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SOME COMPARISON THEOREMS IN FINSLER AND MINKOWSKI SPACES

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Abstract. It is given the survey of comparison theorems for volumes balls and spheres in Finsler and Hilbert spaces.

1. Riemann – Hadamard manifolds

In 1972, in the course of the study of some problems of geometric probability in $\mathbb{H}^2$, L. A. Santaló and I. Yañez [18] proved the following result: *Let* $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ *be a family of compact h–convex domains in* $\mathbb{H}^2$ *which expands over the whole space. Then* $\lim_{t\to 0}\frac{\mathrm{Area}(\Omega(t))}{\mathrm{Length}(\partial\Omega(t))}=1$. A domain in the hyperbolic space $\mathbb{H}^{n+1}$ of sectional curvature -1 (and dimension $n+1$) is a closed subset of $\mathbb{H}^{n+1}$ with interior not empty. An *h-convex domain* (or convex by horoballs in the terminology of [2]) in the hyperbolic space $\mathbb{H}^{n+1}$ of sectional curvature -1 (and dimension $n+1$) is a domain $\Omega\subset H^{n+1}$ with boundary $\partial\Omega$ such that, for every $p\in\partial\Omega$, there is a horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}$ through p such that Ω is contained in the horoball of $\mathbb{H}^{n+1}$ bounded by $\mathcal{H}$. This $\mathcal{H}$ is called a *supporting horosphere* of Ω (and of $\partial\Omega$). We say that *a family of domains* $\{C(t)\}_{t\in\mathbb{R}^+}$ *in* $\mathbb{H}^{n+1}$ *expands over the whole space* (e.o.w.s. in abbreviated notation) if for any $x\in\mathbb{H}^{n+1}$ there is a $t_0\in\mathbb{R}$ such that, for every $t>t_0$, $x\in C(t)$.

Theorem 1 ([5], A. A. Borisenko, V. Miquel, 1997)**.** *Let* $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ *be a family of h–convex domains expanding over the whole Lobachevsky space* $\mathbb{H}^{n+1}$*. Then*

$$\lim_{t\to\infty}\frac{volume(\Omega(t))}{volume(\partial\Omega(t))}=\frac{1}{n}.$$

This result had been generalized for Riemannian – Hadamard manifolds M.

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Definition 1. A C^2 hypersurface $N \subset M$ such that in every point all the normal curvatures are greater or equal than a non-negative λ is called a regular λ–convex hypersurface. When N is the boundary of a domain Ω it is said that Ω is a regular λ–convex domain when its normal curvature with respect to the inward normal direction is greater or equal than λ.

Definition 2. A locally hypersurface N of a Hadamard manifold is said to be h–convex if every point has a locally supporting horosphere.

Theorem 2 ([4], A. A. Borisenko, 2002)**.** *Let M be a $(n+1)$–dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \le K \le -k_1^2, \quad k_1, k_2 > 0$$

Then there are functions $\alpha(r)$, $\alpha_1(r)$ of the inradius and $\beta(R)$, $\beta_1(R)$ of the circumradius such that $\alpha(r), \alpha_1(r) \to 1/(nk_2)$ and $\beta(R), \beta_1(R) \to 1/(nk_1)$ when r and R grow to infinity such that:

(a1) *For compact λ–convex domain Ω in M with $\lambda \le k_2$*

$$\alpha(r)\frac{\lambda}{k_2} \le \frac{\operatorname{vol}(\Omega)}{\operatorname{vol}(\partial\Omega)} \le \beta(R).$$

(a2) *For compact h–compact domain Ω in M*

$$\alpha_1(r) \le \frac{\operatorname{vol}(\Omega)}{\operatorname{vol}(\partial\Omega)} \le \beta_1(R).$$

For a family $\{\Omega(t)\}_{t\in\mathbf{R}^+}$ of compact convex domains expanding over the whole space as a consequence there are true the following results:

(b1) *For compact λ–convex sets $\{\Omega(t)\}_{t\in\mathbf{R}^+}$, $\lambda \le k_2$*

$$\frac{\lambda}{nk_2^2} \le \lim_{t\to\infty}\inf\frac{\operatorname{vol}(\Omega(t))}{\operatorname{vol}(\partial\Omega(t))} \le \lim_{t\to\infty}\sup\frac{\operatorname{vol}(\Omega(t))}{\operatorname{vol}(\partial\Omega(t))} \le \frac{1}{nk_1}.$$

