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GENERALIZED LAGRANGE – HAMILTON SPACES OF ORDER k

IRENA ČOMIĆ

Abstract. In this paper the generalized Lagrange – Hamilton spaces are introduced. The group of transformation is given, further some complicated but useful relations concerning the partial derivatives of variables in new and old coordinate systems are derived. The sprays and antisprays are also studied.

Introduction

The $(k+1)n$ dimensional generalized Lagrange space is an $\operatorname{Osc}^k M$ space supplied with regular Lagrangian $L(x^a, y^{1a}, y^{2a}, \dots, y^{ka})$, where $y^{Aa} = \frac{d^A}{dt^A} x^a$, $A = \overline{1,k}$. They are studied in many papers and books as [2], [18], [19], [20], [16] and others. The K-Hamilton space is $(k+1)n$ dimensional space, where some point of this space has coordinates $(x^a, p_{1a}, \dots, p_{ka})$, where $p_{Aa} (A = \overline{1,k})$ are independent covector fields. If $k = 1$ we have Hamilton space. Such spaces are studied in [1], [3–7], [10–12], [14], [15], [21–23], [25], [26] and many others. If instead of covector fields $p_1, \dots, p_k$ we take independent vector fields $y^1, \dots, y^k$ we obtain K-Lagrange spaces.

The $(k+1)n$ dimensional Hamilton space of order k was introduced by R. Miron in [17]. Some point u of this space has coordinates

$$(x^a, y^{1a}, \dots, y^{(k-1)a}, p_{ka}),$$

where $y^{Aa} = \frac{1}{A!} \frac{d^A}{dt^A} x^a$, $A = \overline{1, k-1}$ and (p_{ka}) is a covector. The space is supplied with regular Hamiltonian $H(x, y^1, \dots, y^{k-1}, p_k)$ from which the metric tensor is derived in the usual manner. The complex structures in the above spaces are introduced in [24].

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The Hamilton spaces of higher order are introduced in [9]. A point u of this $(k+1)n$ dimensional space has coordinates $(x^a, p_{1a}, p_{2a}, \dots, p_{ka})$, where p_1 is a covector and $p_{Aa} = \frac{d^{A-1}}{dt^{A-1}} p_{1a}$, $A = \overline{2,k}$. In the transformation group the expressions $\frac{d^A}{dt^A} \frac{\partial x^{a'}}{\partial x^a}$ appear, which are functions of $y^1, y^2, \dots, y^A$, but they were not written explicitly and were not treated as variables. Such spaces are studied in [10, 13].

Here the $(2k+1) \cdot n$ dimensional generalized Lagrange–Hamilton spaces $(\mathrm{GLH})^{(nk)}$ are introduced, where some point $u \in (\mathrm{GLH})^{(nk)}$ has coordinates

$$(x^a, y^{1a}, \dots, y^{ka}, p_{1a}, p_{2a}, \dots, p_{ka}),$$

where $y^{(A+1)a} = \frac{d^A}{dt^A} y^{1a}$, $p_{(A+1)a} = \frac{d^A}{dt^A} p_{1a}$, $A = \overline{1, k-1}$. The group of transformation is given, further some complicated but useful relations concerning the partial derivatives of variables in new and old coordinate systems are derived. The natural and special adapted bases of $T(\mathrm{GLH})^{(nk)}$ and $T^*(\mathrm{GLH})^{(nk)}$ are established. Using the matrix representation, the duality of bases in T and T^* is proved. The name 'special adapted' comes from the fact that the elements of these bases are transforming as tensors and they have the property that the J structure in the natural and special adapted bases has the same components. By action of the J structure on the vector field dr and 1-form field δr the corresponding Liouville vector and 1-form fields are constructed.

The sprays and antisprays are also studied and interesting results are obtained. Here the metric tensor does not appear. If the regular Lagrangian L and Hamiltonian H is given the metric tensor can be derived by the usual manner. From this the metric connection, the torsion and curvature tensors and the structure equations can be obtained, which will be the subject of next papers. The application of this theory is given in variation calculus in the papers which will be appeared.

Special cases of $(\mathrm{GLH})^{(nk)}$ are Lagrange spaces of order k, $\operatorname{Osc}^k M$ spaces, Finsler spaces, generalized Hamilton spaces and so on.

1. Definitions, group of coordinate transformation

Let us denote by $(\mathrm{LH})^{(n1)}$ the $3n$ dimensional C^∞ manifold in which some point (y,p) has coordinates $(x^a = y^{0a}, y^{1a}, p_{1a})$, $a = \overline{1,n}$.

Some curve c in $(\mathrm{LH})^{(n1)}$ is given by $c\colon t \in [a,b] \to c(t) \in (\mathrm{LH})^{(n1)}$, where in some local chart (U, φ) a point $(y,p) \in c(t)$ has coordinates

$$(x^a(t) = y^{0a}(t), y^{1a}(t), p_{1a}(t)).$$

If in some other chart (U', φ') the same point (y,p) has coordinates

$$(x^{a'}(t) = y^{0a'}(t), y^{1a'}(t), p_{1a'}(t)),$$

then the allowable transformations are given by

$$
\begin{aligned}
&x^{a'} = x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'}), \quad (x^a(t) = x^a(x^{a'}(t)),\\
&y^{1a'} = B^{a'}_a y^{1a}, \quad B^{a'}_a = \frac{\partial x^{a'}}{\partial x^a} = B^{a'}_a(t), \quad p_{1a'} = B^a_{a'} p_{1a}.
\end{aligned}
\tag{1.1}
$$

The first two equations in (1.1) give the coordinate transformation in the Finsler space. Here, in $(\mathrm{LH})^{(n1)}$ the point of the space has three components: $(x) = (x^a)$ - the point in the base manifold M, the contravariant vector field $(y^{(1)}) = (y^{1a})$ and a covariant vector field $(p_{(1)}) = (p_{1a})$.

$(y^{(1)})$ can be interpreted as the velocity vector and $(p_{(1)})$ as the generalized momentum.

Let us denote by $(\mathrm{LH})^{(nk)}$ the $(2k+1)n$ dimensional C^∞ manifold in which a point $(y,p) = (x = y^{(0)}, y^{(1)}, y^{(2)}, \dots, y^{(k)}, p_{(1)}, p_{(2)}, \dots, p_{(k)})$ has coordinates

$$(x^a = y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}, p_{1a}, p_{2a}, \dots, p_{ka}), \quad a = \overline{1,n}.$$

We can interpret the point in $(\mathrm{LH})^{(nk)}$ as a point (x) in the base manifold together with a contravariant vector $(y^{(1)})$, covariant vector $(p_{(1)})$ and their derivatives up to order k.

Some curve $c \in (\mathrm{LH})^{(nk)}$ is given by

$$c\colon t \in [a,b] \to c(t) \in (\mathrm{LH})^{(nk)}.$$

A point $(y,p) \in c(t)$ has coordinates

$$(x^a(t) = y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t)),$$

where

$$
\begin{aligned}
&y^{Aa}(t) = d^A_t y^{0a}(t) \quad A = \overline{1,k} \quad d^A_t = \frac{d^A}{dt^A}\\
&p_{\alpha a}(t) = d^{\alpha-1}_t p_{1a}(t), \quad \alpha = \overline{1,k}, \quad d^{\alpha-1}_t = \frac{d^{\alpha-1}}{dt^{\alpha-1}}.
\end{aligned}
\tag{1.2}
$$

The allowable coordinate transformations are given by

$$
\begin{aligned}
&x^{a'} = x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'})\\
&y^{1a'} = B^{a'}_a y^{1a}, \quad B^{a'}_a = \partial_{0a} x^{a'} = \partial_a x^{a'}, \partial_{Aa} = \frac{\partial}{\partial y^{Aa}} \quad A = \overline{0,k},\\
&y^{2a'} = \binom{1}{0}(d^1_t B^{a'}_a) y^{1a} + \binom{1}{1} B^{a'}_a y^{2a} = d^1_t(B^{a'}_a y^{1a}),\\
&\vdots\\
&y^{ka'} = \binom{k-1}{0}(d^{k-1}_t B^{a'}_a) y^{1a} + \binom{k-1}{1}(d^{k-2}_t B^{a'}_a) y^{2a} + \cdots\\
&\qquad + \binom{k-1}{k-1} B^{a'}_a y^{ka} = d^{k-1}_t(B^{a'}_a y^{1a}),
\end{aligned}
\tag{1.3a}
$$

$$
(1.3b) \quad
\begin{aligned}
&p_{1a'} = B^a_{a'} p_{1a} \quad B^a_{a'} = \partial_{0a'} x^a = \frac{\partial x^a}{\partial x^{a'}} = B^a_{a'}(t),\\
&p_{2a'} = \binom{1}{0}(d^1_t B^a_{a'}) p_{1a} + \binom{1}{1} B^a_{a'} p_{2a} = d^1_t (B^a_{a'} p_{1a}),\\
&\vdots\\
&p_{ka'} = \binom{k-1}{0}(d^{k-1}_t B^a_{a'}) p_{1a} + \binom{k-1}{1}(d^{k-2}_t B^a_{a'}) p_{2a} + \cdots\\
&\quad + \binom{k-1}{k-1} B^a_{a'} p_{ka}.
\end{aligned}
$$

Theorem 1.1. *The transformations of type (1.3) on the common domain form a group.*

The proof is similar to those given in [8] and [9].

Definition 1.1. The generalized Lagrange–Hamilton space of order k, $(\mathrm{GLH})^{(nk)}$ is a $(\mathrm{LH})^{(nk)}$ space, where the allowable coordinate transformations are given by (1.3) and in which a differentiable Lagrangian $L(x, y^{(1)}, y^{(2)}, \dots, y^{(k)})$ and a differentiable Hamiltonian $H(x, p_{(1)}, p_{(2)}, \dots, p_{(k)})$ are given.

From (1.3) it is not obvious that $p_{\alpha a'}$, $\alpha = \overline{1,k}$ are functions of y^{Aa}, $A = \overline{0, \alpha - 1}$. This can be seen if we write:

$$
(1.4) \quad
\begin{aligned}
d^1_t B^{a'}_a &= \partial_{a_1} B^{a'}_a y^{1a_1}, \quad \partial_{a_1} = \frac{\partial}{\partial x^{a_1}}\\
d^2_t B^{a'}_a &= (\partial^2 a_2 a_1 B^{a'}_a) y^{1a_1} y^{1a_2} + (\partial_{a_1} B^{a'}_a) y^{2a_1}, \ \partial^2_{a_2 a_1} = \frac{\partial^2}{\partial y^{0a_2} \partial y^{0a_1}},\\
d^3_t B^{a'}_a &= (\partial^3_{a_3 a_2 a_1} B^{a'}_a) y^{1a_1} y^{1a_2} y^{1a_3} + 3(\partial^2_{a_2 a_1} B^{a'}_a) y^{1a_1} y^{2a_2} + (\partial_{a_1} B^{a'}_a) y^{3a_1},\\
d^4_t B^{a'}_a &= (\partial^4_{a_4 a_3 a_2 a_1} B^{a'}_a) y^{1a_1} y^{1a_2} y^{1a_3} y^{1a_4} + 6(\partial^3_{a_3 a_2 a_1} B^{a'}_a) y^{1a_1} y^{1a_2} y^{2a_3}\\
&\quad + 3(\partial^2_{a_2 a_1} B^{a'}_a) y^{2a_1} y^{2a_2} + 4(\partial^2_{a_2 a_1} B^{a'}_a) y^{1a_1} y^{3a_2} + (\partial_{a_1} B^{a'}_a) y^{4a_1},\\
&\vdots
\end{aligned}
$$

From (1.4) we can obtain another set of formulae if we make the changes $(a', a, a_1, a_2, a_3, a_4) \to (a, a', a'_1, a'_2, a'_3, a'_4)$.

