Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 313–327
www.emis.de/journals
ISSN 1786-0091

VERTICAL LAPLACIAN ON COMPLEX FINSLER BUNDLES

CRISTIAN IDA

ABSTRACT. In this paper we define vertical and horizontal Laplace type operators for functions on the total space of a complex Finsler bundle (E, L). We also define the $\square''^{v}$-Laplacian for (p, q, r, s)-forms with compact support on E and we get the local expression of this Laplacian explicitly in terms of vertical covariant derivatives with respect to the Chern-Finsler linear connection of (E, L).

1. INTRODUCTION

As it is well known, (see [7], [9], [16]), the Laplacian plays an important role in the theory of harmonic integral and Bochner technique both in Riemannian and Kähler manifolds. In recent years some results on Laplacians and their applications in real Finsler spaces have been investigated by [4], [5], [13] and others. Also, in [19] and [20] the h''-horizontal Laplacian on strongly Kähler-Finsler manifolds is studied. Recently, in [18], the complex vertical and horizontal Laplacian on strongly pseudoconvex complex Finsler manifolds was defined and the relationship with the Hodge Laplace operator associated to the holomorphic tangent bundle with respect to the Hermitian metric of Sasaki type naturally induced by the strongly pseudoconvex complex Finsler metric on M, were clarified. The aim of the present paper is to study the Laplace type operators on a complex Finsler bundle (E, L). In our context, we are mainly interested to obtain the vertical part of the Laplacian, first for functions and next for forms with compact support on the total space of a complex Finsler bundle. In the first section of this paper, following [2], [10] and [12], we briefly recall some basic notions on complex Finsler bundles. In the second section, according to [1], [3] we present the Chern-Finsler linear connection as a Hermitian connection in the pull-back bundle and we write the vertical covariant derivatives of contravariant and covariant complex tensor fields. In the third section, using an argument

2000 *Mathematics Subject Classification.* 53B40, 53B35, 53B21.
Key words and phrases. complex Finsler bundles, vertical Laplacian.

similar to that in [19], we define horizontal and vertical Laplace type operators for functions on E. In the last section we define an L^2 - global Hermitian inner product on the space of (p,q,r,s)-forms with compact support on E and we give the local expression of the adjoint operator $\delta^{''v}$ of $d^{''v}$ with respect to this inner product. Thus, we can define the $\square^{''v}$-Laplacian for (p,q,r,s)-forms with compact support on E. Finally we get explicitly the expression of this Laplacian in terms of vertical covariant derivatives of the Chern-Finsler connection.

Let $\pi\colon E\to M$ be a holomorphic vector bundle of rank m over a complex manifold M of $\dim_{\mathbb{C}} M=n$. Let us consider $\mathcal{U}=\{U_\alpha\}$ an open covering set of M and (z^k), $k=1,\ldots,n$ the local complex coordinates in a chart (U,φ) and $s_U=\{s_a\}$, $a=1,\ldots,m$ a local holomorphic frame for the sections of E over U. The covering $\{U,s_U\}_{U\in\mathcal{U}}$ induces a complex coordinate system $u=(z^k,\eta^a)$ on $\pi^{-1}(U)$, where $s=\eta^a s_a$ is a holomorphic section on E_z. If we denote by $g_{UV}\colon U\cap V\to GL(m,\mathbb{C})$ the transition functions, then in $z\in U\cap V$, $g_{UV}(z)$ has a local representation by the complex matrix $M^a_b(z)$ and if $(z^{'k},\eta^{'a})$ are the complex coordinates in $\pi^{-1}(V)$ the transition laws of these coordinates are:

$$z^{'k}=z^{'k}(z)\ ;\ \eta^{'a}=M^a_b(z)\,\eta^b, \tag{1.1}$$

where $(M^a_b(z))$, $a,b=1,\ldots,m$ are holomorphic functions and $\det M^a_b\neq 0$.

As we already know, the total space E has a structure of $(m+n)$-dimensional complex manifold because the transition functions $M^a_b(z)$ are holomorphic. Consider the complexified tangent bundle $T_{\mathbb{C}}E=T^{'}E\oplus T^{''}E$, where $T^{'}E$ and $T^{''}E=\overline{T^{'}E}$ are the holomorphic and antiholomorphic tangent bundles. The vertical holomorphic tangent bundle $VE=\ker\pi_*$ is the relative tangent bundle of the holomorphic projection π. A local frame field on V_uE is $\{\frac{\partial}{\partial\eta^a}\}$, $a=1,\ldots,m$. The vertical distribution V_uE is isomorphic to the sections module of E in u.

A subbundle HE of $T^{'}E$ complementary to VE, i.e. $T^{'}E=VE\oplus HE$, is called a *complex nonlinear connection* on E, briefly (c.n.c). A local base for the horizontal distribution H_uE, called adapted for the (c.n.c) is $\{\frac{\delta}{\delta z^k}=\frac{\partial}{\partial z^k}-N^a_k\frac{\partial}{\partial\eta^a}\}$, $k=1,\ldots,n$, where $N^a_k(z,\eta)$ are the coefficients of the (c.n.c). In the following we consider the abbreviate notations: $\partial_k=\frac{\partial}{\partial z^k}$; $\delta_k=\frac{\delta}{\delta z^k}$; $\dot{\partial}_a=\frac{\partial}{\partial\eta^a}$. Locally, $\{\delta_k\}$ defines an isomorphism of $\pi^*(T^{'}M)$ with HE if and only if they are changed under the rules

$$\delta_k=\frac{\partial z^{'j}}{\partial z^k}\delta^{'}_j\ ;\ \dot{\partial}_b=M^a_b\,\dot{\partial}^{'}_a \tag{1.2}$$

and consequently for its coefficients (see (7.1.9) in [12]) we have that

$$\frac{\partial z^{'k}}{\partial z^j}N^{'a}_k=M^a_bN^b_j-\frac{\partial M^a_b}{\partial z^j}\eta^b. \tag{1.3}$$

The existence of a (c.n.c) is an important ingredient in the "linearization" of the geometry of the total space of a holomorphic vector bundle E. The adapted frames denoted by $\{\delta_{\overline{k}} := \frac{\delta}{\delta \overline{z}^k}\}$ and $\{\dot{\partial}_{\overline{a}} := \frac{\partial}{\partial \overline{\eta}^a}\}$, for the distributions $\overline{HE}$ and $\overline{VE}$ are obtained respectively by conjugation. The adapted coframes are locally given by

$$\{dz^k\}, \{\delta\eta^a = d\eta^a + N^a_k dz^k\}, \{d\overline{z}^k\}, \{\delta\overline{\eta}^a = d\overline{\eta}^a + N^{\overline{a}}_{\overline{k}} d\overline{z}^k\}. \tag{1.4}$$

Definition 1.1. A strictly pseudoconvex complex Finsler structure on E, is a positive real valued smooth function $F^2 = L\colon E \to R_+ \cup \{0\}$, which satisfies the following conditions:

(i) L is smooth on $E - \{0\}$;

(ii) $L(z, \eta) \geq 0$ and $L(z, \eta) = 0$ if and only if $\eta = 0$;

(iii) $L(z, \lambda\eta) = |\lambda|^2 L(z, \eta)$ for any $\lambda \in \mathbb{C}$;

(iv) $(h_{a\overline{b}}) = (\dot{\partial}_a \dot{\partial}_{\overline{b}}\ (L))$ (the complex Levi matrix) is positive defined and determines a Hermitian metric tensor on the fibers of vertical bundle VE.