(b2) *For compact h–convex sets $\{\Omega(t)\}_{t\in\mathbf{R}^+}$, $\lambda \le k_2$*

$$\frac{1}{nk_2} \le \lim_{t\to\infty}\inf\frac{\operatorname{vol}(\Omega(t))}{\operatorname{vol}(\partial\Omega(t))} \le \lim_{t\to\infty}\sup\frac{\operatorname{vol}(\Omega(t))}{\operatorname{vol}(\partial\Omega(t))} \le \frac{1}{nk_1}.$$

(b3) *For compact h–convex sets $\{\Omega(t)\}_{t\in\mathbf{R}^+}$ in Lobachevsky space $\mathbf{H^{n+1}}$ of the sectional curvature – 1*

$$\lim_{t\to\infty}\frac{\operatorname{vol}(\Omega(t))}{\operatorname{vol}(\partial\Omega(t))} = \frac{1}{n}.$$

2. Finsler – Hadamard manifolds

Our goal is to generalize this theorem for Finsler manifolds.

By definition, a Finsler metric on a manifold is a family of Minkowski norms on the tangent spaces. A *Minkowski norm* on a vector space V^n is a nonnegative function $F\colon V^n \to [0,\infty)$ with the following properties:

(1) F is positively homogeneous of degree one, i.e., for any $y \in V^n$ and any $\lambda > 0$, $F(\lambda y) = \lambda F(y)$;
(2) F is C^∞ on $V^n \backslash \{0\}$ and for any vector $y \in V^n$ the following bilinear symmetric form $g_y \colon V^n \times V^n \to \mathbb{R}$ is positive definite,

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial t \partial s}[F^2(y+su+tv)]|_{s=t=0}.$$

Property 2 is also called the *strong convexity property.*

A Minkowski norm is said to be *reversible* if $F(y) = F(-y)$, $y \in V^n$. In this paper, Minkowski norms are not assumed to be reversible.

By 1. and 2., one can show that $F(y) > 0$ for $y \neq 0$ and $F(u+v) \leqslant F(u)+F(v)$.

A vector space V^n with the Minkowski norm is called a *Minkowski space.* Notice that reversible Minkowski spaces are finite-dimensional Banach spaces.

Let (V^n, F) be the Minkowski space. Then the set $I = F^{-1}(1)$ is called the *indicatrix* in the Minkowski space. It is also called the *unit sphere.*

A set $U \subset V^n$ is said to be *strongly convex* if there exist a function F satisfying 2. such that $\partial U = F^{-1}(1)$. Remark that a strong convexity is equivalent to a positivity of all normal curvatures of ∂U for any Euclidean metric on V^n.

Let M^n be an n-dimensional connected C^∞-manifold. Denote by $TM^n = \bigsqcup_{x\in M^n} T_xM^n$ the tangent bundle of M^n, where T_xM^n is the tangent space at x. A *Finsler metric* on M^n is a function $F \colon TM^n \to [0,\infty)$ with the following properties:

(1) F is C^∞ on $TM^n \backslash \{0\}$;
(2) At each point $x \in M^n$, the restriction $F|_{T_xM^n}$ is a Minkowski norm on T_xM^n.

The pair (M^n, F) is called *a Finsler manifold.*

Let (M^n, F) be a Finsler manifold. Let (x^i, y^i) be a standard local coordinate system in TM^n, i.e., y^i are determined by $y = y^i \frac{\partial}{\partial x^i}|_x$. For a non-zero vector $y = y^i \frac{\partial}{\partial x^i}$, put $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}(x,y)$. The induced inner product g_y is given by

$$g_y(u,v) = g_{ij}(x,y)u^iv^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$.

By the homogeneity of F, we have $F(x,y) = \sqrt{g_y(y,y)} = \sqrt{g_{ij}(x,y)y^iy^j}$.

In the Riemannian case g_{ij} are functions of $x \in M^n$ only, and in the Minkowski case g_{ij} are functions of $y \in T_xM^n = V^n$ only.

The notions of length and area are also generalized to Finsler geometry.