From (1.3) and (1.4) it follows

$$
\begin{aligned}
y^{0a'} &= y^{0a'}(y^{0a})\\
y^{1a'} &= y^{1a'}(y^{0a}, y^{1a}), \dots,\\
y^{ka'} &= y^{ka'}(y^{0a}, y^{1a}, \dots, y^{ka}),\\
p_{1a'} &= p_{1a'}(y^{0a}, p_{1a}),\\
p_{2a'} &= p_{2a'}(y^{0a}, y^{1a}, p_{1a}, p_{2a}), \dots,\\
p_{ka'} &= p_{ka'}(y^{0a}, y^{1a}, \dots, y^{(k-1)a}, p_{1a}, p_{2a}, \dots, p_{ka}).
\end{aligned} \tag{1.5}
$$

Theorem 1.2. *The following relation is valid:*

$$
\begin{aligned}
d_t^A B_a^{a'} &= (\partial_a d_t^{A-1} B_b^{a'}) y^{1b} + \binom{A-1}{1} (\partial_a d_t^{A-2} B_b^{a'}) y^{2b} + \cdots\\
&+ \binom{A-1}{A-2} (\partial_a d_t^1 B_b^{a'}) y^{(A-1)b} + \binom{A-1}{A-1} (\partial_a B_b^{a'}) y^{Ab} = \partial_a y^{Aa'}
\end{aligned} \tag{1.6}
$$

for $A = \overline{1, k}$.

Proof. From the relations

$$
\begin{aligned}
(d_t^1 B_a^{a'}) &= (\partial_b B_a^{a'}) y^{1b} = (\partial_a B_b^{a'}) y^{1b},\\
d_t^A B_a^{a'} &= d_t^{A-1}(d_t^1 B_a^{a'}) = d_t^{A-1}[(\partial_a B_b^{a'}) y^{1b}]
\end{aligned}
$$

and the Leibniz rule for differentiation the first part of (1.6) follows. As $y^{0b} = x^b, y^{1b}, \dots, y^{Ab}$ are independent variables, from the right hand side of (1.6) we can take out ∂_a and the comparison with the obtained equation with $y^{Aa'}$ from (1.3) results in the second part of (1.6). □

Theorem 1.3. *The partial derivatives of the variables, $d_t^A B_a^{a'}$, $d_t^\alpha B_{a'}^a$ are connected by the following formulae:*

$$
\begin{aligned}
&\partial_{0a} y^{0a'} = \partial_{1a} y^{1a'} = \cdots = \partial_{ka} y^{ka'} = B_a^{a'}\\
&\partial_a y^{Aa'} = \partial_{0a} y^{Aa'} = d_t^A B_a^{a'} \quad A = \overline{1, k},\\
&\partial_{Aa} y^{(A+B)a'} = \frac{A+B}{A} \partial_{(A-1)a} y^{(A+B-1)a'} = \cdots\\
&\quad = \binom{A+B}{A} d_t^B B_a^{a'} = \binom{A+B}{A} \partial_{0a} y^{Ba'},\\
&\partial^{1a} p_{1a'} = \partial^{2a} p_{2a'} = \cdots \partial^{ka} p_{ka'} = B_{a'}^a,\\
&\partial^{\alpha a} = \frac{\partial}{\partial p_{\alpha a}} \quad \alpha = \overline{1, k},\\
&\partial^{\alpha a} p_{(\alpha+\beta)a'} = \binom{\alpha+\beta-1}{\alpha-1} \partial^{1a} p_{(\beta+1)a'} = \binom{\alpha+\beta-1}{\alpha-1} d_t^\beta B_{a'}^a.
\end{aligned} \tag{1.7}
$$

Proof. From (1.4) it is obvious that $d_t^A B_a^{a'}$ $A = \overline{0,k}$ are functions only of $y^{0a}, y^{1a}, \dots, y^{Aa}$ so $d_t^A\ B_{a'}^a$ are functions only of $y^{0a'}, y^{1a'}, \dots, y^{Aa'}$. From this and the second part of (1.3) we can conclude that $p_{\alpha a'}$ are linear functions of $p_{1a}, p_{2a}, \dots, p_{\alpha a}$ $\alpha = \overline{1,k}$. This fact results in the following equations:

$$
\begin{aligned}
&\partial^{1a} p_{1a'} = B_{a'}^a,\\
&\partial^{1a} p_{2a'} = \binom{1}{0} d_t^1 B_{a'}^a, \quad \partial^{2a} p_{2a'} = \binom{1}{1} B_{a'}^a,\\
&\partial^{1a} p_{3a'} = \binom{2}{0} d_t^2 B_{a'}^a, \quad \partial^{2a} p_{3a'} = \binom{2}{1} d_t^1 B_{a'}^a, \quad \partial^{3a} p_{3a'} = \binom{2}{2} B_{a'}^a, \dots,\\
&\partial^{1a} p_{\alpha a'} = \binom{\alpha-1}{0} d_t^{\alpha-1} B_{a'}^a, \quad \partial^{2a} p_{\alpha a'} = \binom{\alpha-1}{1} d_t^{\alpha-2} B_{a'}^a, \dots,\\
&\partial^{\alpha a} p_{\alpha a'} = \binom{\alpha-1}{\alpha-1} B_{a'}^a, \dots .
\end{aligned}
$$

The above equations are the last three equations from (1.7).

Using (1.2) and (1.5) we can write (1.3a) in the form

$$
\begin{aligned}
y^{1a'} &= (\partial_{0a} y^{0a'}) y^{1a} = B_a^{a'} y^{1a} = d_t^1 y^{0a'}\\
y^{2a'} &= (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a} = d_t^1 y^{1a'}\\
y^{3a'} &= (\partial_{0a} y^{2a'}) y^{1a} + (\partial_{1a} y^{2a'}) y^{2a} + (\partial_{2a} y^{2a'}) y^{3a} = d_t^1 y^{2a'}, \dots,\\
y^{Aa'} &= (\partial_{0a} y^{(A-1)a'}) y^{1a} + (\partial_{1a} y^{(A-1)a'}) y^{2a} + \dots + (\partial_{(A-1)a} y^{(A-1)a'}) y^{Aa}\\
&= d_t^1 y^{(A-1)a'}, \dots,\\
y^{ka'} &= (\partial_{0a} y^{(k-1)a'}) y^{1a} + (\partial_{1a} y^{(k-1)a'}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a'}) y^{ka}\\
&= d_t^1 y^{(k-1)a'}.
\end{aligned} \tag{1.8}
$$

If we compare $y^{1a'}, y^{2a'}, \dots, y^{Aa'}, \dots, y^{ka'}$ from (1.3a) and (1.8) we get

$$y^{1a'} : \partial_{0a} y^{0a'} = B_a^{a'}$$

$$
\begin{aligned}
y^{2a'} : \qquad (\partial_{0a} y^{1a'}) &= \binom{1}{0} d_t^1 B_a^{a'},\\
\partial_{1a} y^{1a'} &= \binom{1}{1} B_a^{a'},
\end{aligned}
$$

$$
\begin{aligned}
y^{Aa'} : \ \partial_{0a} y^{(A-1)a'} &= \binom{A-1}{0} d_t^{A-1} B_a^{a'},\\
\partial_{1a} y^{(A-1)a'} &= \binom{A-1}{1} d_t^{A-2} B_a^{a'}, \dots
\end{aligned}
$$

$$\partial_{(A-1)a}y^{(A-1)a} = \binom{A-1}{A-1}B_a^{a'}, \dots$$

$$\begin{aligned}
y^{ka'}:\ \partial_{0a}y^{(k-1)a'} &= \binom{k-1}{0}d_t^{k-1}B_a^{a'},\\
\partial_{1a}y^{(k-1)a'} &= \binom{k-1}{1}d_t^{k-2}B_a^{a'}, \dots\\
\partial_{(k-1)a}y^{(k-1)a} &= \binom{k-1}{k-1}B_a^{a'}.
\end{aligned}$$

The above equations are the explicit form of the first three equations from (1.7). □

From (1.5) it can be seen that it is reasonable to calculate $\partial_{Aa}p_{\alpha a}$ for $A-1 \leq \alpha$. We have

Theorem 1.4. *The following relation is valid:*

$$\partial_{\alpha a}p_{(\alpha+\beta)a'} = \binom{\alpha+\beta-1}{\beta-1}\partial_{0a}p_{\beta a'}. \tag{1.9}$$

Proof. From (1.5) we have $p_{1a'} = p_{1a'}(y^{0a}, p_{1a})$ and

$$\begin{aligned}
p_{2a'} &= d_t^1 p_{1a'} = (\partial_{0a}p_{1a'})y^{1a} + (\partial^{1a}p_{1a'})p_{2a} = \partial_{0a}(p_{1a'}y^{1a}) + B_{a'}^a p_{2a},\\
p_{3a'} &= d_t^1 p_{2a'} = \partial_{0a}\left[\binom{1}{1}p_{2a'}y^{1a} + \binom{1}{0}p_{1a'}y^{2a}\right]\\
&\quad + \binom{1}{0}d_t^1 B_{a'}^a p_{2a} + \binom{1}{1}B_{a'}^a p_{3a},
\end{aligned}$$

and from

$$\begin{aligned}
&p_{2a'} = p_{2a'}(y^{0a}, y^{1a}, p_{1a}, p_{2a}) \text{ we get}\\
&p_{3a'} = (\partial_{0a}p_{2a'})y^{1a} + (\partial_{1a}p_{2a'})y^{2a} + (\partial^{1a}p_{2a'})p_{2a} + (\partial^{2a}p_{2a'})p_{3a}.
\end{aligned}$$

The comparison of the two expressions for $p_{3a'}$ gives

$$y^{2a}: \partial_{1a}p_{2a'} = \binom{1}{0}\partial_{0a}p_{1a}; \quad p_{2a}: \partial^{1a}p_{2a'} = \binom{1}{0}d_t^1 B_{a'}^a.$$

Further we have

$$\begin{aligned}
p_{4a'} &= d_t^2 p_{2a'} = \partial_{0a}\left[\binom{2}{2}p_{3a'}y^{1a} + \binom{2}{1}p_{2a'}y^{2a} + \binom{2}{0}p_{1a'}y^{3a}\right] +\\
&\qquad \binom{2}{0}(d_t^2 B_a^a)p_{2a} + \binom{2}{1}(d_t^2 B_{a'}^a)p_{3a} + \binom{2}{2}B_{a'}^a p_{4a},\\
p_{4a'} &= d_t^1 p_{3a'} = (\partial_{0a}p_{3a'})y^{1a} + (\partial_{1a}p_{3a'})y^{2a} + (\partial_{2a}p_{3a'})y^{3a} +\\
&\qquad (\partial^{1a}p_{3a'})p_{2a} + (\partial^{2a}p_{3a'})p_{3a} + (\partial^{3a}p_{3a'})p_{4a}.
\end{aligned}$$

The comparison of two above equations gives:

$$y^{2a} : \partial_{1a} p_{3a'} = \binom{2}{1} \partial_{0a} p_{2a'} \qquad y^{3a} : \partial_{2a} p_{3a'} = \binom{2}{0} \partial_{0a} p_{1a'}$$

$$p_2 : \partial^{1a} p_{3a'} = \binom{2}{0} d_t^2 B^a_{a'} \qquad p_{3a} : \partial^{2a} p_{3a'} = \binom{2}{1} d_t^1 B^a_{a'} a',$$

$$p_{4a} : \partial^{3a} p_{3a'} = \binom{2}{2} B^a_{a'}.$$

In the similar way comparing the relations

$$p_{\alpha a'} = \partial_{0a} d_t^{\alpha-2}(p_{1a'} y^{1a}) + d_t^{\alpha-2}(B^a_{a'} p_{2a}),$$
$$p_{\alpha a'} = d_t^1 p_{(\alpha-1)a'}(y^{0a}, y^{1a}, \dots, y^{(\alpha-2)a}, p_{1a}, p_{2a}, \dots, p_{(\alpha-1)a})$$

we obtain (1.9). □

2. The natural and special adapted bases in $T(\mathrm{GLH})^{(nk)}$ and $T^*(\mathrm{GLH})^{(nk)}$

The natural basis, $\bar{B}_{\mathrm{LH}}$ of $T(\mathrm{GLH})^{(nk)}$ as usual consists of partial derivatives of variables, i.e. $\bar{B}_{\mathrm{LH}} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}, \partial^{1a}, \partial^{2a}, \dots, \partial^{ka}\}$,

$$\partial_{0a} = \partial_a = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}, \ \partial_{Aa} = \frac{\partial}{\partial y^{Aa}}, \ A = \overline{1,k}, \ \partial^{\alpha a} = \frac{\partial}{\partial p_{\alpha a}}, \ \alpha = \overline{1,k}. \tag{2.1}$$