Definition 1.2. The pair (E, L) is called a complex Finsler bundle.

According to [2], [12], we have

Proposition 1.1. *A (c.n.c) on (E, L) depending only on the complex Finsler structure L is the Chern-Finsler (c.n.c) locally given by*

$$\overset{CF}{N^a_k} = h^{\overline{c}a} \partial_k\ \dot{\partial}_{\overline{c}}\ (L). \tag{1.5}$$

Proposition 1.2. *([12]) The Lie brackets of the adapted frames from $T_{\mathbb{C}}(E)$, with respect to the Chern-Finsler (c.n.c) are given by*

$$[\delta_j, \delta_k] = 0\,,\ [\delta_j, \delta_{\overline{k}}] = (\delta_{\overline{k}}\ \overset{CF}{N^a_j})\ \dot{\partial}_a - (\delta_j\ \overset{CF}{N^{\overline{a}}_{\overline{k}}})\ \dot{\partial}_{\overline{a}}\,,$$

$$[\delta_j, \dot{\partial}_b] = (\dot{\partial}_b \overset{CF}{N^a_j})\ \dot{\partial}_a\,,\ [\delta_j, \dot{\partial}_{\overline{b}}] = (\dot{\partial}_{\overline{b}} \overset{CF}{N^a_j})\ \dot{\partial}_a\,,\ [\dot{\partial}_a, \dot{\partial}_b] = [\dot{\partial}_a, \dot{\partial}_{\overline{b}}] = 0$$

and their conjugates.

2. The Chern-Finsler linear connection

The notions in this section are defined according to [3]. Let $\pi\colon E \to M$ be a holomorphic vector bundle endowed with a strictly pseudoconvex complex Finsler structure, as in the first section. We also identify the holomorphic local frame fields $s = \{s_1, \ldots, s_m\}$ of E, with the one of the pull-back bundle $\widetilde{E} := \pi^* E$ which is isomorphic to VE by $\dot{\partial}_a \leftrightarrow \pi^* s_a$. Then, $\widetilde{E}$ admits a Hermitian metric h_L defined by

$$h_L(Z, W) = h_{a\overline{b}} Z^a \overline{W}^b,\ Z, W \in \Gamma(\widetilde{E}). \tag{2.1}$$

Let $\nabla\colon \Gamma(\widetilde{E}) \to A^1(\widetilde{E})$ be the Hermitian connection of the bundle $(\widetilde{E}, h_L)$, i.e. $\nabla = \nabla' + \nabla''$ is the unique connection on the bundle $(\widetilde{E}, h_L)$ satisfying the conditions

$$\nabla'' = d''\,;\ dh_L(Z, W) = h_L(\nabla Z, W) + h_L(Z, \nabla W),\ \forall Z, W \in \Gamma(\widetilde{E}). \tag{2.2}$$

Definition 2.1. ([1]) The Hermitian connection ∇ on $(\widetilde{E}, h_L)$ is called the Chern-Finsler linear connection of the bundle (E, L).

The $(1,0)$-connection form $\omega = (\omega_b^a)$ of ∇ with respect to a holomorphic frame $s = \{s_a\}$, $a = 1, \ldots, m$, is defined by

$$\nabla s_b = \omega_b^a \otimes s_a \, , \, \omega_b^a = \Gamma_{bk}^a dz^k + C_{bc}^a d\eta^c, \tag{2.3}$$

where the local coefficients of the connection are given by

$$\Gamma_{bk}^a = h^{\overline{c}a} \partial_k (h_{b\overline{c}}) \, ; \, C_{bc}^a = h^{\overline{d}a} \, \dot{\partial}_c \, (h_{b\overline{d}}). \tag{2.4}$$

Using the adapted frames and coframes from the first section, the $(1,0)$-connection form ω_b^a can be written by

$$\omega_b^a = L_{bk}^a dz^k + C_{bc}^a \delta\eta^c \, ; \, L_{bk}^a = h^{\overline{c}a} \delta_k (h_{b\overline{c}}). \tag{2.5}$$

We note that the vertical coefficients C_{bc}^a satisfy the symmetry relation

$$C_{bc}^a = C_{cb}^a \tag{2.6}$$

and it is easy to check

$$C_{ab}^a = \dot{\partial}_b \, (\ln h), \tag{2.7}$$

where $h = \det(h_{a\overline{b}})$.

In the end of this section, we briefly recall the vertical covariant derivatives with respect to the Chern-Finsler connection for contravariant and covariant tensor fields.

If $T^{I_p \overline{J_q} A_r \overline{B_s}}(z, \eta)$ are the components of a contravariant complex tensor field of (p, q, r, s)-type on E, where

$$I_p = (i_1 \ldots i_p); \; J_q = (j_1 \ldots j_q); \; A_r = (a_1 \ldots a_r); \; B_s = (b_1 \ldots b_s),$$

then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

$$\nabla_{\dot{\partial}_a} T^{I_p \overline{J_q} A_r \overline{B_s}} = \dot{\partial}_a \, (T^{I_p \overline{J_q} A_r \overline{B_s}}) + \sum_{k=1}^{r} T^{I_p \overline{J_q} a_1 \ldots a_{k-1} \lambda a_{k+1} \ldots a_r \overline{B_s}} C_{\lambda a}^{a_k}, \tag{2.8}$$

$$\nabla_{\dot{\partial}_{\overline{a}}} T^{I_p \overline{J_q} A_r \overline{B_s}} = \dot{\partial}_{\overline{a}} \, (T^{I_p \overline{J_q} A_r \overline{B_s}}) + \sum_{k=1}^{s} T^{I_p \overline{J_q} A_r \overline{b_1} \ldots \overline{b_{k-1}} \, \overline{\lambda} \, \overline{b_{k+1}} \ldots \overline{b_s}} C_{\overline{\lambda} \, \overline{a}}^{\overline{b_k}}. \tag{2.9}$$

If $T_{I_p \overline{J_q} A_r \overline{B_s}}(z, \eta)$ are the components of a covariant complex tensor field of (p, q, r, s)-type on E, then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

$$\nabla_{\dot{\partial}_a} T_{I_p \overline{J_q} A_r \overline{B_s}} = \dot{\partial}_a \, (T_{I_p \overline{J_q} A_r \overline{B_s}}) - \sum_{k=1}^{r} T_{I_p \overline{J_q} a_1 \ldots a_{k-1} \lambda a_{k+1} \ldots a_r \overline{B_s}} C_{a_k a}^{\lambda}, \tag{2.10}$$

$$\nabla_{\dot{\partial}_{\overline{a}}} T_{I_p\overline{J_q}A_r\overline{B_s}} = \dot{\partial}_{\overline{a}} \left(T_{I_p\overline{J_q}A_r\overline{B_s}}\right) - \sum_{k=1}^{s} T_{I_p\overline{J_q}A_r\overline{b_1}\ldots\overline{b_{k-1}}\,\overline{\lambda}\,\overline{b_{k+1}}\ldots\overline{b_s}} C^{\overline{\lambda}}_{\overline{b_k}\,\overline{a}}. \tag{2.11}$$

We remark that if we combine the formulas above, we get the vertical covariant derivatives for mixed contravariant and covariant complex tensor fields. Also, it is easy to check

$$\nabla_{\dot{\partial}_c} h_{a\overline{b}} = \nabla_{\dot{\partial}_{\overline{c}}} h_{a\overline{b}} = \nabla_{\dot{\partial}_c} h^{\overline{b}a} = \nabla_{\dot{\partial}_{\overline{c}}} h^{\overline{b}a} = 0. \tag{2.12}$$

3. Vertical and horizontal Laplace type operators for functions on E

In this section we define vertical and horizontal Laplace type operators for smooth functions on the total space of a complex Finsler bundle.