Given a Finsler metric F on a manifold M^n.

Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM^n and $\{\theta^i\}_{i=1}^n$ the dual basis for $T_x^*M^n$. The set

$$B_x^n = \left\{(y^i) \in \mathbb{R}^n : F(x, y^ie_i) < 1\right\}$$

is an open strongly convex open subset in $\mathbb{R}^n$, bounded by the indicatrix in T_xM^n. Then define

$$dV_F = \sigma_F(x)\theta^1 \wedge \ldots \wedge \theta^n,$$

where

$$\sigma_F(x) := \frac{\mathrm{Vol}_E(\mathbb{B}^n)}{\mathrm{Vol}_E(B_x^n)}.$$

Here $\mathrm{Vol}_E(A)$ denotes the Euclidean volume of A, and $\mathbb{B}^n$ is the standard unit ball in $\mathbb{R}^n$.

The volume form dV_F determines a regular measure $\mathrm{Vol}_F = \int dV_F$ and is called the *Busemann–Hausdorff volume form.*

For any Riemannian metric $g_{ij}(x)u^iv^j$ the Busemann–Hausdorff volume form is the standard Riemannian volume form

$$dV_g = \sqrt{\det(g_{ij})}\theta^1 \wedge \ldots \wedge \theta^n.$$

Let $\varphi\colon N^{n-1} \to M^n$ be a hypersurface in (M^n, F).

The Finsler metric F determines a local normal vector field as follows. A vector n_x is called the *normal vector* to N^{n-1} at $x \in N^{n-1}$ if $n_x \in T_{\varphi(x)}M^n$ and $g_{n_x}(y, n_x) = 0$ for all $y \in T_xN^{n-1}$. Notice that in general non-symmetric case the vector $-n_x$ is not a normal vector.

Define now an induced volume form on N^{n-1}. Let n be a unit normal vector field along N^{n-1}. Let $\overline{F} = \varphi^*F$ be the induced Finsler metric on N^{n-1} and $dV_{\overline{F}}$ be the Busemann–Hausdorff volume form of $\overline{F}$. For $x \in N^{n-1}$ we define

$$\zeta(x, n_x) := \frac{\mathrm{Vol}_E(\mathbb{B}^n)}{\mathrm{Vol}_E(B_x^n)} \frac{\mathrm{Vol}_E(B_x^{n-1}(n_x))}{\mathrm{Vol}_E(\mathbb{B}^{n-1})}.$$

Here $B_x^n = \big\{(y^i) \in \mathbb{R}^n : F(y^ie_i) < 1\big\}$. To define $B_x^{n-1}(n_x)$ we take a basis $\{e_i\}_{i=1}^n$ for $T_{\varphi(x)}M^n$ such that $e_1 = n_x$ and $\{e_i\}_{i=2}^n$ is a basis for T_xN^{n-1}. Then $B_x^{n-1}(n_x) = \big\{(y^j) \in \mathbb{R}^{n-1} : F(y^je_j) < 1\big\}$, where the index j passes from 2 to n.

Note that if F is Riemannian, then $\zeta \equiv 1$.

Set

$$dA_F := \zeta(x, n_x)dV_{\overline{F}}.$$

The form dA_F is called the *induced volume form* of dV_F with respect to n [19].

The sense of defining such volume form is given by the *co-area formula* [19]. We shall need the co-area formula in one simple case for metric balls:

$$\mathrm{Vol}(B(r,p)) = \int_0^r \mathrm{Vol}(S(t,p))dt.$$

Here $\mathrm{Vol}(S(t,p))$ is the induced volume on $S(t,p)$ [19].

Finally, we introduce some more functions which are called *non-Riemannian curvatures.* These curvatures all vanish for Riemannian spaces. We shall need only one of this curvatures, which is closely connected to the volume form.

Let (M^n, F) be a Finsler space. Consider the Busemann-Hausdorff volume form dV_F with the density σ_F. We define

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)}, y \in T_x M^n.$$

τ is called the *distortion* of (M^n, F). The condition $\tau \equiv const$ implies F is a Riemannian metric [19].