Theorem 2.1. *The elements of $\bar{B}_{\mathrm{LH}}$ are transforming in the following way:*

$$\begin{aligned}
\partial_{0a} &= (\partial_{0a} y^{0a'})\partial_{0a'} + (\partial_{0a} y^{1a'})\partial_{1a'} + \dots + (\partial_{0a} y^{ka'})\partial_{ka'} \\
&\quad + (\partial_{0a} p_{1a'})\partial^{1a'} + (\partial_{0a} p_{2a'})\partial^{2a'} + \dots + (\partial_{0a} p_{ka'})\partial^{ka'}, \\
\partial_{1a} &= (\partial_{1a} y^{1a'})\partial_{1a'} + (\partial_{1a} y^{2a'})\partial_{2a'} + \dots + (\partial_{1a} y^{ka'})\partial_{ka'} \\
&\quad + (\partial_{1a} p_{2a'})\partial^{2a'} + (\partial_{1a} p_{3a'})\partial^{3a'} + \dots + (\partial_{1a} p_{ka'})\partial^{ka'}, \\
\partial_{2a} &= (\partial_{2a} y^{2a'})\partial_{2a'} + (\partial_{2a} y^{3a'})\partial_{3a'} + \dots + (\partial_{2a} y^{ka'})\partial_{ka'} \\
&\quad + (\partial_{2a} p_{3a'})\partial^{3a'} + \dots + (\partial_{2a} p_{ka'})\partial^{ka'}, \dots \\
\partial_{ka} &= (\partial_{ka} y^{ka'})\partial_{ka'}
\end{aligned} \tag{2.2a}$$

$$\begin{aligned}
\partial^{1a} &= (\partial^{1a} p_{1a'})\partial^{1a'} + (\partial^{1a} p_{2a'})\partial^{2a'} + \dots + (\partial^{1a} p_{ka'})\partial^{ka'}, \\
\partial^{2a} &= (\partial^{2a} p_{2a'})\partial^{2a'} + (\partial^{2a} p_{3a'})\partial^{3a'} + \dots + (\partial^{2a} p_{ka'})\partial^{ka'}, \\
\partial^{3a} &= (\partial^{3a} p_{3a'})\partial^{3a'} + \dots + (\partial^{3a} p_{ka'})\partial^{ka'}, \dots \\
\partial^{ka} &= (\partial^{ka} p_{ka'})\partial^{ka'}.
\end{aligned} \tag{2.2b}$$

The proof follows from (1.5).

Let us introduce the notations

$$[\partial_{Aa} y]_{1,k+1} = [\partial_{0a} \partial_{1a} \dots \partial_{ka}], \quad [\partial^{\alpha a} p]_{1,k} = [\partial^{1a} \partial^{2a} \dots \partial^{ka}] \tag{2.3}$$

$$(2.4)\qquad [A_a^{a'}]_{k+1,k+1} = \begin{bmatrix} \partial_{0a}y^{0a'} & 0 & 0 & \cdots & 0 \\ \partial_{0a}y^{1a'} & \partial_{1a}y^{1a'} & 0 & \cdots & 0 \\ \partial_{0a}y^{2a'} & \partial_{1a}y^{2a'} & \partial_{2a}y^{2a'} & \cdots & 0 \\ \vdots & & & & \\ \partial_{0a}y^{ka'} & \partial_{1a}y^{ka'} & \partial_{2a}y^{ka'} & \cdots & \partial_{ka}y^{ka'} \end{bmatrix}$$

$$(2.5)\qquad [B_{aa'}]_{k,k+1} = \begin{bmatrix} \partial_{0a}p_{1a'} & 0 & 0 & \cdots & 0 & 0 \\ \partial_{0a}p_{2a'} & \partial_{1a}p_{2a'} & 0 & \cdots & 0 & 0 \\ \partial_{0a}p_{3a'} & \partial_{1a}p_{3a'} & \partial_{2a}p_{3a'} & \cdots & 0 & 0 \\ \vdots & & & & & \\ \partial_{0a}p_{ka'} & \partial_{1a}p_{ka'} & \partial_{2a}p_{ka'} & \cdots & \partial_{(k-1)a}p_{ka'} & 0 \end{bmatrix}$$

$$[0] = [0]_{k+1,k}$$

$$(2.6)\qquad [C_{a'}^{a}]_{k,k} = \begin{bmatrix} \partial^{1a}p_{1a'} & 0 & \cdots & 0 \\ \partial^{1a}p_{2a'} & \partial^{2a}p_{2a'} & \cdots & 0 \\ \vdots & & & \\ \partial^{1a}p_{ka'} & \partial^{2a}p_{ka'} & \cdots & \partial^{ka}p_{ka'} \end{bmatrix}.$$

Using the above notations (2.2) can be written in the form:

$$(2.7)\qquad [\partial_{Aa}y\partial^{\alpha a}p]_{1,2k+1} = [\partial_{a'}y\partial^{a'}p]_{1,2k+1} \begin{bmatrix} A_a^{a'} & 0 \\ B_{aa'} & C_{a'}^{a} \end{bmatrix}_{2k+1,2k+1}$$

Using (1.7) and (1.9) the elements of matrices $[A_a^{a'}]$, $[B_{aa'}]$, $[C_{a'}^{a}]$ can be written in the form

$$(2.8)\qquad [A_a^{a'}]_{k+1,k+1} = \begin{bmatrix} \binom{0}{0}B_a^{a'} & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}d_t^1B_a^{a'} & \binom{1}{1}B_a^{a'} & 0 & 0 & \cdots & 0 \\ \binom{2}{0}d_t^2B_a^{a'} & \binom{2}{1}d_t^1B_a^{a'} & \binom{2}{2}B_a^{a'} & 0 & \cdots & 0 \\ \binom{3}{0}d_t^3B_a^{a'} & \binom{3}{1}d_t^2B_a^{a'} & \binom{3}{2}d_t^1B_a^{a'} & \binom{3}{3}B_a^{a'} & \cdots & 0 \\ \vdots & & & & & \\ \binom{k}{0}d_t^kB_a^{a'} & \binom{k}{1}d_t^{k-1}B_a^{a'} & \binom{k}{2}d_t^{k-2}B_a^{a'} & & \cdots & \binom{k}{k}B_a^{a'} \end{bmatrix}$$

$$[B_{aa'}]_{k,k+1} =$$

$$(2.9)\qquad = \begin{bmatrix} \partial_{0a}p_{1a'} & 0 & 0 & 0\cdots & 0 & 0 \\ \partial_{0a}p_{2a'} & \binom{1}{0}\partial_{0a}p_{1a'} & 0 & 0 & 0 & 0 \\ \partial_{0a}p_{3a'} & \binom{2}{1}\partial_{0a}p_{2a'} & \binom{2}{0}\partial_{0a}p_{1a'} & 0 & 0 & 0 \\ \vdots & & & & & \\ \partial_{0a}p_{ka'} & \binom{k-1}{k-2}\partial_{0a}p_{(k-1)a'} & \binom{k-1}{k-3}\partial_{0a}p_{(k-2)a'} & \cdots & \binom{k-1}{0}\partial_{0a}p_{1a'} & 0 \end{bmatrix}$$

$$(2.10)\qquad [C^a_{a'}]_{k,k} = \begin{bmatrix} \binom{0}{0}B^a_{a'} & 0 & 0 & \cdots & 0 \\ \binom{1}{0}d^1_tB^a_{a'} & \binom{1}{1}B^a_{a'} & 0 & \cdots & 0 \\ \binom{2}{0}d^2_tB^a_{a'} & \binom{2}{1}d^1_tB^a_{a'} & \binom{2}{2}B^a_{a'} & & 0 \\ \vdots & & & & \\ \binom{k-1}{0}d^{k-1}_tB^a_{a'} & \binom{k-1}{1}d^{k-2}_tB^a_{a'} & \binom{k-1}{2}d^{k-3}_tB^a_{a'} & & \binom{k-1}{k-1}B^a_{a'} \end{bmatrix}$$

The natural basis of $T^*(\mathrm{GLH})^{(nk)}$ is

$$\bar{B}^*_{\mathrm{LH}} = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}, dp_{1a}, dp_{2a}, \dots, dp_{ka}\}.$$

Theorem 2.2. *The elements of $\bar{B}^*_{\mathrm{LH}}$ are transforming in the following way:*

$$(2.11)\qquad \begin{aligned} dy^{0a'} &= (\partial_{0a}y^{0a'})dy^{0a} \\ dy^{1a'} &= (\partial_{0a}y^{1a'})dy^{0a} + (\partial_{1a}y^{1a'})dy^{1a}, \dots, \\ dy^{ka'} &= (\partial_{0a}y^{ka'})dy^{0a} + (\partial_{1a}y^{ka'})dy^{1a} + \cdots + (\partial_{ka}y^{ka'})dy^{ka}, \\ dp_{1a'} &= (\partial_{0a}p_{1a'})dy^{0a} + (\partial^{1a}p_{1a'})dp_{1a}, \\ dp_{2a'} &= (\partial_{0a}p_{2a'})dy^{0a} + (\partial_{1a}p_{2a'})dy^{1a} + (\partial^{1a}p_{2a'})dp_{1a} + (\partial^{2a}p_{2a'})dp_{2a}, \\ &\ \ \vdots \\ dp_{ka'} &= (\partial_{0a}p_{ka'})dy^{0a} + (\partial_{1a}p_{ka'})dy^{1a} + \cdots + (\partial_{(k-1)a}p_{ka'})dy^{(k-1)a} + \\ &\quad (\partial^{1a}p_{ka'})dp_{1a} + \cdots + (\partial^{ka}p_{ka'})dp_{ka}. \end{aligned}$$

Let us introduce the notations

$$(2.12)\qquad [dy^a]_{k+1,1} = \begin{bmatrix} dy^{0a} \\ dy^{1a} \\ \vdots \\ dy^{ka} \end{bmatrix}, \quad [dp_a]_{k,1} = \begin{bmatrix} dp_{1a} \\ dp_{2a} \\ \vdots \\ dp_{ka} \end{bmatrix}.$$

Using notations (2.8), (2.9) and (2.10) we can write (2.11) in the form

$$\begin{bmatrix} dy^{a'} \\ dp_{a'} \end{bmatrix}_{2k+1,1} = \begin{bmatrix} A_a^{a'} & 0 \\ B_{aa'} & C_{a'}^a \end{bmatrix}_{2k+1,2k+1} \begin{bmatrix} dy^a \\ dp_a \end{bmatrix}. \tag{2.13}$$

Theorem 2.3. *If the bases $\bar{B}^*_{\mathrm{LH}}$ and $\bar{B}_{\mathrm{LH}}$ are dual to each other, then the bases $\bar{B}^{*'}_{\mathrm{LH}}$ and $\bar{B}'_{\mathrm{LH}}$ are also dual to each other.*

Proof. Under duality of two bases we understand as usual the following relations

$$\begin{aligned} &\langle dy^{Aa}, \partial_{Bb}\rangle = \delta^A_B \delta^a_b \quad \langle dp_{Aa}, \partial^{Bb}\rangle = \delta^B_A \delta^b_a \\ &\langle dy^{Aa}, \partial^{Bb}\rangle = 0 \quad \langle dp_{Aa}, \partial_{Bb}\rangle = 0 \end{aligned}$$

or

$$\begin{bmatrix} dy^a \\ dp_a \end{bmatrix} [\partial_b y \partial^b p] = I_{2k+1,2k+1}. \tag{2.14}$$

We want to prove that if (2.14) is valid, then the same relation is valid if a is everywhere substituted by a', i.e., that (2.14) is coordinate invariant. If we introduce the notation

$$T = \begin{bmatrix} A_a^{a'} & 0 \\ B_{aa'} & C_{a'}^a \end{bmatrix},$$

then (2.7) and (2.13) can be written in the form

$$[\partial_b y \partial^b p] = [\partial_{b'} y \partial^{b'} p] T \tag{2.15}$$

$$\begin{bmatrix} dy^{a'} \\ dp_{a'} \end{bmatrix} = T \begin{bmatrix} dy^a \\ dp_a \end{bmatrix} \Rightarrow \begin{bmatrix} dy^a \\ dp_a \end{bmatrix} = T^{-1} \begin{bmatrix} dy^{a'} \\ dp_{a'} \end{bmatrix}. \tag{2.16}$$

The substitution of (2.15) and (2.16) into (2.14) results in

$$T^{-1} \begin{bmatrix} dy^{a'} \\ dp_{a'} \end{bmatrix} [\partial_{b'} y \partial^{b'} p] T = I \Rightarrow \begin{bmatrix} dy^{a'} \\ dp_{a'} \end{bmatrix} [\partial_{b'} y \partial^{b'} p] = TIT^{-1} = I. \qquad \square$$

From (2.2) and (2.11) it is obvious that under coordinate transformation (1.3) the elements of natural bases $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$ are not transforming as tensors. Now we shall construct a new so-called special adapted bases B_{LH} and B^*_{LH}, whose elements transform as tensors and in which the J structure (which will be defined later) has the same components as in the natural bases.