We recall that for a smooth function f on E, its exterior derivative is given by

$$df = \delta_k(f)dz^k + \dot{\partial}_a\,(f)\delta\eta^a + \delta_{\overline{k}}(f)d\overline{z}^k + \dot{\partial}_{\overline{a}}\,(f)\delta\overline{\eta}^a.$$

In the sequel we assume that the base manifold of the complex Finsler bundle (E, L) is a Hermitian manifold (M, g). Then, due to [6], in natural manner we can consider the following Hermitian metric structure on E

$$G = g_{j\overline{k}}(z)dz^j \otimes d\overline{z}^k + h_{a\overline{b}}(z,\eta)\delta\eta^a \otimes \delta\overline{\eta}^b, \tag{3.1}$$

where $g_{j\overline{k}}(z)$ is a Hermitian metric on the base manifold M, and $h_{a\overline{b}}$ is the Hermitian metric tensor defined by the complex Finsler structure L, and the adapted coframes are considered with respect to the Chern-Finsler (c.n.c). Thus, (E, G) can be considered as an $(m+n)$-dimensional Hermitian manifold.

Let

$$\Phi = i(g_{j\overline{k}}dz^j \wedge d\overline{z}^k + h_{a\overline{b}}\delta\eta^a \wedge \delta\overline{\eta}^b) = \Phi^h + \Phi^v \tag{3.2}$$

be the associated Kähler form of G.

Then, the volume form associated to the Hermitian metric G of E is

$$dV_E = \frac{1}{(n+m)!}\Phi^{n+m} = \frac{1}{n!}(\Phi^h)^n \wedge \frac{1}{m!}(\Phi^v)^m. \tag{3.3}$$

If we denote by $g = \det(g_{j\overline{k}})$ and $h = \det(h_{a\overline{b}})$, then

$$(\Phi^h)^n = i^n(-1)^{\frac{n(n-1)}{2}} n! g dz^1 \wedge \ldots \wedge dz^n \wedge d\overline{z}^1 \wedge \ldots \wedge d\overline{z}^n,$$
$$(\Phi^v)^m = i^m(-1)^{\frac{m(m-1)}{2}} m! h \delta\eta^1 \wedge \ldots \wedge \delta\eta^m \wedge \delta\overline{\eta}^1 \wedge \ldots \wedge \delta\overline{\eta}^m$$

and therefore we get

$$dV_E = i^{m^2+n^2} g h dz \wedge d\overline{z} \wedge \delta\eta \wedge \delta\overline{\eta}, \tag{3.4}$$

where $dz = dz^1 \wedge \ldots \wedge dz^n$ and $\delta\eta = \delta\eta^1 \wedge \ldots \wedge \delta\eta^m$.

Let $X = X^k\delta_k + X^a \dot{\partial}_a + X^{\overline{k}}\delta_{\overline{k}} + X^{\overline{a}} \dot{\partial}_{\overline{a}} \in \Gamma(T_{\mathbb{C}}E)$ be a complex vector field on E. Then the divergence of X is defined by the equation

$$\mathcal{L}_X dV_E = (\operatorname{div} X)dV_E, \tag{3.5}$$

where $\mathcal{L}_X$ denotes the Lie derivative.

Corresponding to the decomposition of the complex vector field X, the divergence of X admits the decomposition

$$\operatorname{div} X = \operatorname{div}_h X + \operatorname{div}_v X + \operatorname{div}_{\overline{h}} X + \operatorname{div}_{\overline{v}} X$$

where $\operatorname{div}_h X = \operatorname{div} X^h$, $\operatorname{div}_v X = \operatorname{div} X^v$, $\operatorname{div}_{\overline{h}} X = \operatorname{div} X^{\overline{h}}$ and $\operatorname{div}_{\overline{v}} X = \operatorname{div} X^{\overline{v}}$.

Proposition 3.1. *If* $X = X^k\delta_k + X^a \dot{\partial}_a + X^{\overline{k}}\delta_{\overline{k}} + X^{\overline{a}} \dot{\partial}_{\overline{a}} \in \Gamma(T_{\mathbb{C}}E)$, *then*

$$\operatorname{div}_h X = X^k\delta_k(\ln g) + X^k\delta_k(\ln h) - X^k \dot{\partial}_b (\overset{CF}{N^b_k}) + \delta_k X^k;$$

$$\operatorname{div}_{\overline{h}} X = X^{\overline{k}}\delta_{\overline{k}}(\ln g) + X^{\overline{k}}\delta_{\overline{k}}(\ln h) - X^{\overline{k}} \dot{\partial}_{\overline{b}} (\overset{CF}{N^{\overline{b}}_{\overline{k}}}) + \delta_{\overline{k}} X^{\overline{k}};$$

$$\operatorname{div}_v X = X^a \dot{\partial}_a (\ln h) + \dot{\partial}_a X^a; \ \operatorname{div}_{\overline{v}} X = X^{\overline{a}} \dot{\partial}_{\overline{a}} (\ln h) + \dot{\partial}_{\overline{a}} X^{\overline{a}}.$$

Proof. Using the local expressions of the Lie brackets in Proposition 1.2, a straightforward calculus similar to that in [19] lead us to

$$(\operatorname{div} X)gh = X(gh) + [\delta_k X^k + \delta_{\overline{k}} X^{\overline{k}} - (X^k \dot{\partial}_b \overset{CF}{N^b_k} + X^{\overline{k}} \dot{\partial}_{\overline{b}} \overset{CF}{N^{\overline{b}}_{\overline{k}}} - \dot{\partial}_a X^a - \dot{\partial}_{\overline{a}} X^{\overline{a}})]gh.$$

Thus we have

$$\begin{aligned}\operatorname{div}_h X &= \frac{1}{gh} X^k \delta_k(gh) + \delta_k X^k - X^k \dot{\partial}_a \overset{CF}{N^a_k} \\ &= X^k\delta_k(\ln g) + X^k\delta_k(\ln h) - X^k \dot{\partial}_a \overset{CF}{N^a_k} + \delta_k X^k\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}_v X &= \frac{1}{gh} X^a \dot{\partial}_a (gh) + \dot{\partial}_a X^a \\ &= X^a \dot{\partial}_a (\ln h) + \dot{\partial}_a X^a.\end{aligned}$$