To measure the rate of changes of the distortion along geodesics, we define

$$S(x, y) = \frac{d}{dt} \left[\tau(c(t), \dot{c}(t))\right]|_{t=0}, y \in T_x M^n$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. S is called the *S-curvature*. It is also called the *mean covariation* and *mean tangent curvature*. One can easily show that $S = 0$ for any Riemannian metric.

A Finsler metric F is said to be of *constant S-curvature* δ if

$$S(x, y) = \delta F(x, y)$$

for all $y \in T_x M^n \backslash \{0\}$ and $x \in M^n$. The upper and lower bounds of S-curvature are defined by the same way.

For a given vector $y \in T_x M^n \backslash \{0\}$ denote by Y its extension to a geodesic field in a neighborhood of x. Let ∇ denote the Chern connection, $\tilde{\nabla}$ denote the Levi-Civita connection of the induced Riemannian metric $\tilde{g} = \mathbf{g}_Y$. For a vector $v \in T_x M^n$ define

$$\mathbf{T}_y(v) = \mathbf{g}_y(\nabla_v V, y) - \tilde{g}(\tilde{\nabla}_v V, y)$$

where V is a vector field such that $V_x = v$.

The function $\mathbf{T}_y(v)$, $y \in T_x M^n \backslash \{0\}$ is called $\mathbf{T}$-*curvature*.

$\mathbf{T}$-curvature is said to be bounded above $\mathbf{T} \geqslant -\delta$ if [19, p. 223]

$$\mathbf{T}_y(u) \geqslant -\delta \left[\mathbf{g}_y(u, u) - \mathbf{g}_y\left(u, \frac{y}{F(y)}\right)^2\right] F(y)$$

The upper bound is defined at the same manner.

Notice that the $\mathbf{T}$-curvature vanish for Berwald metrics; the converse is also true [19, p. 155].

Theorem 3 ([6], A. A. Borisenko, E. A. Olin, 2007)**.** *Let (M^{n+1}, F) be an $(n+1)$-dimensional Finsler-Hadamard manifold that satisfies the following conditions:*

(1) *Flag curvature satisfies the inequalities $-k_2^2 \leqslant K \leqslant -k_1^2$, $k_1, k_2 > 0$,*
(2) *S-curvature satisfies the inequalities $n\delta_1 \leqslant S \leqslant n\delta_2$ such that $\delta_i < k_i$.*

Let $B_r^{n+1}(p)$ be the metric ball of radius r in M^{n+1} with the center at a point $p \in M^{n+1}$, $S_r^n(p) = \partial B_r^{n+1}(p)$ be the metric sphere. Let $\mathrm{Vol} = \int dV_F$ be the measure of Busemann-Hausdorff, $Area(S_r^n(p)) = \int dA_F$ is the induced measure on $S_r^n(p)$.

Then there exist functions $f(r)$ and $\mathcal{F}(r)$ such that $f(r) \to 1/(n(k_2-\delta_2))$ and $\mathcal{F}(r) \to 1/(n(k_1-\delta_1))$ as r goes to infinity and that

$$f(r) \leqslant \frac{\mathrm{Vol}(B_r^{n+1}(p))}{\mathrm{Area}(S_r^n(p))} \leqslant \mathcal{F}(r).$$

Here

$$\begin{aligned} f(r) &= \frac{1}{(1-e^{-2k_2r})^n}\times \\ &\times\left(\frac{1}{n(k_2-\delta_2)} - \frac{n}{n(k_2-\delta_2)-2k_2}(e^{-2k_2r}-e^{-nr(k_2-\delta_2)})\right) \\ \mathcal{F}(r) &= \frac{1}{n(k_1-\delta_1)}(1-e^{-nr(k_1-\delta_1)}). \end{aligned}$$

As a consequence, for a family $\{B_r^{n+1}(p)\}_{r\geqslant 0}$ we have

$$\begin{aligned} \frac{1}{n(k_2-\delta_2)} &\leqslant \lim_{r\to\infty}\inf \frac{\mathrm{Vol}(B_r^{n+1}(p))}{\mathrm{Area}(S_r^n(p))} \\ &\leqslant \lim_{r\to\infty}\sup \frac{\mathrm{Vol}(B_r^{n+1}(p))}{\mathrm{Area}(S_r^n(p))} \leqslant \frac{1}{n(k_1-\delta_1)}. \end{aligned}$$

If (M^{n+1},F) is a space of constant flag curvature $K=-k^2$ and S-curvature $S=n\delta$, $\delta<k$, we have

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r^{n+1}(p))}{\mathrm{Area}(S_r^n(p))} = \frac{1}{n(k-\delta)}$$

For a Riemannian space $S=0$ and Theorem 2 is a special case of Theorem 1.