Definition 2.1. The special adapted basis B_{LH} of $T(\mathrm{GLH})^{(nk)}$

$$B_{\mathrm{LH}} = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}, \delta^{1a}, \delta^{2a}, \dots, \delta^{ka}\} \tag{2.17}$$

is defined by

$$
\begin{aligned}
\delta_{0a} &= \binom{0}{0}\partial_{0a} - \binom{1}{0}N_{0a}^{1b}\partial_{1b} - \cdots - \binom{k}{0}N_{0a}^{kb}\partial_{kb}\\
&\quad - \binom{0}{0}N_{0a1b}\partial^{1b} - \binom{1}{0}N_{0a2b}\partial^{2b} - \cdots - \binom{k-1}{0}N_{0akb}\partial^{kb}\\
\delta_{1a} &= \binom{1}{1}\partial_{1a} - \binom{2}{1}N_{0a}^{1b}\partial_{2b} - \cdots - \binom{k}{1}N_{0a}^{(k-1)b}\partial_{kb}\\
&\quad - \binom{1}{1}N_{0a1b}\partial^{2b} - \binom{2}{1}N_{0a2b}\partial^{3b} - \cdots - \binom{k-1}{1}N_{0a(k-1)b}\partial^{kb}\\
\delta_{2a} &= \binom{2}{2}\partial_{2a} - \binom{3}{2}N_{0a}^{1b}\partial_{3b} - \cdots - \binom{k}{2}N_{0a}^{(k-2)b}\partial_{kb}\\
&\quad - \binom{2}{2}N_{0a1b}\partial^{3b} - \cdots - \binom{k-1}{2}N_{0a(k-2)b}\partial^{kb}, \ldots\\
\delta_{ka} &= \binom{k}{k}\partial_{ka}
\end{aligned}
\tag{2.18a}
$$

$$
\begin{aligned}
\delta^{1a} &= \binom{0}{0}\partial^{1a} - \binom{1}{0}N_{2b}^{0a}\partial^{2b} - \binom{2}{0}N_{3b}^{0a}\partial^{3b} - \cdots - \binom{k}{0}N_{kb}^{0a}\partial^{kb}\\
\delta^{2a} &= \binom{1}{1}\partial^{2a} - \binom{2}{1}N_{2b}^{0a}\partial^{3b} - \cdots - \binom{k-1}{1}N_{(k-1)b}^{0a}\partial^{kb} \ldots\\
\delta^{kb} &= \binom{k-1}{k-1}\partial^{kb}.
\end{aligned}
\tag{2.18b}
$$

Let us introduce the notations

$$
\begin{aligned}
&[\delta_{Aa}(y)]_{1,k+1} = [\delta_{0a}\delta_{1a}\ldots\delta_{ka}], \quad A = \overline{0,k}\\
&[\delta^{\alpha a}(p)]_{1,k} = [\delta^{1a}\delta^{2a}\ldots\delta^{ka}], \quad \alpha = \overline{1,k}\\
&[N_{0a}^{Bb}]_{k+1,k+1} = \begin{bmatrix} \binom{0}{0}\delta_a^b & 0 & \cdots & 0 & 0\\ -\binom{1}{0}N_{0a}^{1b} & \binom{1}{1}\delta_a^b & \cdots & 0 & 0\\ \vdots & \vdots & & & \vdots\\ -\binom{k}{0}N_{0a}^{kb} & -\binom{k}{1}N_{0a}^{(k-1)b} & \cdots & \binom{k}{k-1}N_{0a}^{1b} & \delta_a^b \end{bmatrix}\\
&[N_{0a\beta b}]_{k,k+1} = \begin{bmatrix} -\binom{0}{0}N_{0a1b} & 0 & \cdots & 0 & 0\\ -\binom{1}{0}N_{0a2b} & -\binom{1}{1}N_{0a1b} & \cdots & 0 & 0\\ \vdots & & & & \vdots\\ -\binom{k-1}{0}N_{0akb} & -\binom{k-1}{1}N_{0a(k-1)b} & \cdots & \binom{k-1}{k-1}N_{0a1b} & 0 \end{bmatrix}
\end{aligned}
\tag{2.19a}
$$

$$\text{(2.19b)}\quad [N^{0a}_{\beta b}]_{k,k} = \begin{bmatrix} \binom{0}{0}\delta^a_b & 0 & 0 & \cdots & 0 \\ -\binom{1}{0}N^{0a}_{2b} & \binom{1}{1}\delta^a_b & 0 & \cdots & 0 \\ -\binom{2}{0}N^{0a}_{3b} & -\binom{2}{1}N^{0a}_{2b} & \binom{2}{2}\delta^a_b & \cdots & 0 \\ \vdots & & & & \\ -\binom{k}{0}N^{0a}_{kb} & -\binom{k-1}{1}N^{0a}_{(k-1)b} & -\binom{k-2}{2}N^{0a}_{(k-2)b} & \cdots & \binom{k-1}{k-1}\delta^a_b \end{bmatrix}.$$

Using the above notations (2.18) can be written in the matrix form as follows:

$$\text{(2.20)}\qquad [\delta_{Aa}(y)\delta^{\alpha a}(p)]_{1,2k+1} = [\partial_{Bb}(y)\partial^{\beta b}(p)]_{1,2k+1}\cdot N_{2k+1,2k+1},$$

where

$$\text{(2.21)}\qquad N = \begin{bmatrix} [N^{Bb}_{0a}]_{k+1,k+1} & [0]_{k+1,k} \\ [N_{0a\beta b}]_{k,k+1} & [N^{0a}_{\beta b}]_{k,k} \end{bmatrix}.$$

The first request to the adapted basis B_{LH} is that their elements transform as tensors, i.e.,

$$\text{(2.22)}\qquad \delta_{Aa'} = B^a_{a'}\delta_{Aa}\quad A=\overline{0,k},\qquad \delta^{\alpha a'} = B^{a'}_a\delta^{\alpha a}\quad \alpha=\overline{1,k}.$$

Theorem 2.4. *The elements of the special adapted basis B_{LH} are transforming as tensor if and only if the following relations are valid:*

$$\text{(2.23)}\qquad \begin{aligned} &[N^{Bb'}_{0a'}]_{k+1,k+1} = [A^{b'}_b N^{Bb}_{0a}B^a_{a'}]_{k+1,k+1} \\ &[N_{0a'\beta b'}]_{k,k+1} = [(B_{bb'}N^{Bb}_{0a} + C^b_{b'}N_{0a\beta b})B^a_{a'}]_{k,k+1} \\ &[N^{0a'}_{\beta b'}] = [C^b_{b'}N^{0a}_{\beta b}B^{a'}_a]_{k,k}. \end{aligned}$$

Proof. If we denote by $[B^a_{a'}]_{k+1,k+1}$ the diagonal matrix, whose elements are all equal to zero except the diagonal elements, which are equal to $B^a_{a'}$, similar for $[B^{a'}_a]_{k,k}$, then using (2.3)-(2.7), further (2.19)-(2.21) we can write (2.22) in the form

$$\text{(2.24)}\qquad \begin{aligned} [\delta_{Aa'}(y)\delta^{\alpha a'}(p)]_{1,2k+1} &= [\delta_{Aa}(y)\delta^{\alpha a}(p)]_{1,2k+1}\begin{bmatrix} [B^a_{a'}]_{k+1,k+1} & [0]_{k+1,k} \\ [0]_{k,k+1} & [B^{a'}_a]_{k,k} \end{bmatrix} \\ &= [\partial_{Bb}(y)\partial^{\beta b}(p)]_{1,2k+1}N_{2k+1,2k+1}\cdot B_{2k+1,2k+1} \\ &= [\partial_{Bb'}(y)\partial^{\beta b'}(p)]\begin{bmatrix} A^{b'}_b & 0 \\ B_{bb'} & C^b_{b'} \end{bmatrix}_{2k+1,2k+1}\cdot N_{2k+1,2k+1}\cdot B_{2k+1,2k+1}. \end{aligned}$$

On the other side from (2.20) we get

$$\text{(2.25)}\qquad [\delta_{Aa'}(y)\delta^{\alpha a'}(p)] = [\partial_{Bb'}(y)\partial^{\beta b'}(p)]\cdot\begin{bmatrix} [N^{Bb'}_{0a'}]_{k+1,k+1} & [0]_{k+1,k} \\ [N_{0a'\beta b'}]_{k,k+1} & [N^{0a'}_{\beta b'}]_{k,k} \end{bmatrix}.$$

The comparison of (2.23) and (2.24) gives

$$\begin{bmatrix} [N^{Bb'}_{0a'}]_{k+1,k+1} & [0]_{k+1,k} \\ [N_{0a'\beta b'}]_{k,k+1} & [N^{0a'}_{\beta b'}]_{k,k} \end{bmatrix}$$

$$= \begin{bmatrix} [A_b^{b'}]_{k+1,k+1} & [0]_{k+1,k} \\ [B_{bb'}]_{k,k+1} & [C_{b'}^b]_{k,k} \end{bmatrix} \cdot \begin{bmatrix} [N_{0a}^{Bb}]_{k+1,k+1} & [0]_{k+1,k} \\ [N_{0a\beta b}]_{k,k+1} & N_{\beta bk,k}^{0a} \end{bmatrix} \cdot B =$$

$$= \begin{bmatrix} [A_b^{b'} N_{0a}^{Bb} + 0]_{k+1,k+1} & [0]_{k+1,k} + [0]_{k+1,k} \\ [B_{bb'} N_{0a}^{Bb} + C_{b'}^b N_{0a\beta b}]_{k,k+1} & [0_{k,k} + C_{b'}^b N_{\beta b}^{0a}]_{k,k} \end{bmatrix} \cdot B =$$

$$= \begin{bmatrix} [A_b^{b'} N_{0a}^{Bb} B_{a'}^a]_{k+1,k+1} & [0]_{k+1,k} \\ [(B_{bb'} N_{0a}^{Bb} + C_{b'}^b N_{0a\beta b}) B_{a'}^a]_{k,k+1} & [C_{b'}^b N_{\beta b}^{0a} B_a^{a'}]_{k,k} \end{bmatrix}.$$

From the above it follows (2.23). □

Definition 2.2. The special adapted basis B^*_{LH} of $T^*(\text{GLH})^{(nk)}$ is

$$B^*_{\text{LH}} = \{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}, \delta p_{1a}, \delta p_{2a}, \dots, \delta p_{ka}\}, \tag{2.26}$$

where

$$\begin{aligned} \delta y^{0a} &= dy^{0a} = dx^a \\ \delta y^{1a} &= \binom{1}{1} dy^{1a} + \binom{1}{0} M_{0b}^{1a} dy^{0b} \\ \delta y^{2a} &= \binom{2}{2} dy^{2a} + \binom{2}{1} M_{0b}^{1a} dy^{1b} + \binom{2}{0} M_{0b}^{2a} dy^{0b}, \dots \\ \delta y^{ka} &= \binom{k}{k} dy^{ka} + \binom{k}{k-1} M_{0b}^{1a} dy^{(k-1)b} + \dots + \binom{k}{0} M_{0b}^{ka} dy^{0b} \end{aligned} \tag{2.27a}$$

$$\begin{aligned} \delta p_{1a} &= \binom{0}{0} M_{0a1b} dy^{0b} + \binom{0}{0} dp_{1a}, \\ \delta p_{2a} &= \binom{1}{0} M_{0a2b} dy^{0b} + \binom{1}{1} M_{0a1b} dy^{1b} + \binom{1}{0} M_{0a}^{1b} dp_{1b} + \binom{1}{1} dp_{2a}, \\ &\vdots \\ \delta p_{ka} &= \binom{k-1}{0} M_{0akb} dy^{0b} + \dots + \binom{k-1}{k-1} M_{0a1b} dy^{(k-1)b} \\ &\quad + \binom{k-1}{0} M_{0a}^{(k-1)b} dp_{1b} + \dots + \binom{k-1}{k-1} dp_{ka}. \end{aligned} \tag{2.27b}$$