The relation between $\operatorname{div}_{\overline{h}} X$ and $\operatorname{div}_{\overline{v}} X$ follows in the similar manner. $\square$

According to (2.7) and $\nabla_{\dot{\partial}_a} X^a = \dot{\partial}_a X^a + X^a C^b_{ba}$ we get

$$\operatorname{div}_v X = \nabla_{\dot{\partial}_a} X^a. \tag{3.6}$$

Note that the gradient of f is defined by

$$G(X, \operatorname{grad} f) = Xf, \ \forall X \in \Gamma(T_{\mathbb{C}}E). \tag{3.7}$$

Thus, for $X = X^k\delta_k + X^a \dot{\partial}_a + X^{\overline{k}}\delta_{\overline{k}} + X^{\overline{a}} \dot{\partial}_{\overline{a}} \in \Gamma(T_{\mathbb{C}}E)$ we have

$$\operatorname{grad} f = \operatorname{grad}_h f + \operatorname{grad}_v f + \operatorname{grad}_{\overline{h}} f + \operatorname{grad}_{\overline{v}} f,$$

where

$$\mathrm{grad}_h f = g^{\overline{k}l}\delta_{\overline{k}}(f)\delta_l\,,\ \mathrm{grad}_v f = h^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ (f)\ \dot{\partial}_a,$$
$$\mathrm{grad}_{\overline{h}} f = g^{\overline{l}k}\delta_k(f)\delta_{\overline{l}}\,,\ \mathrm{grad}_{\overline{v}} f = h^{\overline{b}a}\ \dot{\partial}_a\ (f)\ \dot{\partial}_{\overline{b}}\ .$$

We define the Laplace type operator for smooth functions on E by

$$\Box f = (\mathrm{div}\circ\mathrm{grad})f \tag{3.8}$$

Using the decompositions of the divergence and of the gradient in the relation (3.8) we get

$$\Box f = \Box^{'h} f + \Box^{'v} f + \Box^{''h} f + \Box^{''v} f. \tag{3.9}$$

where

$$\Box^{'h} f = (\mathrm{div}_h\circ\mathrm{grad}_h)f\,,\ \Box^{'v} f = (\mathrm{div}_v\circ\mathrm{grad}_v)f,$$
$$\Box^{''h} f = (\mathrm{div}_{\overline{h}}\circ\mathrm{grad}_{\overline{h}})f\,,\ \Box^{''v} f = (\mathrm{div}_{\overline{v}}\circ\mathrm{grad}_{\overline{v}})f.$$

We have

Proposition 3.2. *Let f be a smooth function on E. Then we have*

$$\Box^{'h} f = \frac{1}{g}\delta_l(gg^{\overline{k}l}\delta_{\overline{k}}f) + g^{\overline{k}l}[\delta_l(\ln h) - \dot{\partial}_b \overset{CF}{N_l^b}]\delta_{\overline{k}}f,$$
$$\Box^{''h} f = \frac{1}{g}\delta_{\overline{l}}(gg^{\overline{l}k}\delta_k f) + g^{\overline{l}k}[\delta_{\overline{l}}(\ln h) - \dot{\partial}_{\overline{b}} \overset{CF}{N_{\overline{l}}^{\overline{b}}}]\delta_k f,$$
$$\Box^{'v} f = \frac{1}{h}\ \dot{\partial}_a\ (hh^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ f)\,;\ \Box^{''v} f = \frac{1}{h}\ \dot{\partial}_{\overline{a}}\ (hh^{\overline{a}b}\ \dot{\partial}_b\ f).$$

Proof. By direct computations, we have

$$\begin{aligned}\Box^{'h} f &= \mathrm{div}_h(\mathrm{grad}_h f)\\ &= \mathrm{div}_h[g^{\overline{k}l}\delta_{\overline{k}}(f)\delta_l]\\ &= g^{\overline{k}l}\delta_{\overline{k}}(f)\delta_l(\ln g) + g^{\overline{k}l}\delta_{\overline{k}}(f)\delta_l(\ln h) + \delta_l[g^{\overline{k}l}\delta_{\overline{k}}(f)] - g^{\overline{k}l}\delta_{\overline{k}}(f)\ \dot{\partial}_b \overset{CF}{N_l^b}\\ &= \frac{1}{g}\delta_l[gg^{\overline{k}l}\delta_{\overline{k}}(f)] - g^{\overline{k}l}[\delta_l(\ln h) - \dot{\partial}_b \overset{CF}{N_l^b}]\delta_{\overline{k}}(f)\end{aligned}$$

and

$$\begin{aligned}\Box^{'v} f &= \mathrm{div}_v(\mathrm{grad}_v f)\\ &= \mathrm{div}_v[h^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ (f)\ \dot{\partial}_a]\\ &= h^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ (f)\ \dot{\partial}_a\ (\ln h) + \dot{\partial}_a\ [h^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ (f)]\\ &= \frac{1}{h}\ \dot{\partial}_a\ [hh^{\overline{b}a}\ \dot{\partial}_{\overline{b}}\ (f)].\end{aligned}$$

The relation between $\Box^{''h} f$ and $\Box^{''v} f$ follows in the similar manner. $\Box$

Remark 3.1. In terms of the vertical covariant derivatives with respect to the Chern-Finsler connection, we have

$$\Box'^{v} f = h^{\overline{b}a}\nabla_{\dot{\partial}_a}\nabla_{\dot{\partial}_{\overline{b}}} f. \tag{3.10}$$

Proposition 3.3. *If $Y = Y^a\ \dot{\partial}_a \in \Gamma(VE)$, then*

$$(\mathrm{div}\ Y)dV_E = d[i(Y)dV_E]\,;\ (\mathrm{div}\ \overline{Y})dV_E = d[i(\overline{Y})dV_E],$$

where $i(Y)$ denotes the interior product.

Proof. We have

$$i(Y)dV_E = i^{m^2+n^2}\sum_{a=1}^{m}(-1)^{a-1}Y^a gh dz\wedge d\overline{z}\wedge\delta\eta^1\wedge\ldots\wedge\widehat{\delta\eta^a}\wedge\ldots\wedge\delta\eta^m\wedge\delta\overline{\eta}.$$

Using

$$d(\delta\eta^a) = \delta_{\overline{l}}(\overset{CF}{N_k^a})d\overline{z}^l\wedge dz^k + \dot{\partial}_b\ (\overset{CF}{N_k^a})\delta\eta^b\wedge dz^k + \dot{\partial}_{\overline{b}}\ (\overset{CF}{N_k^a})\delta\overline{\eta}^b\wedge dz^k$$

we get

$$\begin{aligned} d[i(Y)dV_E] &= i^{m^2+n^2}\sum_{b=1}^{m}\sum_{a=1}^{m}(-1)^{a-1}\ \dot{\partial}_b\ (Y^a gh)\delta\eta^b\wedge dz\wedge d\overline{z}\wedge\delta\eta^1\wedge\ldots\wedge \\ &\qquad \wedge\widehat{\delta\eta^a}\wedge\ldots\wedge\delta\eta^m\wedge\delta\overline{\eta} \\ &= i^{m^2+n^2}\ \dot{\partial}_a\ (Y^a gh)dz\wedge d\overline{z}\wedge\delta\eta\wedge\delta\overline{\eta}. \end{aligned}$$