Let (M^{n+1},F) be a Finsler manifold. Then the exponential speed of the volume growth of a ball of radius $t>0$ is called the *volume growth entropy* of (M^{n+1},F). The explicit expression for the volume growth entropy is given by

$$\lim_{t\to\infty} \frac{\ln(\mathrm{Vol}(B_t^{n+1}(p))}{t}.$$

In this section we estimate the volume growth entropy of a Finsler-Hadamard manifold with the pinched flag curvature and the S-curvature.

Theorem ([6]). *Let (M^{n+1},F) be an $(n+1)$-dimensional Finsler-Hadamard manifold that satisfies the following conditions:*

1. *Flag curvature satisfies the inequalities $-k_2^2 \leqslant K \leqslant -k_1^2$, $k_1,k_2>0$,*
2. *S-curvature satisfies the inequalities $n\delta_1 \leqslant S \leqslant n\delta_2$ such that $\delta_i<k_i$.*

Then we have

$$n(k_1-\delta_1) \leqslant \lim_{t\to\infty} \frac{\ln(\mathrm{Vol}(B_t^{n+1}(p)))}{t} \leqslant n(k_2-\delta_2).$$

If (M^{n+1}, F) *is a space of constant flag curvature* $K = -k^2$ *and S-curvature* $S = n\delta$, $\delta < k$, *we have*

$$\lim_{t\to\infty} \frac{\ln(\mathrm{Vol}(B_t^{n+1}(p)))}{t} = n(k-\delta).$$

Let M be a complete Finsler manifold. Then

(1) A set A is said to be *convex*, if each shortest path with endpoints in A is entirely contained in A.
(2) A set A is said to be *locally convex*, if each point $P \in A$ has a neighborhood U_P in M such that the set $A \cap U_P$ is convex.

Hadamard proved the following theorem.

Theorem ([20])**.** *Let* φ *be an immersion of compact n-dimensional oriented manifold* M *in Euclidean space* E^{n+1}, $n \geqslant 2$ *with everywhere positive Gaussian curvature. Then* $\varphi(M)$ *is a convex hypersurface.*

Chern and Lashof generalized this theorem.

Theorem ([20])**.** *Let* φ *be an immersion of compact n-dimensional oriented manifold* M *in Euclidean space* E^{n+1}, $n \geqslant 2$. *Then the following two assertions are equivalent.*

(1) *The degree of the spherical mapping equals* ± 1 *and the Gaussian curvature does not change its sign;*
(2) $\varphi(M)$ *is a convex hypersurface.*

A topological immersion $f\colon N^n \to M^{n+1}$ of a manifold N^n into a Riemannian manifold M^{n+1} is called *locally convex* at a point $x \in N^n$ if x has a neighborhood U such that $f(U)$ is a part of the boundary of a convex set in M^{n+1}.

Heijenoort proved the following theorem.

Theorem ([21])**.** *Let* $f\colon N^n \to E^{n+1}$, $n \geqslant 2$ *be a topological immersion of a connected manifold* N^n. *If* f *is locally convex at all points and has at least one point of local strict support and* N^n *is complete in the metric induced by this immersion, then* f *is an embedding and* $f(N^n)$ *is the boundary of a convex body.*

S. Alexander [1] (see also A. A. Borisenko [3]) generalized Hadamard's theorem for compact immersions when an ambient space is a complete simply connected manifold of non-positive curvature (*Hadamard* manifold).

Theorem ([1], [3])**.** *Let* $f\colon N^n \to M^{n+1}$, $n \geqslant 2$ *be an immersion of compact connected manifold* N^n *in a complete simply connected Riemannian manifold* M^{n+1} *of non-positive sectional curvature. If the immersion* f *is locally convex, then* f *is an embedding,* $f(N^n)$ *is the boundary of a convex set in* M^{n+1}, *and* $f(N^n)$ *is homeomorphic to the sphere* $\mathbb{S}^n$.