We introduce the notations:

$$[\delta y^{Aa}]_{k+1,1} = \begin{bmatrix} \delta y^{0a} \\ \delta y^{1a} \\ \vdots \\ \delta y^{ka} \end{bmatrix}, \quad [\delta p_{\alpha a}]_{k,1} = \begin{bmatrix} \delta p_{1a} \\ \delta p_{2a} \\ \vdots \\ \delta p_{ka} \end{bmatrix} \tag{2.28a}$$

(2.28b)
$$[M^{Aa}_{0b}]_{k+1,k+1} = \begin{bmatrix} \binom{0}{0}\delta^a_b & 0 & 0\ 0\cdots & 0 \\ \binom{1}{0}M^{1a}_{0b} & \binom{1}{1}\delta^a_b & 0\ 0 & 0 \\ 0 & & 0 & \\ \vdots & & & \\ \binom{k}{0}M^{ka}_{0b} & \binom{k}{1}M^{(k-1)b}_{0b} & & \binom{k}{k}\delta^a_b \end{bmatrix}$$

(2.28c)
$$[M_{0aBb}]_{k,k+1} = \begin{bmatrix} \binom{0}{0}M_{0b1a} & 0 & 0 & \cdots & 0 \\ \binom{1}{0}M_{0a2b} & \binom{1}{1}M_{0a1b} & 0 & \cdots & 0 \\ \vdots & & & & \\ \binom{k-1}{0}M_{0akb} & \binom{k-1}{1}M_{0a(k-1)b} & \cdots & \binom{k-1}{k-1}M_{0a1b} & 0 \end{bmatrix}$$

(2.28d)
$$[M^{\beta b}_{0a}]_{k,k} = \begin{bmatrix} \binom{0}{0}\delta^b_a & 0 & 0\cdots & 0 \\ \binom{1}{0}M^{1b}_{0a} & \binom{1}{1}\delta^b_a & 0\cdots & 0 \\ \vdots & & & \\ \binom{k-1}{0}M^{(k-1)b}_{0k} & \binom{k-1}{1}M^{(k-2)b}_{0a} & & \binom{k-1}{k-1}\delta^b_a \end{bmatrix}$$

Using the notations (2.28) we can write (2.27) in the form

$$\begin{bmatrix}\delta y^{Aa} \\ \delta p_{\alpha a}\end{bmatrix} = M \begin{bmatrix} dy^a \\ dp_a\end{bmatrix} = \begin{bmatrix} [M^{Aa}_{0b}]_{k+1,k+1} & [0]_{k+1,k} \\ [M_{0aBb}]_{k,k+1} & [M^{\beta b}_{0a}]_{k,k}\end{bmatrix} \begin{bmatrix} dy^{Bb} \\ dp_{\beta b}\end{bmatrix}. \tag{2.29}$$

The first request to B^*_{LH} is that their elements are transforming as tensors, i.e.,

$$\delta y^{Aa'} = B^{a'}_a \delta y^{Aa}, \quad \delta p_{\alpha a'} = B^a_{a'}\delta p_{\alpha a}, \quad A = \overline{0,k},\ \alpha = \overline{1,k}. \tag{2.30}$$

Theorem 2.5. *The elements of the special adapted basis B^*_{LH} are transforming as tensors (i.e., (2.30) are satisfied) if and only if the following relations are satisfied:*

$$\begin{aligned} [M^{Aa}_{0b}] &= [B^a_{a'}][M^{aa'}_{0b'}][A^{b'}_b] \\ [M_{0aBb}] &= [B^{a'}_a][M_{0a'Bb'}][A^{b'}_b] + [B^{a'}_a][M^{\beta b'}_{0a'}][B_{bb'}] \\ [M^{\beta b}_{0a}] &= [B^{a'}_a][M^{\beta b'}_{0a'}][C^b_{b'}]. \end{aligned} \tag{2.31}$$

Proof. (2.30) can be written in the matrix form as follows:

$$\begin{bmatrix}\delta y^{Aa'} \\ \delta p_{\alpha a'}\end{bmatrix}_{2k+1,1} = \begin{bmatrix} [B^{a'}_a]_{k+1,k+1} & 0 \\ 0 & [B^a_{a'}]_{k,k}\end{bmatrix} \begin{bmatrix}\delta y^{Aa} \\ \delta p_{\alpha a}\end{bmatrix} = B' \cdot M \cdot \begin{bmatrix} dy^a \\ dp_a\end{bmatrix}, \tag{2.32}$$

where (2.29) was used. On the other side using (2.29) and (2.16) we have

$$\begin{bmatrix}\delta y^{Aa'}\\ \delta p_{\alpha a'}\end{bmatrix} = M'\begin{bmatrix}dy^{a'}\\ dp_{a'}\end{bmatrix} = M'T\begin{bmatrix}dy^{a}\\ dp_{a}\end{bmatrix}. \tag{2.33}$$

The comparison of (2.30) and (2.33) gives

$$M'T = B'M \Rightarrow BM'T = M, \tag{2.34}$$

where B is $(B')^{-1}$.

The explicit form of the above equation is

$$\begin{aligned}
\begin{bmatrix}[M^{Aa}_{0b}]_{k+1,k+1} & [0]_{k+1,k}\\ [M_{0a\beta b}]_{k,k+1} & [M^{\beta b}_{0a}]_{k,k}\end{bmatrix} &= \begin{bmatrix}[B^{a}_{a'}]_{k+1,k+1} & [0]_{k+1,k}\\ 0 & [B^{a'}_{a}]_{k,k}\end{bmatrix}\times\\
&\times\begin{bmatrix}[M^{Aa'}_{0b'}]_{k+1,k+1} & [0]_{k+1,k}\\ [M_{0a'Bb'}]_{k,k+1} & [M^{\beta b'}_{0a'}]_{k,k}\end{bmatrix}\cdot\begin{bmatrix}[A^{b'}_{b}]_{k+1,k+1} & 0\\ [B_{bb'}]_{k,k+1} & [C^{b}_{b'}]_{k,k}\end{bmatrix}\\
&= \begin{bmatrix}[B^{a}_{a'}M^{Aa'}_{0b'}]_{k+1,k+1} & [0]_{k+1,k}\\ [B^{a'}_{a}M_{0a'Bb'}]_{k,k+1} & [B^{a'}_{a}M^{\beta b'}_{0a'}]_{k,k}\end{bmatrix}\cdot\begin{bmatrix}[A^{b'}_{b}]_{k+1,k+1} & [0]_{k+1,k}\\ [B_{bb'}]_{k,k+1} & [C^{b}_{b'}]_{k,k}\end{bmatrix}\\
&= \begin{bmatrix}[B^{a}_{a'}M^{aa'}_{0b'}A^{b'}_{b}]_{k+1,k+1} & [0]_{k+1,k}\\ [B^{a'}_{a}M_{0a'Bb'}A^{b'}_{b} + B^{a'}_{a}M^{\beta b'}_{0a'}B_{bb'}]_{k,k+1} & [B^{a'}_{a}M^{\beta b'}_{0a'}C^{b}_{b'}]_{k,k}\end{bmatrix}.
\end{aligned}$$

From the above equation it follows (2.31). □

The other request to B^*_{LH} and B_{LH} is to be inverse to each other.
The following theorem gives answer on this condition.

Theorem 2.6. *The necessary and sufficient condition that the special adapted basis B^*_{LH} be dual to B_{LH}, when the natural basis $\bar{B}_{\mathrm{LH}}$ is dual to $\bar{B}_{\mathrm{LH}}$ is*

$$MN = I. \tag{2.35}$$

Proof. From

$$\begin{bmatrix}\delta y^{Aa}\\ \delta p_{\alpha a}\end{bmatrix} = M\begin{bmatrix}dy^{b}\\ dp_{b}\end{bmatrix},$$

$$[\delta_{Aa}(y)\delta^{\alpha a}(p)] = [\partial_{Bb}(y)\partial^{\beta b}(p)]\cdot N,$$

it follows

$$\begin{bmatrix}\delta y^{Aa}\\ \delta p_{\alpha a}\end{bmatrix}_{2k+1,1}[\delta_{Aa}(y)\delta^{\alpha a}(p)]_{1,2k+1} = I_{2k+1,2k+1},$$

$$M\begin{bmatrix}dy^{b}\\ dp_{b}\end{bmatrix}[\partial_{Bb}(y)\partial^{\beta b}(p)]N = M\cdot I_{2k+1,2k+1},\cdot N = MN.$$

From the above it follows (2.35) □

3. The J structure in $(\mathrm{GLH})^{(nk)}$

Definition 3.1. The k-tangent structure J is a $\mathcal{F}$ linear mapping ($\mathcal{F}$ denotes the modul of C^∞ functions on $(\mathrm{GLH})^{(nk)}$)

$$J\colon T(\mathrm{GLH})^{(nk)} \to T(\mathrm{GLH})^{(nk)}$$

defined by

$$\begin{gathered} J\partial_{0a} = \partial_{1a}, J\partial_{1a} = 2\partial_{2a}, \dots, J\partial_{(k-1)a} = k\partial_{ka}, J\partial_{ka} = 0, \\ J\partial^{1a} = \partial^{2a}, J\partial^{2a} = 2\partial^{3a}, \dots, J\partial^{(k-1)a} = (k-1)\partial^{ka}, J\partial^{ka} = 0. \end{gathered} \tag{3.1}$$

From Definition 3.1 it follows

Theorem 3.1. *The structure J defined by (3.1) in the natural basis $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$ can be expressed in the matrix form as follows:*

$$J = [\partial_{Aa}(y)\partial^{\alpha a}(p)][J] \otimes \begin{bmatrix} dy^{Bb} \\ dp_{\beta b} \end{bmatrix}, \tag{3.2}$$

where

$$[J] = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{3.3}$$

$$J_1 = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 1\delta^a_b & 0 & \cdots & & 0 \\ 0 & 2\delta^a_b & \cdots & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & k\delta^a_b & 0 \end{bmatrix}_{k+1,k+1} \qquad J_2 = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 1\delta^b_a & 0 & \cdots & & 0 \\ 0 & 2\delta^b_a & \cdots & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & (k-1)\delta^b_a & 0 \end{bmatrix}_{k,k}. \tag{3.4}$$

From (3.4) it follows

$$J_1^k = 0, \quad J_2^{k-1} = 0 \tag{3.5}$$

and from (3.3) and (3.5) it follows $J^k = 0$.