But

$$\begin{aligned} &\dot{\partial}_a\ (Y^a gh) = g\ \dot{\partial}_a\ (Y^a h) = gh[\frac{1}{h}\ \dot{\partial}_a\ (Y^a h)] \\ &\quad = gh[\dot{\partial}_a\ Y^a + Y^a\frac{1}{h}\ \dot{\partial}_a\ (h)] = gh[\dot{\partial}_a\ Y^a + Y^a\ \dot{\partial}_a\ (\ln h)] = gh\ \mathrm{div}_v\, Y. \quad \Box \end{aligned}$$

Using (3.6) and Proposition 3.3. we get

Theorem 3.1. *Let (E, L) be a complex Finsler bundle over a Hermitian manifold (M, g). If $X = X^a\ \dot{\partial}_a \in \Gamma(VE)$ is with compact support on E, then*

$$\int_E \nabla_{\dot{\partial}_a} X^a dV_E = 0\ \ ;\int_E \nabla_{\dot{\partial}_{\overline{a}}} X^{\overline{a}} dV_E = 0. \tag{3.11}$$

In the following if we consider a smooth function f on E and a vertical vector field $X = X^a\ \dot{\partial}_a$ with compact support on E and $X^a = h^{\overline{b}a}\nabla_{\dot{\partial}_{\overline{b}}} f$ then, according to (2.6), the divergence of X is

$$\mathrm{div}\ X = \nabla_{\dot{\partial}_a}(h^{\overline{b}a}\nabla_{\dot{\partial}_{\overline{b}}} f) = h^{\overline{b}a}\nabla_{\dot{\partial}_a}\nabla_{\dot{\partial}_{\overline{b}}} f = \Box'^{v} f. \tag{3.12}$$

Thus, we have

Corollary 3.1. *If (E, L) is a complex Finsler bundle over a Hermitian manifold (M, g), then for any smooth function f with compact support on E, we have*

$$\int_E \Box'^v f dV_E = 0; \int_E h^{\overline{b}a} \nabla_{\dot{\partial}_a} \nabla_{\dot{\partial}_{\overline{b}}} f dV_E = 0. \tag{3.13}$$

4. $\Box''^v$-Laplacian for forms with compact support on E

In this section we define $\Box''^v$-Laplacian for (p, q, r, s)-forms with compact support on E. We first define an L^2-global Hermitian inner product on the space $\mathcal{A}_c^{p,q,r,s}(E)$ of forms with compact support on E. Secondly, we give the local expression of the adjoint operator δ''^v of d''^v with respect to this inner product and thus we can define the $\Box''^v$-Laplacian for these forms. Finally, we get the local expression of this Laplacian, explicitly in terms of vertical covariant derivatives of Chern-Finsler connection.

Let φ, $\psi \in \mathcal{A}_c^{p,q,r,s}$ be two forms with compact support on E locally given by

$$\varphi = \frac{1}{p!q!r!s!} \varphi_{I_p \overline{J}_q A_r \overline{B}_s} dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{A_r} \wedge \delta\overline{\eta}^{B_s},$$
$$\psi = \frac{1}{p!q!r!s!} \psi_{I_p \overline{J}_q A_r \overline{B}_s} dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{A_r} \wedge \delta\overline{\eta}^{B_s},$$

where

$$I_p = (i_1 \dots i_p),\ J_q = (j_1 \dots j_q),\ A_r = (a_1 \dots a_r),\ B_s = (b_1 \dots b_s),$$
$$dz^{I_p} = dz^{i_1} \wedge \dots \wedge dz^{i_p},\ d\overline{z}^{J_q} = d\overline{z}^{j_1} \wedge \dots \wedge d\overline{z}^{j_q},\ \delta\eta^{A_r} = \delta\eta^{a_1} \wedge \dots \wedge \delta\eta^{a_r}$$

and $\delta\overline{\eta}^{B_s} = \delta\overline{\eta}^{b_1} \wedge \dots \wedge \delta\overline{\eta}^{b_s}$.

We define

$$< \varphi, \psi >= \frac{1}{p!q!r!s!} \varphi_{I_p \overline{J}_q A_r \overline{B}_s} \overline{\psi^{\overline{I}_p J_q \overline{A}_r B_s}} = \sum \varphi_{I_p \overline{J}_q A_r \overline{B}_s} \overline{\psi^{\overline{I}_p J_q \overline{A}_r B_s}}. \tag{4.1}$$

Here the sum is after $i_1 < \dots < i_p$, $j_1 < \dots < j_q$, $a_1 < \dots < a_r$, $b_1 < \dots < b_s$ and

$$\psi^{\overline{I}_p J_q \overline{A}_r B_s} = \psi_{K_p \overline{L}_q C_r \overline{D}_s} g^{\overline{I}_p K_p} g^{\overline{L}_q J_q} h^{\overline{A}_r C_r} h^{\overline{D}_s B_s},$$

where $g^{\overline{I}_p K_p} = g^{\overline{i}_1 k_1} \dots g^{\overline{i}_p k_p}$ and $h^{\overline{A}_r C_r} = h^{\overline{a}_1 c_1} \dots h^{\overline{a}_r c_r}$.

We note that the above inner product is independent of the local coordinates on E, thus $< \varphi, \psi >$ is a global inner product on E. In particular we have

$$|\varphi|^2 =< \varphi, \varphi >= \frac{1}{p!q!r!s!} \varphi_{I_p \overline{J}_q A_r \overline{B}_s} \overline{\varphi^{\overline{I}_p J_q \overline{A}_r B_s}}.$$

Using (4.1), we define a global inner product on $\mathcal{A}_c^{p,q,r,s}(E)$ by

$$(\varphi, \psi) = \int_E < \varphi, \psi > dV_E,\ \|\varphi\|^2 = (\varphi, \varphi). \tag{4.2}$$

According to [14], the vertical differential operators for (p,q,r,s) -forms are defined by

$$d^{'v}\colon \mathcal{A}^{p,q,r,s}\to \mathcal{A}^{p,q,r+1,s},\ d^{''v}\colon \mathcal{A}^{p,q,r,s}\to \mathcal{A}^{p,q,r,s+1},$$

where

$$(d^{'v}\varphi)_{I_p\overline{J}_qa_1\ldots a_{r+1}\overline{B}_s}=(-1)^{p+q}\sum_{i=1}^{r+1}(-1)^{i-1}\,\dot{\partial}_{a_i}\,(\varphi_{I_p\overline{J}_qa_1\ldots\widehat{a}_i\ldots a_{r+1}\overline{B}_s}),\tag{4.3}$$

$$(d^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_1\ldots\overline{b}_{s+1}}=(-1)^{p+q+r}\sum_{i=1}^{s+1}(-1)^{i-1}\,\dot{\partial}_{\overline{b}_i}\,(\varphi_{I_p\overline{J}_qA_r\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_{s+1}}).\tag{4.4}$$