The goal of this paper is to generalize this theorem to an immersion of compact manifold into a complete simply connected Finsler manifold of non-positive curvature (*Finsler–Hadamard* manifold).

Theorem 4 ([8], A. A. Borisenko, E. A. Olin, 2008)**.** *Let $f\colon N^n \to M^{n+1}$, $n \geqslant 2$ be an immersion of a compact connected manifold N^n in a complete simply connected Finsler manifold M^{n+1}. Let N^n and M^{n+1} satisfy the following conditions:*

(1) *The flag curvature $K \leqslant -k^2$, $k \geqslant 0$;*
(2) *The $\mathbf{T}$-curvature $|\mathbf{T}| \leqslant \delta$, where $0 \leqslant \delta < k$;*
(3) *All the normal curvatures of N^n*

$$\mathbf{k_n} > 2\delta.$$

Then f is an embedding, $f(N^n)$ is the boundary of a convex set in M^{n+1} which is homeomorphic to the ball, and $f(N^n)$ is homeomorphic to the sphere $\mathbb{S}^n$.

We also show that the theorem of S. Alexander holds for Berwald spaces without any additional restrictions.

Note, that if all the flag curvatures vanish (the case when $k = 0$) then $\mathbf{T} = 0$ and we obtain the generalization of Hadamard's theorem to Minkowski spaces.

3. Asymptotic Properties of Hilbert geometry

Consider a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures in $\mathbb{R}^n$ equipped with a Euclidean norm $\|\cdot\|$.

For given two distinct points p and q in U, let p_1 and q_1 be the corresponding intersection point of the half line $p + \mathbb{R}_-(q-p)$ and $p + \mathbb{R}_+(q-p)$ with ∂U. Then consider the following distance function

$$d_U(p,q) = \frac{1}{2}\ln\frac{\|q-q_1\|}{\|q-p_1\|} \times \frac{\|p-p_1\|}{\|p-q_1\|}$$
$$d_U(p,p) = 0$$

The obtained metric space (U, d_U) is called Hilbert geometry and is a complete noncompact geodesic metric space with the $\mathbb{R}^n$-topology and in which the affine open segments joining two points are geodesics [11].

The distance function is associated in a natural way with the Finsler metric F_U on U. For a point $p \in U$ and a tangent vector $v \in T_pU = \mathbb{R}^n$

$$F_U(p,v) = \frac{1}{2}\|v\|\left(\frac{1}{\|p-p_-\|} + \frac{1}{\|p-p_+\|}\right)$$

where p_- and p_+ is the intersection point of the half-lines $p + \mathbb{R}_- v$ and $p + \mathbb{R}_+ v$ with ∂U.

Then $d_U(p,q) = \inf \int_I F_U(c(t), \dot{c}(t))dt$ when $c(t)$ ranges over all smooth curves joining p to q.

In is known (see for example [19]) that Hilbert metrics are the metrics of constant flag curvature -1.

When $U = B_r^n$ then we obtain the Klein model of the n-dimensional Lobachevsky space $\mathbb{H}^n$ and the Finsler metric has the explicit expression

$$F_{B_r^n}(p,v) = \sqrt{\frac{\|v\|^2}{r - \|p\|^2} + \frac{< v, p >^2}{(r^2 - \|p\|^2)^2}}$$

It is proved in [11] that the balls of arbitrary radii are convex sets in Hilbert geometry.

The asymptotic properties of Hilbert geometry have been obtained lately. All this properties mean that Hilbert geometry is “almost” Riemannian at infinity. It is proved in [14] that Hilbert metric “tends” to Riemannian metric as follows.

Theorem ([14])**.** *Let $\mathcal{C} \in \mathbb{R}^n$ be a bounded open convex domain whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex. For any $p \in \mathcal{C}$ let $\delta(p) > 0$ be the Euclidean distance from p to $\partial\mathcal{C}$. Then there exists a family $(\vec{l}_p)_{p\in\mathcal{C}}$ of linear transformations in $\mathbb{R}^n$ such that*

$$\lim_{\delta(p)\to 0} \frac{F_C(p,v)}{\|\vec{l}_p(v)\|} = 1$$

uniformly in $v \in \mathbb{R}^n \backslash \{0\}$

This means that the unit sphere in the tangent space of given Hilbert metric tends to ellipsoid in continuous topology as the tangent point goes to the absolute.