The explicit form of (3.2) is

$$\begin{aligned} J = {}& \partial_{1a} \otimes dy^{0a} + 2\partial_{2a} \otimes dy^{1a} + \dots + k\partial_{ka} \otimes dy^{(k-1)a} \\ & + \partial^{2a} \otimes dp_{1a} + 2\partial^{3a} \otimes dp_{2a} + + \dots + (k-1)\partial^{ka} \otimes dp_{(k-1)a}. \end{aligned} \tag{3.6}$$

Proof. From the duality of $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$ it follows:

$$\begin{aligned} &J\partial_{0b} = \langle \partial_{1a} \otimes dy^{0a}, \partial_{0b} \rangle = \partial_{1a}\delta^a_b = \partial_{1b} \\ &J\partial_{1b} = \langle 2\partial_{2a} \otimes dy^{1a}, \partial_{1b} \rangle = 2\partial_{2a}\delta^a_b = 2\partial_{2b}, \dots \\ &J\partial^{1b} = \langle \partial^{2a} \otimes dp_{1a}, \partial^{1b} \rangle = \partial^{2a}\delta^b_a = \partial^{2b}, \dots. \quad \square \end{aligned}$$

Theorem 3.2. *If the structure J defined by (3.1), or equivalently by (3.2)-(3.4) is expressed in the special adapted bases B_{LH} and B^*_{LH} then it is determined by the same matrix J as (3.3), (3.4), or*

$$J = [\delta_{Aa}(y)\delta^{\alpha a}(p)][J] \otimes \begin{bmatrix} \delta y^{Bb} \\ \delta p_{\beta b} \end{bmatrix}. \tag{3.7}$$

The explicit form of (3.7) is

$$\begin{aligned} J = {} & \delta_{1a} \otimes \delta y^{0a} + 2\delta_{2a} \otimes \delta y^{1a} + \cdots + k\delta_{ka} \otimes \delta y^{(k-1)a} \\ & + \delta^{2a} \otimes \delta p_{1a} + 2\delta^{3a} \otimes \delta p_{2a} + \cdots + (k-1)\delta^{ka} \otimes \delta p_{(k-1)a}. \end{aligned} \tag{3.8}$$

from which follows

$$\begin{aligned} & J\delta_{0a} = \delta_{1a}, J\delta_{1a} = 2\delta_{2a}, \dots, J\delta_{(k-1)a} = k\delta_{ka}, J\delta_{ka} = 0, \\ & J\delta^{1a} = \delta^{2a}, J\delta^{2a} = 2\delta^{3a}, \dots, J\delta^{(k-1)}_{a} = (k-1)\delta^{ka}, J\delta^{ka} = 0. \end{aligned} \tag{3.9}$$

Proof. We shall prove only some of the above equations, the others can be proved using the same method and the mathematical induction. From (2.18) and (3.1) we get

$$\begin{aligned} J\delta_{0a} = {} & \binom{0}{0}\partial_{1a} - \binom{1}{0}N^{1b}_{0a} \cdot 2\partial_{2a} - \cdots - \binom{k-1}{0}N^{(k-1)b}_{0a}k\partial_{ka} \\ & - \binom{0}{0}N_{0a1b}\partial^{2b} - \cdots - \binom{k-2}{0}N_{0a(k-1)b}(k-1)\partial^{kb} = \delta_{1a} \\ J\delta_{1a} = {} & \binom{1}{1}2\partial_{2a} - \binom{2}{1}N^{1b}_{0a}3\partial_{3a} - \cdots - \binom{k-1}{1}N^{(k-2)b}_{0a}k\partial_{ka} \\ & - \binom{1}{1}N_{0a1b}2\partial^{3b} - \binom{2}{1}N_{0a2b}3\partial^{4a} - \cdots - \binom{k-2}{1}N_{0a(k-2)b}(k-1)\partial^{kb} \\ = {} & 2\delta_{2a}, \dots \\ J\delta_{ka} = {} & 0 \\ J\delta^{1a} = {} & \binom{0}{0}\partial^{2a} - \binom{1}{0}N^{0a}_{2b}2\partial^{3b} - \cdots - \binom{k-1}{0}N^{0a}_{(k-1)b}(k-1)\partial^{kb} = \delta^{2a} \\ J\delta^{2a} = {} & \binom{1}{1}2\partial^{3a} - \binom{2}{1}N^{0a}_{2b}3\partial^{4a} - \cdots - \binom{k-2}{1}N^{0a}_{(k-2)b}(k-1)\partial^{ka} = 2\delta^{3a}, \dots \\ J\delta^{ka} = {} & 0. \end{aligned}$$

□

Theorem 3.3. *For the k-tangent structure J determined by (3.6) we have*

$$\begin{aligned} & dy^{0b}J = 0, dy^{1b}J = dy^{0b}, dy^{2b}J = 2dy^{1b}, \dots, dy^{ka}J = kdy^{(k-1)a} \\ & dp_{1b}J = 0, dp_{2b}J = dp_{1b}, dp_{3b}J = 2dp_{2b}, \dots, dp_{kb}J = (k-1)dp_{(k-1)b}. \end{aligned} \tag{3.10}$$

The proof follows from (3.6) and the duality of $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$.

Theorem 3.4. *For the k-tangent structure J determined by (3.8) we have*

$$\delta y^{0b}J=0,\delta y^{1b}J=\delta y^{0b},\delta y^{2b}J=2\delta y^{1b},\dots,\delta y^{kb}J=k\delta^{(k-1)b},$$
$$\delta p_{1b}J=0,\delta p_{2b}J=\delta p_{1b},\delta p_{3b}J=2p_{2b},\dots,\delta p_{kb}J=(k-1)\delta p_{(k-1)b}.$$

Proof. The proof follows from (3.8) and the duality of B^*_{LH} and B_{LH}. It can be proved directly using (3.10), (2.27) and the duality of $\bar{B}^*_{\mathrm{LH}}$ and $\bar{B}_{\mathrm{LH}}$. We have for instance

$$\delta y^{0a}J=dy^{0b}J=0$$
$$\delta y^{1a}J=\binom{1}{1}dy^{0a}=\delta y^{0a}$$
$$\delta y^{2a}J=\binom{2}{2}2dy^{1a}+\binom{2}{1}M^{1a}_{0b}dy^{0b}=2\delta y^{1a}$$
$$\delta y^{3a}J=\binom{3}{3}3dy^{2a}+\binom{3}{2}M^{1a}_{0b}2dy^{1b}+\binom{3}{1}M^{2a}_{0b}dy^{0a}=3\delta y^{2a},\dots,$$
$$\delta p_{1a}J=\binom{0}{0}M_{0a1b}dy^{0b}J+\binom{0}{0}dp_{1b}J=0$$
$$\delta p_{2a}J=\binom{1}{1}M_{0a1b}dy^{1b}J+\binom{1}{1}dp_{2a}J=\binom{1}{1}M_{0a1b}dy^{0b}+\binom{1}{1}dp_{1a}=\delta p_{1a}$$
$$\delta p_{3a}J=\binom{2}{1}M_{0a2b}dy^{0b}+\binom{2}{2}M_{0a1b}2dy^{1b}+\binom{2}{1}M^{1b}_{0a}dp_{1b}+\binom{2}{2}2dp_{2b}=2\delta p_{2a}$$
$$\begin{aligned}\delta p_{4a}J=&\binom{3}{1}M_{0a3b}dy^{0b}+\binom{3}{2}M_{0a2b}\cdot 2dy^{1b}+\binom{3}{3}M_{0a1b}\cdot 3dy^{2b}+\\&\binom{3}{1}M^{2b}_{0a}dp_{1b}+\binom{3}{2}M^{1b}_{0a}\cdot 2dp_{2b}+\binom{3}{3}\cdot 3dp_{3b}=3\delta p_{3a},\dots.\end{aligned}$$

□

Definition 3.2. The k-tangent structure $\bar{J}=J^T$ is the adjoint map of the k-tangent structure J and it is a $\mathcal{F}$ linear mapping

$$\bar{J}:T^*(\mathrm{GLH})^{(nk)}\to T^*(\mathrm{GLH})^{(nk)}.$$

The following relations are valid:

$$\begin{aligned}&\bar{J}dy^{0a}=0,\bar{J}dy^{1a}=dy^{0a},\bar{J}dy^{2a}=2dy^{1a},\dots,\bar{J}dy^{ka}=kdy^{(k-1)a}\\&\bar{J}dp_{1a}=0,\bar{J}dp_{2a}=dp_{1a},\bar{J}dp_{3a}=2dp_{2a},\dots,\bar{J}dp_{ka}=(k-1)dp_{(k-1)a}.\end{aligned}\tag{3.11}$$

Theorem 3.5. *The k-tangent structure $\bar{J}$ in the natural bases $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$ can be expressed in the matrix form as follows:*

$$\bar{J}=[dy^{Aa}dp_{\alpha a}][\bar{J}]\otimes\begin{bmatrix}\partial_{Bb}(y)\\ \partial^{\beta b}(p)\end{bmatrix},\tag{3.12}$$

where

$$[\bar{J}] = \begin{bmatrix} \bar{J}_1 & \\ & \bar{J}_2 \end{bmatrix}, \tag{3.13}$$

$$\bar{J}_1 = \begin{bmatrix} 0 & 1\delta_a^b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2\delta_a^b & 0 & & 0 \\ 0 & 0 & 0 & 3\delta_a^b & & 0 \\ \vdots & & & & & k\delta_a^b \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}_{k+1,k+1} \tag{3.14a}$$

$$\bar{J}_2 = \begin{bmatrix} 0 & 1\delta_b^a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2\delta_b^a & 0 & & 0 \\ 0 & 0 & 0 & 3\delta_b^a & & 0 \\ \vdots & & & & & (k-1)\delta_b^a \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}_{k,k} \tag{3.14b}$$

or in the explicit form

$$\begin{aligned} \bar{J} = {} & dy^{0a} \otimes \partial_{1a} + 2dy^{1a} \otimes \partial_{2a} + \cdots + k dy^{(k-1)a} \otimes \partial_{ka} \\ & + dp_{1a} \otimes \partial^{2a} + 2dp_{2a} \otimes \partial^{3a} + \cdots + (k-1) dp_{(k-1)a} \otimes \partial^{ka}. \end{aligned} \tag{3.15}$$

Proof. From the duality of $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$ and (3.15) it follows (3.11). From (3.14) it follows $\bar{J}_1{}^k = 0$, $J_2^{k-1} = 0$ and from (3.13) we have $\bar{J}^k = 0$, which gives the name of the k-tangent structure. □

Theorem 3.6. *For the k-tangent structure $\bar{J}$ determined by (3.11) we have:*

$$\begin{aligned} &\partial_{0a}\bar{J} = \partial_{1a}, \quad \partial_{1a}\bar{J} = 2\partial_{2a}, \ldots, \partial_{(k-1)a}\bar{J} = k\partial_{ka}, \quad \partial_{ka}\bar{J} = 0, \\ &\partial^{1a}\bar{J} = \partial^{2a}, \quad \partial^{2a}\bar{J} = 2\partial^{3a}, \ldots, \partial^{(k-1)a}\bar{J} = (k-1)\partial^{ka}, \quad \partial^{ka}\bar{J} = 0. \end{aligned} \tag{3.16}$$

Proof. The proof follows from (3.15) and the duality of $\bar{B}_{\mathrm{LH}}$ and $\bar{B}^*_{\mathrm{LH}}$. □

Theorem 3.7. *For the k-tangent structure $\bar{J}$ the following relations are valid:*

$$\begin{aligned} &\bar{J}\delta y^{0a} = 0, \quad \bar{J}\delta y^{1a} = \delta y^{0a}, \ldots, \bar{J}\delta y^{ka} = k\delta y^{(k-1)a}, \\ &\bar{J}\delta p_{1a} = 0, \quad \bar{J}\delta p_{2a} = \delta p_{1a}, \ldots, \bar{J}\delta p_{ka} = (k-1)\delta p_{(k-1)a}. \end{aligned} \tag{3.17}$$

Proof. We have

$$\bar{J}\begin{bmatrix} \delta y^{Aa} \\ \delta p_{\alpha a} \end{bmatrix} = \begin{bmatrix} \bar{J}\delta y^{Aa} \\ \bar{J}\delta p_{\alpha a} \end{bmatrix} = \bar{J}M \begin{bmatrix} dy^{Bb} \\ dp_{\beta b} \end{bmatrix},$$

where

$$\bar{J}M = \begin{bmatrix} \bar{J}_1 & 0 \\ 0 & \bar{J}_2 \end{bmatrix} \begin{bmatrix} M_{0b}^{Aa} & 0 \\ M_{0aBb} & M_{0a}^{\beta b} \end{bmatrix} = \begin{bmatrix} \bar{J}_1 M_{0b}^{Aa} & 0 \\ \bar{J}_2 M_{0aBb} & \bar{J}_2 M_{0a}^{\beta b} \end{bmatrix},$$

$$[\bar{J}_1 M_{0b}^{Aa}] = \begin{bmatrix} \binom{1}{0} M_{0b}^{1a} & \binom{1}{1}\delta_b^a & 0 & \cdots & 0 \\ 2\binom{2}{0} M_{0b}^{2a} & 2\binom{2}{1} M_{0b}^{1a} & 2\binom{2}{2}\delta_b^a & \cdots & 0 \\ \vdots & & & & \vdots \\ k\binom{k-1}{0} M_{0akb} & k\binom{k-1}{1} M_{0a}^{(k-2)b} & & & k\binom{k-1}{k-1}\delta_b^a \\ 0 & 0 & & & 0 \end{bmatrix},$$

$$[\bar{J}_2 M_{0aBb}] = \begin{bmatrix} \binom{1}{0} M_{0a2b} & \binom{1}{1}\delta_a^b & 0 & \cdots & 0 \\ 2\binom{2}{0} M_{0a3b} & 2\binom{2}{1} M_{0a2b} & 2\binom{2}{2}\delta_a^b & \cdots & 0 \\ \vdots & & & & \vdots \\ (k-1)\binom{k-1}{0} M_{0a(k-1)b} & & & & (k-1)\binom{k-1}{k-1}\delta_a^b \\ 0 & 0 & 0 & & 0 \end{bmatrix},$$

$$[\bar{J}_2 M_{0a}^{\beta b}] = \begin{bmatrix} \binom{1}{0} M_{0a}^{1b} & \binom{1}{1}\delta_a^b & 0 & \cdots & 0 \\ 2\binom{2}{0} M_{0a}^{2b} & 2\binom{2}{1} M_{0a}^{1b} & 2\binom{2}{2}\delta_a^b & \cdots & 0 \\ \vdots & & & & \vdots \\ (k-1)\binom{k-1}{0} M_{0a}^{(k-1)b} & & & & (k-1)\binom{k-1}{k-1}\delta_a^b \\ 0 & 0 & 0 & & 0 \end{bmatrix}.$$