Because of the symmetry relation (2.6) and the skew symmetry of $\varphi_{I_p\overline{J}_qA_r\overline{B}_s}$ we can replace in (4.3) and (4.4) the vertical derivatives with the vertical covariant derivatives with respect to the Chern-Finsler connection. Thus we get

$$(d^{'v}\varphi)_{I_p\overline{J}_qa_1\ldots a_{r+1}\overline{B}_s}=(-1)^{p+q}\sum_{i=1}^{r+1}(-1)^{i-1}\nabla_{\dot{\partial}_{a_i}}(\varphi_{I_p\overline{J}_qa_1\ldots\widehat{a}_i\ldots a_{r+1}\overline{B}_s}),\tag{4.5}$$

$$(d^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_1\ldots\overline{b}_{s+1}}=(-1)^{p+q+r}\sum_{i=1}^{s+1}(-1)^{i-1}\nabla_{\dot{\partial}_{\overline{b}_i}}(\varphi_{I_p\overline{J}_qA_r\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_{s+1}}).\tag{4.6}$$

Let $\delta^{'v}$ and $\delta^{''v}$ be the adjoint operators of $d^{'v}$ and $d^{''v}$ respectively, with respect to the inner product defined in (4.2). That means

$$\delta^{'v}\colon \mathcal{A}_c^{p,q,r,s}\to \mathcal{A}_c^{p,q,r-1,s},\ (d^{'v}\varphi,\psi)=(\varphi,\delta^{'v}\psi),$$
$$\delta^{''v}\colon \mathcal{A}_c^{p,q,r,s}\to \mathcal{A}_c^{p,q,r,s-1},\ (d^{''v}\varphi,\psi)=(\varphi,\delta^{''v}\psi).$$

In the sequel, using an argument similar to that in [19] and [20], we obtain the local expression of the adjoint operator $\delta^{''v}$.

If (E,L) is a complex Finsler bundle over a Hermitian manifold (M,g), then according to (3.11) if $\varphi\in\mathcal{A}_c^{p,q,r,s}$ and $\psi\in\mathcal{A}_c^{p,q,r,s+1}$ we have

$$\int_E\nabla_{\dot{\partial}_{\overline{b}}}(\varphi_{I_p\overline{J}_qA_r\overline{B}_s}\overline{\psi^{\overline{I}_pJ_q\overline{A}_rbB_s}})dV_E=0.$$

Thus

$$\int_E\nabla_{\dot{\partial}_{\overline{b}}}(\varphi_{I_p\overline{J}_qA_r\overline{B}_s})\overline{\psi^{\overline{I}_pJ_q\overline{A}_rbB_s}}dV_E=-\int_E\varphi_{I_p\overline{J}_qA_r\overline{B}_s}(\overline{\nabla_{\dot{\partial}_b}\psi^{\overline{I}_pJ_q\overline{A}_rbB_s}})dV_E.$$

Using (2.12), we have

$$\begin{aligned}&\int_E\nabla_{\dot{\partial}_{\overline{b}}}(\varphi_{I_p\overline{J}_qA_r\overline{B}_s})\overline{\psi^{\overline{I}_pJ_q\overline{A}_rbB_s}}dV_E\\&\quad=-\int_E\varphi_{I_p\overline{J}_qA_r\overline{B}_s}g^{\overline{K}_pI_p}g^{\overline{J}_qL_q}h^{\overline{C}_rA_r}h^{\overline{B}_sD_s}\overline{h^{\overline{d}b}(\nabla_{\dot{\partial}_b}\psi_{K_p\overline{L}_qC_r\overline{d}\,\overline{D}_s})}dV_E.\end{aligned}\tag{4.7}$$

The term on the left-hand side of (4.7) reduces to

$$(-1)^{p+q+r}\int_E \nabla_{\dot{\partial}_{\overline{b}}}(\varphi_{I_p\overline{J}_qA_r\overline{B}_s})\overline{\psi^{b\overline{I}_pJ_q\overline{A}_rB_s}}dV_E = (d^{''v}\varphi,\psi)$$

and by setting

$$(\delta^{''v}\psi)_{K_p\overline{L}_qC_r\overline{D}_s} = -(-1)^{p+q+r}h^{\overline{d}b}\nabla_{\dot{\partial}_b}\psi_{K_p\overline{L}_qC_r\overline{d}\,\overline{D}_s} \tag{4.8}$$

we get from (4.7) that $(d^{''v}\varphi,\psi) = (\varphi,\delta^{''v}\psi)$. Thus, $\delta^{''v}$ from (4.8) is the adjoint of $d^{''v}$ with respect to the inner product defined by (4.2).

Remark 4.1. By the same calculus as above, we can prove that for $\varphi\in\mathcal{A}_c^{p,q,r+1,s}$ the local expression of $\delta^{'v}$ is given by

$$(\delta^{'v}\varphi)_{K_p\overline{L}_qC_r\overline{D}_s} = -(-1)^{p+q}h^{\overline{d}b}\nabla_{\dot{\partial}_{\overline{d}}}\varphi_{K_p\overline{L}_qbC_r\overline{D}_s}.$$

We define

$$\Box^{''v} = d^{''v}\delta^{''v} + \delta^{''v}d^{''v} : \mathcal{A}_c^{p,q,r,s}\to\mathcal{A}_c^{p,q,r,s} \tag{4.9}$$

which is a partial differential operator, called $\Box^{''v}$-*Laplacian* for forms with compact support on E. We have

Theorem 4.1. *Let (E,L) be a complex Finsler bundle over a Hermitian manifold (M,g). If $\varphi\in\mathcal{A}_c^{p,q,r,s}$, then*

$$\begin{aligned}(\Box^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{B}_s} &= -\sum_{c,d=1}^m h^{\overline{d}c}\nabla_{\dot{\partial}_c}\nabla_{\dot{\partial}_{\overline{d}}}\varphi_{I_p\overline{J}_qA_r\overline{B}_s}\\ &\quad+\sum_{c,d=1}^m\sum_{i=1}^s(-1)^{i-1}h^{\overline{d}c}[\nabla_{\dot{\partial}_c},\nabla_{\dot{\partial}_{\overline{b}_i}}]\varphi_{I_p\overline{J}_qA_r\overline{d}\,\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_s}.\end{aligned} \tag{4.10}$$