B. Colbois and P. Verovic proved in [14] that the balls in an $(n+1)$-dimensional Hilbert geometry have the same volume growth entropy as those in $\mathbb{H}^{n+1}$, namely n. We obtain the analogous result for the spheres in Hilbert geometry. At theorem 5 we used the Busemann–Hausdorff volume form of sphere S_t^n.

Theorem 5 ([7], A. A. Borisenko, E.A. Olin, 2008)**.** *Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures. Then we have*

$$\lim_{t\to\infty} \frac{\ln(\mathrm{Vol}(S_t^n))}{t} = n$$

Theorem 6 ([7], A. A. Borisenko, E.A. Olin, 2008)**.** *Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U \in \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures. Fix a point $o \in U$, we will consider this point as the origin and the center of all the considered balls. Denote by $\omega(u)\colon \mathbb{S}^n \to \mathbb{R}_+$ the radial function for ∂U, i.e. the mapping $\omega(u)u$, $u \in \mathbb{S}^n$ is a parametrization of ∂U, and by $\iota\colon \mathbb{R}^{n+1} \to \mathbb{S}^n$ the mapping such that $\iota(p) = \frac{u_p}{||u_p||}$, where u_p is the radius-vector of a point p, dp is Euclidean volume form of boundary ∂U.*

Denote by K and k the maximum and minimum normal curvature of ∂U, $c = \max_{u\in\mathbb{S}^n} \frac{\omega(u)}{\omega(-u)}$, $\omega_0 = \min_{u\in\mathbb{S}^n}\omega(u)$, $\omega_1 = \max_{u\in\mathbb{S}^n}\omega(u)$. Then we have

$$\lim_{\rho\to\infty}\sup\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \leqslant \frac{1}{n}c^{\frac{n}{2}}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{1}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp}$$

$$\lim_{\rho\to\infty}\inf\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \geqslant \frac{1}{n}\frac{1}{c^{\frac{n}{2}}}\left(\frac{k}{K}\right)^{\frac{n}{2}}(k\omega_0)^{\frac{n}{2}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp}$$

or, more simple expression

$$\lim_{\rho\to\infty}\sup\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \leqslant \frac{1}{n}\left(\frac{K}{k}\right)^{\frac{n}{2}}\left(\frac{\omega_1}{\omega_0}\right)^{n+1}\left(\frac{\omega_1}{k}\right)^{\frac{n}{2}}\frac{1}{k\omega_1}\frac{\mathrm{Vol}_E(\mathbb{S}^n)}{\mathrm{Vol}_E(\partial U)}$$

$$\lim_{\rho\to\infty}\inf\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \geqslant \frac{1}{n}\left(\frac{k}{K}\right)^{\frac{n}{2}}\left(\frac{\omega_0}{\omega_1}\right)^{\frac{n}{2}}\omega_0^n(k\omega_0)^{\frac{n}{2}}\frac{\mathrm{Vol}_E(\mathbb{S}^n)}{\mathrm{Vol}_E(\partial U)}$$

If U is a symmetric domain with respect to o then we have

$$\lim_{\rho\to\infty}\sup\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \leqslant \frac{1}{n}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{\omega_1^n}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\mathrm{Vol}_E(\mathbb{S}^n)}{\mathrm{Vol}_E(\partial U)}$$

$$\lim_{\rho\to\infty}\inf\frac{\mathrm{Vol}(B_\rho^{n+1})}{\mathrm{Vol}(S_\rho^n)} \geqslant \frac{1}{n}\left(\frac{k}{K}\right)^{\frac{n}{2}}(k\omega_0)^{\frac{n}{2}}\omega_0^n\frac{\mathrm{Vol}_E(\mathbb{S}^n)}{\mathrm{Vol}_E(\partial U)}.$$

For Lobachevsky space these limits are equal $\frac{1}{n}$ and we obtain exact formula.

Notice that in this theorem the ratio of the volume of the ball to the *internal* volume of the sphere is considered, unlike theorem, where the *induced* volume is used.

It should be noticed here that if we calculate the Hausdorff measure for the submanifold in a Finsler manifold with the symmetric metric then we will obtain the *internal* volume on submanifold in the metric induced from the ambient space.