If we calculate

$$\bar{J} M \begin{bmatrix} dy^{Bb} \\ dp_{\beta b} \end{bmatrix}$$

and compare the obtained relations with (2.27) we obtain (3.17). □

In the similar way it can be proved:

Theorem 3.8. *For the k-tangent structure $\bar{J}$ determined by (3.11) the following relations are valid:*

$$\begin{aligned} &\delta_{0a}\bar{J} = \delta_{1a}, \delta_{1a}\bar{J} = 2\delta_{2a}, \dots, \delta_{(k-1)a}\bar{J} = k\delta_{ka}, \delta_{ka}\bar{J} = 0 \\ &\delta^{1a}\bar{J} = \delta^{2a}, \delta^{2a}\bar{J} = 2\delta^{3a}, \dots, \delta^{(k-1)a}\bar{J} = (k-1)\delta^{ka}, \delta^{ka}\bar{J} = 0. \end{aligned} \tag{3.18}$$

4. The Liouville vector and 1-form field

If

$$M(y^{0a}, y^{1a}, \dots, y^{ka}, p_{1a}, p_{2a}, \dots, p_{ka})$$

and

$$M'(y^{0a} + dy^{0a}, y^{1a} + dy^{1a}, \dots, y^{ka} + dy^{ka}, p_{1a} + dp_{1a}, p_{2a} + dp_{2a}, \dots, p_{ka} + dp_{ka})$$

are two points in $(\mathrm{GLH})^{(nk)}$, then the vector MM' expressed in the natural basis $T(\mathrm{GLH})^{(nk)}$ has the form

$$\begin{aligned}MM' = dr &= dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka}\\ &\quad + dp_{1a}\partial^{1a} + dp_{2a}\partial^{2a} + \cdots + dp_{ka}\partial^{ka}\\ &= [dy^{Aa} dp_{\alpha a}] \begin{bmatrix}\partial_{Aa}(y)\\ \partial^{\alpha a}(p)\end{bmatrix}.\end{aligned}\tag{4.1}$$

Theorem 4.1. *The vector dr is coordinate invariant.*

Proof. Using (2.20), (2.29) and (2.35) we have

$$\begin{bmatrix}\delta y^{Aa}\\ \delta p_{\alpha a}\end{bmatrix} = M \begin{bmatrix}dy^{Aa}\\ dp_{\alpha a}\end{bmatrix} \Rightarrow [\delta y^{Aa}\delta p_{\alpha a}] = [dy^{Aa} dp_{\alpha a}]M^T$$

$$[\delta_{Aa}\delta^{\alpha a}] = [\partial_{Aa}\partial^{\alpha a}]N \Rightarrow \begin{bmatrix}\delta_{Aa}\\ \delta^{\alpha a}\end{bmatrix} = N^T \begin{bmatrix}\partial_{Aa}\\ \partial^{\alpha a}\end{bmatrix}$$

so we have

$$\begin{aligned}[\delta y^{Aa}\delta p_{\alpha a}] \begin{bmatrix}\delta_{Aa}(y)\\ \delta^{\alpha a}(p)\end{bmatrix} &= [dy^{Aa} dp_{\alpha a}]M^T \cdot N^T \begin{bmatrix}\partial_{Aa}\\ \partial^{\alpha a}\end{bmatrix}\\ &= [dy^{Aa} dp_{\alpha a}] \begin{bmatrix}\partial_{Aa}\\ \partial^{\alpha a}\end{bmatrix} = dr,\end{aligned}\tag{4.2}$$

because $M^T N^T = (NM)^T = (MN)^T = I^T = I$ (T-transposed).

The comparison of (4.1) and (4.2) results

$$\begin{aligned}dr &= dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka}\\ &\quad + dp_{1a}\partial^{1a} + dp_{2a}\partial^{2a} + \cdots + dp_{ka}\partial^{ka}\\ &= \delta y^{0a}\delta_{0a} + \delta y^{1a}\delta_{1a} + \cdots + \delta y^{ka}\delta_{ka}\\ &\quad + \delta p_{1a}\delta^{1a} + \delta p_{2a}\delta^{2a} + \cdots + \delta p_{ka}\delta^{ka}.\end{aligned}\tag{4.3}$$

Let us consider

$$\delta r = dr^T = [\partial_{Aa}(y)\partial^{\alpha a}(p)] \begin{bmatrix}dy^{Aa}\\ dp_{\alpha a}\end{bmatrix},\tag{4.4}$$

the transpose of dr, as a "one-form field" on $T^*(\mathrm{GLH})^{(nk)}$. Using (4.2) and (4.3) δr can be written in the form:

$$\begin{aligned}\delta r &= \partial_{0a}dy^{0a} + \partial_{1a}dy^{1a} + \cdots + \partial_{ka}dy^{ka}\\ &\quad + \partial^{1a}dp_{1a} + \partial^{2a}dp_{2a} + \cdots + \partial^{ka}dp_{ka}\\ &= \delta_{0a}\delta y^{0a} + \delta_{1a}\delta y^{1a} + \cdots + \delta_{ka}\delta y^{ka}\\ &\quad + \delta^{1a}\delta p_{1a} + \delta^{2a}\delta p_{2a} + \cdots + \delta^{ka}\delta p_{ka}.\end{aligned}\tag{4.5}$$

□

In the first part of (4.5) δr is expressed in $\bar{B}^*(\mathrm{GLH})^{(nk)}$, the natural basis of $T^*(\mathrm{GLH})^{(nk)}$ in the second part δr is presented in $B^*(\mathrm{GLH})^{(nk)}$, the special adapted basis of $T^*(\mathrm{GLH})^{(nk)}$. The components of δr are not functions, but differential operators, but they are transforming in the same way as components of covector, so the name "one-form field".

Remark 4.1. From (4.3) and (4.5) it is obvious that dr and δr have the same components in natural and special adapted bases. The same property have the structures J and $\bar{J}$. This fact allows us that the action of J on δr and $\bar{J}$ on dr can be written by the equations of the same from in both coordinate system.

In $(\mathrm{GLH})^{(nk)}$ it is difficult to construct vector fields and 1-form fields, but using dr, δr, the structure J and $\bar{J}$ it can be construct one family of Liouville vector and 1-form fields.

Definition 4.1. The Liouville vector fields $\bar{\Gamma}_0, \bar{\Gamma}_1, \bar{\Gamma}_2, \dots, \bar{\Gamma}_k$ are defined by

$$\begin{aligned} &\bar{\Gamma}_k = dr, \quad \bar{J}\bar{\Gamma}_A = \bar{\Gamma}_A\bar{J} = (k-(A-1))\bar{\Gamma}_{A-1}, \\ &\bar{A} = \overline{1,k}, \quad \bar{J}\bar{\Gamma}_0 = \bar{\Gamma}_0\bar{J} = 0. \end{aligned} \tag{4.6}$$

The above definition is in papers [18, 19, 20, 16, 21, 23] theorem, but it seems to be the most natural definition.

From (4.6) it follows

$$\begin{aligned} &\bar{J}\bar{\Gamma}_0 = \bar{\Gamma}_0 J = 0, \quad \bar{J}\bar{\Gamma}_1 = \bar{\Gamma}_1\bar{J} = k\bar{\Gamma}_0, \quad \bar{J}\bar{\Gamma}_2 = \bar{\Gamma}_2\bar{J} = (k-1)\bar{\Gamma}_1, \\ &\dots, \bar{J}\bar{\Gamma}_{k-1} = \bar{\Gamma}_{k-1}\bar{J} = 2\bar{\Gamma}_{k-2}, \quad \bar{J}\bar{\Gamma}_k = \bar{\Gamma}_k\bar{J} = \bar{\Gamma}_{k-1}. \end{aligned} \tag{4.7}$$

Theorem 4.2. *The Liouville vector fields $\bar{\Gamma}_0, \bar{\Gamma}_1, \dots, \bar{\Gamma}_k$ from $(\mathrm{GLH})^{(nk)}$ expressed in the special adapted basis B of $T(\mathrm{GLH})^{(nk)}$ have the form*

$$\begin{aligned} \bar{\Gamma}_0 &= \binom{k}{0}\delta y^{0a}\delta_{ka}, \\ \bar{\Gamma}_1 &= \binom{k}{1}\delta y^{1a}\delta_{ka} + \binom{k-1}{0}\delta y^{0a}\delta_{(k-1)a} \\ &\quad + \binom{k-1}{0}\delta p_{1a}\delta^{ka}, \\ \bar{\Gamma}_2 &= \binom{k}{2}\delta y^{2a}\delta_{ka} + \binom{k-1}{1}\delta y^{1a}\delta_{(k-1)a} + \binom{k-2}{0}\delta y^{0a}\delta_{(k-2)a} \\ &\quad + \binom{k-1}{1}\delta p_{2a}\delta^{ka} + \binom{k-2}{0}\delta p_{1a}\delta^{(k-1)a}, \dots \end{aligned} \tag{4.8a}$$

(4.8b)

$$\bar{\Gamma}_{k-1} = \binom{k}{k-1}\delta y^{(k-1)a}\delta_{ka} + \binom{k-1}{k-2}\delta y^{(k-2)a}\delta_{(k-1)a} + \cdots$$
$$+ \binom{2}{1}\delta y^{1a}\delta_{2a} + \binom{1}{0}\delta y^{0a}\delta_{1a}$$
$$+ \binom{k-1}{k-2}\delta p_{(k-1)a}\delta^{ka} + \binom{k-2}{k-3}\delta p_{(k-2)a}\delta^{(k-1)a} + \cdots + \binom{1}{0}\delta p_{1a}\delta^{2a},$$
$$\bar{\Gamma}_{k} = \binom{k}{k}\delta y^{ka}\delta_{ka} + \binom{k-1}{k-1}\delta y^{(k-1)a}\delta_{(k-1)A} + \cdots$$
$$+ \binom{1}{1}\delta y^{1a}\delta_{1a} + \binom{0}{0}\delta y^{0a}\delta_{0a}$$
$$+ \binom{k-1}{k-1}\delta p_{ka}\delta^{ka} + \binom{k-2}{k-2}\delta p_{(k-1)a}\delta^{(k-1)a} + \cdots + \binom{0}{0}\delta p_{1a}\delta^{1a}.$$

The proof follows from (4.6).

Theorem 4.3. *The Liouville vector fields* $\bar{\Gamma}_0, \bar{\Gamma}_1, \dots, \bar{\Gamma}_k$ *in* $(\mathrm{GLH})^{(nk)}$ *in the natural basis* $\bar{B}$ *of* $T(\mathrm{GLH})^{(nk)}$ *have the form obtained from (4.7) if*

$$\delta y^{Aa}, \delta p_{\alpha a}, \delta_{Aa}, \delta^{\alpha a},$$

are substituted by $dy^{Aa}, dp_{\alpha a}, \partial_{Aa}, \partial^{\alpha a}$ *respectively for every* $A = \overline{0,k}$, $\alpha = \overline{1,k}$.

