $\bar{g}(x) = k \cdot g(x)$, for all $x \in U$.

Proof. Let us suppose that the assumptions of Lemma 4.1 hold in neighbourhood U. Then the equations (2) hold and we get from them the system of Cauchy type (8). This set is closed system of partial differential equations of Cauchy type with respect to unknown functions $\bar{g}_{ij}(x)$.

For the initial condition:

$$\bar{g}_{ij}(x_o) = k \cdot g_{ij}(x_o), \tag{9}$$

where $x_o \in U$ there is no more than one unique solution.

On the other hand, a trivial solution of equations (8) is $\bar{g} = k \cdot g$, $k = \text{const}$, and it satisfies initial conditions (9). The given mapping is homothetic. □

The Lemma 4.1 implies:

Theorem 4.2. *Let f be a geodesic mapping between (pseudo-) Riemannian spaces V_n and $\bar{V}_n$ with the condition $\bar{g}(f(\bar{x}_o)) = k \cdot g(x_o)$, where g and $\bar{g}$ are* *metrics of V_n and $\bar{V}_n$, respectively. If the Weyl tensor of projective curvature is not vanishing at the point $x_o \in V_n$ then f is a homothetic mapping, i.e.*

$$\bar{g} = k \cdot g, \quad k = \text{const.} \tag{10}$$

Proof. Let f be a geodesic mapping between (pseudo-) Riemannian spaces V_n and $\bar{V}_n$. We suppose, that $V_n, \bar{V}_n \in C^3$ and $\bar{g}(f(x_o)) = k \cdot g(x_o)$.

Because $W(x_o) \neq 0$, then there exists a neighbourhood U at the point x_o, so that $W(x) \not\equiv 0$, for $\forall x \in U$. It follows from Lemma 4.1 that there is only one solution of Levi-Civita equation in the form:

$$\bar{g}(x) \ = k \cdot g(x),\ \ k = \text{const},\ \ \forall x \in U. \tag{11}$$

It means that the system of equations (5) has in the neighbourhood U solution:

$$\bar{g}_{ij}(x) = k \cdot g_{ij}(x),\ \ \psi_i(x) = 0,\ \ \mu(x) = 0,\ \ \text{for } x \in U. \tag{12}$$

These conditions imply that the system of equations at the point x_o fulfils the initial condition (6). If equations (5) fulfil the initial condition (6) at the point $x_o \in U$ then Riemannian spaces V_n and $\bar{V}_n$ are homothetics. It follows from this that the initial conditions globally generate only trivial solutions (10). Equations (10) characterize a homothetic mapping. □

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Josef Mikeš
Department of Algebra and Geometry,
Faculty of Science,
Palacky University of Olomouc,
17. Listopadu 12, 77900 Olomouc,
Czech Republic
E-mail address: mikes@inf.upol.cz

Hana Chudá,
Department of Mathematics,
Faculty of Applied Informatics,
Thomas Bata University in Zlín,
Nad Stráněmi 4511, Zlín 760 05,
Czech Republic
E-mail address: chuda@fai.utb.cz