Proof. Using (2.12), (4.6) and (4.8) we get

$$\begin{aligned}(\delta^{''v}d^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{B}_s} &= (-1)^{p+q+r+1}\sum_{c,d=1}^m h^{\overline{d}c}[\nabla_{\dot{\partial}_c}(d^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{d}\,\overline{B}_s}]\\ &= -\sum_{c,d=1}^m h^{\overline{d}c}\nabla_{\dot{\partial}_c}\nabla_{\dot{\partial}_{\overline{d}}}\varphi_{I_p\overline{J}_qA_r\overline{B}_s}\\ &\quad+\sum_{c,d=1}^m\sum_{i=1}^s(-1)^{i-1}h^{\overline{d}c}\nabla_{\dot{\partial}_c}\nabla_{\dot{\partial}_{\overline{b}_i}}\varphi_{I_p\overline{J}_qA_r\overline{d}\,\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_s}\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}(4.12)\qquad &(d^{''v}\delta^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{B}_s}\\ &= (-1)^{p+q+r+1}\sum_{i=1}^{s}(-1)^{i-1}\nabla_{\dot{\partial}_{\overline{b}_i}}(\delta^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_s}\\ &= -\sum_{c,d=1}^{m}\sum_{i=1}^{s}(-1)^{i-1}h^{\overline{d}c}\nabla_{\dot{\partial}_{\overline{b}_i}}\nabla_{\dot{\partial}_c}\varphi_{I_p\overline{J}_qA_r\overline{d}\,\overline{b}_1\ldots\widehat{\overline{b}}_i\ldots\overline{b}_s}.\end{aligned}$$

Now, summing the relations (4.11) and (4.12) we get (4.10). □

In the following, as in Kähler geometry, we can define an analogue operator $\Lambda^v\colon \mathcal{A}_c^{p,q,r,s}(E)\to\mathcal{A}_c^{p,q,r-1,s-1}(E)$ by

$$\begin{aligned}(4.13)\qquad \Lambda^v\varphi = &\frac{i}{p!q!(r-1)!(s-1)!}h^{\overline{d}c}\varphi_{c\overline{d}I_p\overline{J}_qa_2\ldots a_r\overline{b}_2\ldots\overline{b}_s}dz^{I_p}\wedge d\overline{z}^{J_q}\wedge\\ &\wedge\delta\eta^{a_2}\wedge\ldots\wedge\delta\eta^{a_r}\wedge\delta\overline{\eta}^{b_2}\wedge\ldots\wedge\delta\overline{\eta}^{b_s}.\end{aligned}$$

Thus

$$\begin{aligned}(4.14)\qquad (\Lambda^v\varphi)_{I_p\overline{J}_qa_2\ldots a_r\overline{b}_2\ldots\overline{b}_s} &= ih^{\overline{d}c}\varphi_{c\overline{d}I_p\overline{J}_qa_2\ldots a_r\overline{b}_2\ldots\overline{b}_s}\\ &= (-1)^{r-1}ih^{\overline{d}c}\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s}.\end{aligned}$$

We have

Proposition 4.1. *Let (E,L) be a complex Finsler bundle over a Hermitian manifold (M,g). Then*

$$(4.15)\qquad d^{'v}\Lambda^v-\Lambda^vd^{'v}=-i\delta^{''v}\,;\; d^{''v}\Lambda^v-\Lambda^vd^{''v}=i\delta^{'v},$$

$$(4.16)\qquad \delta^{''v}\Lambda^v-\Lambda^v\delta^{''v}=0\,;\;\delta^{'v}\Lambda^v-\Lambda^v\delta^{'v}=0.$$

Proof. We will prove only the first relation of (4.15) and of (4.16) and the other two follow in the similar manner.

Let $\varphi\in\mathcal{A}_c^{p,q,r,s}$. According to (4.14), we have

$$(\Lambda^v\varphi)_{I_p\overline{J}_qa_2\ldots a_r\overline{b}_2\ldots\overline{b}_s} = (-1)^{r-1}ih^{\overline{d}c}\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s}.$$

By applying $d^{'v}$ from (4.5) and taking into account (2.12) we get

$$\begin{aligned}(4.17)\qquad &(d^{'v}\Lambda^v\varphi)_{I_p\overline{J}_qA_r\overline{b}_2\ldots\overline{b}_s}\\ &= (-1)^{p+q+r-1}ih^{\overline{d}c}\sum_{i=1}^{r}(-1)^{i-1}\nabla_{\dot{\partial}_{a_i}}(\varphi_{I_p\overline{J}_qca_1\ldots\widehat{a}_i\ldots a_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s}).\end{aligned}$$

On the other hand,

$$(d^{'v}\varphi)_{I_p\overline{J}_qa_0A_r\overline{B}_s} = (-1)^{p+q}\sum_{i=0}^{r}(-1)^i\nabla_{\dot{\partial}_{a_i}}(\varphi_{I_p\overline{J}_qa_0\ldots\widehat{a}_i\ldots a_r\overline{B}_s})$$

and by applying Λ^v we have

$$(4.18)\quad \begin{aligned}(\Lambda^v d^{'v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_2\ldots\overline{b}_s} &= (-1)^{p+q}(-1)^r ih^{\overline{d}c}\nabla_{\dot{\partial}_c}(\varphi_{I_p\overline{J}_qA_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s})\\ &+ (-1)^{p+q}(-1)^r ih^{\overline{d}c}\sum_{i=1}^{r}(-1)^i\nabla_{\dot{\partial}_{a_i}}(\varphi_{I_p\overline{J}_qca_1\ldots\widehat{a_i}\ldots a_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s}).\end{aligned}$$

Thus, from (4.17) and (4.18) we get

$$\begin{aligned}(d^{'v}\Lambda^v\varphi - \Lambda^v d^{'v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_2\ldots\overline{b}_s} &= (-1)^{p+q+r}ih^{\overline{d}c}\nabla_{\dot{\partial}_c}(\varphi_{I_p\overline{J}_qA_r\overline{d}\,\overline{b}_2\ldots\overline{b}_s})\\ &= -i(\delta^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_2\ldots\overline{b}_s}\end{aligned}$$

i.e. the first relation of (4.15).

For the first relation of (4.16), applying $\delta^{''v}$ to the relation (4.14) we have

$$(4.19)\quad \begin{aligned}&(\delta^{''v}\Lambda^v\varphi)_{I_p\overline{J}_qa_2\ldots a_r\overline{b}_3\ldots\overline{b}_s}\\ &\quad= -(-1)^{p+q+r-1}(-1)^{r-1}ih^{\overline{e}f}\nabla_{\dot{\partial}_f}(h^{\overline{d}c}\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{e}\,\overline{d}\,\overline{b}_3\ldots\overline{b}_s})\\ &\quad= -(-1)^{p+q}ih^{\overline{e}f}h^{\overline{d}c}\nabla_{\dot{\partial}_f}(\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{e}\,\overline{d}\,\overline{b}_3\ldots\overline{b}_s}).\end{aligned}$$

On the other hand, applying Λ^v to

$$(\delta^{''v}\varphi)_{I_p\overline{J}_qA_r\overline{b}_2\ldots\overline{b}_s} = -(-1)^{p+q+r}h^{\overline{e}f}\nabla_{\dot{\partial}_f}(\varphi_{I_p\overline{J}_qA_r\overline{e}\,\overline{b}_2\ldots\overline{b}_s})$$

we get

$$(4.20)\quad \begin{aligned}&(\Lambda^v\delta^{''v}\varphi)_{I_p\overline{J}_qa_2\ldots a_r\overline{b}_3\ldots\overline{b}_s}\\ &\quad= -(-1)^{p+q+r}(-1)^{r-1}ih^{\overline{d}c}h^{\overline{e}f}\nabla_{\dot{\partial}_f}(\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{d}\,\overline{e}\,\overline{b}_3\ldots\overline{b}_s})\\ &\quad= (-1)^{p+q}ih^{\overline{d}c}h^{\overline{e}f}\nabla_{\dot{\partial}_f}(\varphi_{I_p\overline{J}_qca_2\ldots a_r\overline{d}\,\overline{e}\,\overline{b}_3\ldots\overline{b}_s}).\end{aligned}$$