Proof. The proof follows from Remark 4.1. The first equations obtained in such way have the form

$$\bar{\Gamma}_0 = \binom{k}{0} dy^{0a}\partial_{ka} = dt\left[\binom{k}{0} y^{1a}\delta_{ka}\right] = \bar{\Gamma}'_0 dt$$
$$\bar{\Gamma}_1 = \binom{k}{1} dy^{1a}\partial_{ka} + \binom{k-1}{0} dy^{0a}\partial_{(k-1)a} + \binom{k-1}{0} dp_{1a}\partial^{ka}$$
$$= dt\left[\binom{k}{1} y^{2a}\partial_{ka} + \binom{k-1}{0} y^{1a}\partial_{(k-1)a} + \binom{k-1}{0} p_{2a}\partial^{ka}\right] = \bar{\Gamma}'_1 dt, \dots.$$

In such a way using the relations $dy^{Aa} = y^{(A+1)a}dt$, $dp_{\alpha a} = p_{(\alpha+1)a}dt$, $A = \overline{0,k-1}$, $\alpha = \overline{1,k-1}$, we obtain easily the following relations:

$$\bar{\Gamma}_A = \bar{\Gamma}'_A dt, \quad A = \overline{0,k},$$

where $\bar{\Gamma}'_A$ - are the Liouville vector fields given in the form which is wellknown ([9, 13, 10, 18, 19, 20, 16, 21, 23]). ☐

The difference arises from the fact, that here the notation $y^{Aa} = \frac{d^A y^{0a}}{dt^A}$ is used instead of usual $y^{Aa} = \frac{1}{A!}\frac{d^A y^{0a}}{dt^A}$.

Definition 4.2. The Liouville 1-form fields $\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_k$ are defined by

$$\begin{gathered}\Gamma_k = \delta r \quad J\Gamma_A = \Gamma_A J = (k-(A-1))\Gamma_{A-1}, \quad A = \overline{1,k} \\ J\Gamma_0 = \Gamma_0 J = 0.\end{gathered} \tag{4.9}$$

From (4.9) it follows

$$\begin{gathered}J\Gamma_0 = \Gamma_0 J = 0, \quad J\Gamma_1 = \Gamma_1 J = k\Gamma_0, \dots, \\ J\Gamma_{k-1} = \Gamma_{k-1} J = 2\Gamma_{k-2}, \quad J\Gamma_k = \Gamma_k J = \Gamma_{k-1}.\end{gathered} \tag{4.10}$$

Theorem 4.4. *The Liouville 1-form fields $\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_k$ from $(\mathrm{GLH})^{(nk)}$ expressed in the special adapted basis B^* of $T(\mathrm{GLH})^{(nk)}$ have the form:*

(4.11)

$$\begin{aligned}
\Gamma_0 &= \binom{k}{0}\delta_{ka}\delta y^{0a}, \\
\Gamma_1 &= \binom{k}{1}\delta_{ka}\delta y^{1a} + \binom{k-1}{0}\delta_{(k-1)a}\delta y^{0a} + \binom{k-1}{0}\delta^{ka}\delta p_{1a}, \\
\Gamma_2 &= \binom{k}{2}\delta_{ka}\delta y^{2a} + \binom{k-1}{1}\delta_{(k-1)a}\delta y^{1a} + \binom{k-2}{0}\delta_{(k-2)a}\delta y^{0a} \\
&\quad + \binom{k-1}{1}\delta^{ka}\delta p_{2a} + \binom{k-2}{0}\delta^{(k-1)a}\delta p_{1a}, \dots, \\
\Gamma_{k-1} &= \binom{k}{k-1}\delta_{ka}\delta y^{(k-1)a} + \binom{k-1}{k-2}\delta_{(k-1)a}\delta^{(k-2)a} + \cdots \\
&\quad + \binom{2}{1}\delta_{2a}\delta y^{1a} + \binom{1}{0}\delta_{1a}\delta y^{0a} \\
&\quad + \binom{k-1}{k-2}\delta^{ka}\delta p_{(k-1)a} + \binom{k-2}{k-3}\delta^{(k-1)a}\delta p_{(k-2)a} + \cdots + \binom{1}{0}\delta^{2a}\delta p_{1a}, \\
\Gamma_k &= \binom{k}{k}\delta_{ka}\delta y^{ka} + \binom{k-1}{k-1}\delta_{(k-1)a}\delta y^{(k-1)a} + \cdots \\
&\quad + \binom{1}{1}\delta_{1a}\delta y^{1a} + \binom{0}{0}\delta_{0a}\delta y^{0a} \\
&\quad + \binom{k-1}{k-1}\delta^{ka}\delta p_{ka} + \binom{k-2}{k-2}\delta^{(k-1)a}\delta p_{(k-1)a} + \cdots + \binom{0}{0}\delta^{1a}\delta p_{1a}.
\end{aligned}$$

Theorem 4.5. *The Liouville 1-form fields $\Gamma_0, \Gamma_1, \dots, \Gamma_k$ in $(\mathrm{GLH})^{(nk)}$ in the natural basis $\bar{B}^*$ of $T^*(\mathrm{GLH})^{(nk)}$ have the form obtained from (4.18) if δ_{Aa}, $\delta^{\alpha a}$, δy^{Aa}, $\delta p_{\alpha a}$ are substituted by ∂_{Aa}, $\partial^{\alpha a}$, dy^{Aa}, $dp_{\alpha a}$ respectively for every $A = \overline{0,k}$, $\alpha = \overline{1,k}$.*

The proof follows from Remark 4.1.

5. The sprays and antisprays in $(\mathrm{GLH})^{(nk)}$

The tangent vector to the curve

$$c(t) = (y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), p_{2a}(t), \dots, p_{ka}(t)) \tag{5.1}$$

is the vector dr given by (4.3) and the tangent '1-form' of the curve $c(t)$ is δr defined by (4.5).

Definition 5.1. A k-spray on $(\mathrm{GLH})^{(nk)}$ is a vector field $\bar{S} \in T(\mathrm{GLH})^{(nk)}$ with the property

$$\bar{S}\bar{J} = \bar{J}\bar{S} = \bar{\Gamma}_{k-1}, \tag{5.2}$$

where (see 4.8)

$$\begin{aligned}\bar{\Gamma}_{k-1} = \delta y^{0a}\delta_{1a} + 2\delta y^{1a}\delta_{2a} + \dots + k\delta y^{(k-1)a}\delta_{ka} + \\ \delta p_{1a}\delta^{2a} + \dots + (k-1)\delta p_{(k-1)a}\delta^{ka}.\end{aligned}$$

Theorem 5.1. *The vector field $\bar{S}$ given by*

$$\bar{S} = \bar{\Gamma}_k + \alpha\delta y^{0a}\delta_{ka} + \beta\delta p_{1a}\delta^{ka} \tag{5.3}$$

(α, β real numbers) is a k-spray on $(\mathrm{GLH})^{(nk)}$.

Proof.

$$\begin{aligned}\bar{J}\bar{S} &= \bar{J}\bar{\Gamma}_k + \alpha(\bar{J}\delta y^{0a})\delta_{ka} + \beta(\bar{J}\delta p_{1a})\delta^{ka} = \bar{J}\bar{\Gamma}_k = \bar{\Gamma}_{k-1}\\ \bar{S}\bar{J} &= \bar{\Gamma}_k\bar{J} + \alpha\delta y^{0a}(\delta_{ka}\bar{J}) + \beta\delta p_{1a}(\delta^{ka}\bar{J}) = \bar{\Gamma}_k\bar{J} = \bar{\Gamma}_{k-1}.\end{aligned}$$

□

From the above it follows that $\bar{S}$ given by (5.3) satisfy (5.2), i.e., it is a k-spray.

Theorem 5.2. *The vector field $\bar{S}$ is a tangent vector to the curve $c(t)$ given by (5.1) if in (5.3) $\alpha = 0$, $\beta = 0$, i.e.*

$$\bar{S} = \bar{\Gamma}_k = dr.$$

The vector field $\bar{S}$ in the natural basis $\bar{B}$ has the form

$$\bar{S} = dy^{0a}\partial_{0a} + \dots + dy^{ka}\partial_{ka} + dp_{1a}\partial^{1a} + \dots + dp_{ka}\partial^{ka}.$$

If we take

$$\begin{aligned}dy^{ka} &= -kG^{ka}(y^{0a}, y^{1a}, \dots, y^{(k-1)a})dt\\ dp_{ka} &= -kH_{ka}(y^{0a}, y^{1a}, \dots, y^{(k-1)a}, p_{1a}, \dots, p_{ka})dt,\end{aligned}$$

where the functions G^{ka} and H_{ka} have the same transformation laws as dy^{ka} and dp_{ka} respectively, then the integral curve of $\bar{S}$ is the solution of SODE

$$\begin{aligned}\frac{dy^{ka}}{dt} + kG^{ka}(y^{0a}, y^{1a}, \dots, y^{(k-1)a}) = 0\\ \frac{dp_{ka}}{dt} + kH_{ka}(y^{0a}, \dots, y^{(k-1)a}, p_{1a}, \dots, p_{ka}) = 0\end{aligned} \tag{5.4}$$

with corresponding initial conditions.

Theorem 5.3. *The vector fields* $\bar{S}, \bar{\Gamma}_0, \bar{\Gamma}_1, \ldots, \bar{\Gamma}_k$ *and the structure* $\bar{J}$ *are connected by*

$$\bar{J}\bar{S} = \bar{S}\bar{J} = \bar{\Gamma}_{k-1},\ \ \bar{J}^2\bar{S} = \bar{S}\bar{J}^2 = 2!\bar{\Gamma}_{k-2}, \ldots$$
$$\bar{J}^k\bar{S} = \bar{S}\bar{J}^k = k!\bar{\Gamma}_0,\ \ \bar{J}^{k+1}\bar{S} = \bar{S}\bar{J}^{k+1} = 0.$$

Definition 5.2. A k-antispray field on $(GHL)^{(nk)}$ is a 1-form field $S \in T^*(\mathrm{GLH})^{(nk)}$ with the property

$$SJ = JS = \Gamma_{k-1}, \tag{5.5}$$

where

$$\begin{aligned}\Gamma_{k-1} = \delta_{1a}\delta y^{0a} + 2\delta_{2a}\delta^{1a} + \cdots + (k-1)\delta_{(k-1)a}\delta^{(k-2)a} + k\delta_{ka}\delta y^{(k-1)a} \\ +\delta^{2a}\delta p_{1a} + \cdots + (k-2)\delta^{(k-1)a}\delta p_{(k-2)a} + (k-1)\delta^{ka}\delta p_{(k-1)a}.\end{aligned}$$

Theorem 5.4. *The 1-form field S given by*

$$S = \Gamma_k + \alpha\delta_{ka}\delta y^{0a} + \beta\delta^{ka}\delta p_{1a} \tag{5.6}$$

is a k-antispray on $(\mathrm{GLH})^{(nk)}$.

Proof. We have

$$\begin{aligned}JS &= J\Gamma_k + \alpha(J\delta_{ka})\delta y^{0a} + \beta(J\delta^{ka})\delta p_{1a} = J\Gamma_k = \Gamma_{k-1} \\ SJ &= \Gamma_k J + \alpha\delta_{ka}(\delta_{0a}J) + \beta\delta^{ka}(\delta p_{1a}J) = \Gamma_k J = \Gamma_{k-1}.\end{aligned}$$

From the above it follows that S given by (5.6) satisfy (5.5), i.e., it is k-antispray on $(\mathrm{GLH})^{(nk)}$. □

Theorem 5.5. *The k-antispray field S on* $(\mathrm{GLH})^{(nk)}$ *is parallel to the tangent 1-form of the curve $c(t)$ denoted by δr if in (5.6) we take $\alpha = 0$, $\beta = 0$, i.e. $S = \Gamma_k$.*

Proof. From the last equation of (4.10) and (4.5) it follows $\Gamma_k = \delta r$. □

Theorem 5.6. *The k-antispray S, the Liouville 1-form fields $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ and the structure J are connected by*

$$\begin{array}{ll} SJ = JS = \Gamma_{k-1} & SJ^i = J^iS = i!\Gamma_{k-i}, \ldots \\ SJ^2 = J^2S = 2!\Gamma_{k-2} & SJ^k = J^kS = k!\Gamma_0, \\ SJ^3 = J^3S = 3!\Gamma_{k-3}, \ldots & SJ^{k+1} = J^{k+1}S = 0. \end{array}$$

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University of Novi Sad,
Faculty of Technical Sciences,
21000 Novi Sad,
Serbia
E-mail address: irena.comic@gmail.com