From (4.19) and (4.20) we have $(\delta^{''v}\Lambda^v\varphi - \Lambda^v\delta^{''v}\varphi)_{I_p\overline{J}_qa_2\ldots a_r\overline{b}_3\ldots\overline{b}_s} = 0.$ $\square$

If we define $\square^{'v} = d^{'v}\delta^{'v} + \delta^{'v}d^{'v}$ then, we have

Proposition 4.2. *Let (E, L) be a complex Finsler bundle over a Hermitian manifold (M, g). Then*

$$(4.21)\quad \square^{'v} = \square^{''v}.$$

Proof. According to the above proposition we have

$$\begin{aligned}\square^{'v} &= d^{'v}\delta^{'v} + \delta^{'v}d^{'v} = -id^{'v}(d^{''v}\Lambda^v - \Lambda^v d^{''v}) - i(d^{''v}\Lambda^v - \Lambda^v d^{''v})d^{'v}\\ &= -i(d^{'v}d^{''v}\Lambda^v - d^{'v}\Lambda^v d^{''v} + d^{''v}\Lambda^v d^{'v} - \Lambda^v d^{''v}d^{'v})\end{aligned}$$

and

$$\begin{aligned}\square^{''v} &= d^{''v}\delta^{''v} + \delta^{''v}d^{''v} = id^{''v}(d^{'v}\Lambda^v - \Lambda^v d^{'v}) + i(d^{'v}\Lambda^v - \Lambda^v d^{'v})d^{''v}\\ &= i(d^{''v}d^{'v}\Lambda^v - d^{''v}\Lambda^v d^{'v} + d^{'v}\Lambda^v d^{''v} - \Lambda^v d^{'v}d^{''v}).\end{aligned}$$

Taking into account that $d^{'v}d^{''v}+d^{''v}d^{'v}=0$ (see [14]), we obtain $\Box^{'v}=\Box^{''v}$. $\Box$

Proposition 4.3. $\Box^{''v}\varphi=0$ *if and only if* $\delta^{''v}\varphi=d^{''v}\varphi=0$.

Proof. Follows by direct calculus

$$\begin{aligned}(\Box^{''v}\varphi,\varphi)&=(d^{''v}\delta^{''v}\varphi,\varphi)+(\delta^{''v}d^{''v}\varphi,\varphi)\\&=(\delta^{''v}\varphi,\delta^{''v}\varphi)+(d^{''v}\varphi,d^{''v}\varphi)\\&=||\delta^{''v}\varphi||^2+||d^{''v}\varphi||^2.\end{aligned}$$
$\Box$

If f is a smooth function with compact support on E, we have $\delta^{''v}f=0$, then

$$\Box^{''v}f=\delta^{''v}d^{''v}f=h^{\overline{b}a}\nabla_{\dot{\partial}_a}\nabla_{\dot{\partial}_{\overline{b}}}f.$$

Denoting by $\mathcal{H}^{p,q,r,s}_c(E)=\ker\{\Box^{''v}\colon\mathcal{A}^{p,q,r,s}_c(E)\to\mathcal{A}^{p,q,r,s}_c(E)\}$ the set of $v^{''}$-harmonic forms with compact support on E, then using (2.12) it is easy to check

$$\Box^{''v}\Phi^v=0.$$

Acknowledgement

The author is grateful to the referees and would like to thank them for useful suggestions which improved this paper.

References

[1] M. Abate and G. Patrizio. *Finsler metrics—a global approach*, volume 1591 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. With applications to geometric function theory.

[2] T. Aikou. Finsler geometry on complex vector bundles. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 83–105. Cambridge Univ. Press, Cambridge, 2004.

[3] T. Aikou. The Chern-Finsler connection and Finsler-Kähler manifolds. In *Finsler geometry, Sapporo 2005—in memory of Makoto Matsumoto*, volume 48 of *Adv. Stud. Pure Math.*, pages 343–373. Math. Soc. Japan, Tokyo, 2007.

[4] P. L. Antonelli and B. C. Lackey, editors. *The theory of Finslerian Laplacians and applications*, volume 459 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1998.

[5] D. Bao and B. Lackey. A Hodge decomposition theorem for Finsler spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(1):51–56, 1996.

[6] J.-G. Cao and P.-M. Wong. Finsler geometry of projectivized vector bundles. *J. Math. Kyoto Univ.*, 43(2):369–410, 2003.

[7] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[8] S. Kobayashi. Negative vector bundles and complex Finsler structures. *Nagoya Math. J.*, 57:153–166, 1975.

[9] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5.

[10] S. Kobayashi. Complex Finsler vector bundles. In *Finsler geometry (Seattle, WA, 1995)*, volume 196 of *Contemp. Math.*, pages 145–153. Amer. Math. Soc., Providence, RI, 1996.
[11] R. Miron and M. Anastasiei. *The geometry of Lagrange spaces: theory and applications*, volume 59 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1994.
[12] G. Munteanu. *Complex spaces in Finsler, Lagrange and Hamilton geometries*, volume 141 of *Fundamental Theories of Physics*. Kluwer Academic Publishers, Dordrecht, 2004.
[13] O. Munteanu. Weitzenböck formulas for horizontal and vertical Laplacians. *Houston J. Math.*, 29(4):889–900 (electronic), 2003.
[14] G. Pitis and G. Munteanu. V-cohomology of complex Finsler manifolds. *Studia Univ. Babeş-Bolyai Math.*, 43(3):75–82, 1998.
[15] H. Shima. Vanishing theorems for compact Hessian manifolds. *Ann. Inst. Fourier (Grenoble)*, 36(3):183–205, 1986.
[16] I. Vaisman. *Cohomology and differential forms*. Marcel Dekker Inc., New York, 1973. Translation editor: Samuel I. Goldberg, Pure and Applied Mathematics, 21.
[17] P.-M. Wong. A survey of complex Finsler geometry. In *Finsler geometry, Sapporo 2005—in memory of Makoto Matsumoto*, volume 48 of *Adv. Stud. Pure Math.*, pages 375–433. Math. Soc. Japan, Tokyo, 2007.
[18] C. Zhong. A vanishing theorem on Kaehler Finsler manifolds. *Differential Geom. Appl.*, 27(4):551–565, 2009.
[19] C. Zhong and T. Zhong. Horizontal $\overline{\partial}$-Laplacian on complex Finsler manifolds. *Sci. China Ser. A*, 48(suppl.):377–391, 2005.
[20] C. Zhong and T. Zhong. Hodge decomposition theorem on strongly Kähler Finsler manifolds. *Sci. China Ser. A*, 49(11):1696–1714, 2006.

Department of Algebra, Geometry and Differential Equations,
Transilvania University of Braşov,
Address: Braşov 500091, Str. Iuliu Maniu 50
E-mail address: cristian.ida@unitbv.ro