4. Curvature of the curves in Minkowski geometry

In Finsler space it is possible to define covariant derivative [19, p. 88] For geodesic line $\gamma(s)$ of Finsler space M^{n+1}

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where s is length parameter on the curve in Finsler space. We define the curvature of the curve L in M^{n+1} as

$$k = \|\nabla_{\dot X}\dot X\| \tag{4.1}$$

where $X = X(s)$ — parametrization of the curve, $\|\cdot\|$ is a Finsler norm of vector.

For Finsler space and Minkowski space there are another definitions of curvature [15], [10], [12], [16], [17]. Finsler definition of the curvature of the curves [15] coincides with the of Cartan definition [12].

$$\frac{1}{\rho^2} = k^2 = g_{ij}\left(x, \frac{dX}{ds}\right)\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2} \quad [17]$$

Rund definition of the curvature [16]

$$\frac{1}{r^2} = k^2 = g_{ij}\left(x, \frac{d^2X}{ds^2}\right)\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2} \quad [17]$$

coincides with the definition of the curvature at this article.

Busemann definition of the curvature of the curves in Minkowski space [10] is different from these definitions.

For Minkowski plane different definition had been used in [13].

Let in Minkowski space M^{n+1} we take auxiliary Euclidean metric and some rectangular Cartesian coordinates $y^1, \ldots, y^{n+1}$, $X = X(s)$ is the smooth parametrization of the curve s is Minkowski length of the curve. At this case the formulae (4.1) rewrite in the following way:

$$k_M = \left\|\frac{d^2X}{ds^2}\right\| \tag{4.2}$$

where $\|\cdot\|$ is Minkowski norm. A Minkowski norm on M^{n+1} is nonnegative function on a linear space $F\colon V \to [0, \infty)$ which has the following properties

1) F is C^∞ on M^{n+1}
2) $F(\lambda y) = \lambda F(y)$, for all $\lambda > 0$ and $y \in M^{n+1}$
3) For any $M^{n+1}\big|_0$, the symmetric bilinear form g_y on M^{n+1}

$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}a^i a^j$ is positive definite.

Indicatrix (unit sphere) of Minkowski space is called compact convex hypersurface $F_0 = \{y^1, \ldots, y^{n+1}, F(y^1, \ldots, y^{n+1}) = 1\}$. The ball of the radius R is the set of points such that $F(y^1, \ldots, y^n) \leq R$. Assume that normal curvatures of the indicatrix F_0 in Euclidean space E^{n+1} satisfies inequality:

$$0 < k_1 \leq k \leq k_2$$

Let $\tau(s)$ be a Minkowski unit tangent vector to the curve L in M^{n+1}. The $\tau = \tau(s)$ is the curve $I \subset F_0$. It is called tangent indicatrix.

If $\bar{s}$ is the Minkowski length of the tangent indicatrix then $k_M = \frac{d\bar{s}}{ds}$; and total curvature of the curve L then

$$w_M = \int\limits_L k_M ds$$

is equal the length of the tangent indicatrix.

For Minkowski plane it is possible to define another curvature. Let n be normal to the plane curve L. The unit normal n is a radius-vector of indicatrix

F_0 at the point with tangent vector $\tau(s)$. We take indicatrix of unit normals to an arc of the curve L, $\bar{s}$ is the length on indicatrix F_0. And define

$$k_n = \lim_{\Delta s \to 0} \frac{\Delta \bar{s}}{\Delta s}$$

and

$$k_n = \frac{k_L}{k_{F_0}}, \tag{4.3}$$

where k_L, k_{F_0} is Euclidean curvatures of the curves L, F_0 at the points where tangent vectors are parallel. At Minkowski plane this curvature doesn't coincide with the curvature which was defined by formula (4.2). These two curvatures coincide only in Euclidean plane. The curvature (4.3) was used in [13].

First question is how connect the Minkowski curvature k_M of the curve L, which we define by formula (4.2), and curvature k_e of the curve L, as the curve in Euclidean space. It is true inequalities

$$k_e r \left(\frac{k_1}{k_2}\right) \leq k_M \leq k_e r \left(\frac{k_2}{k_1}\right)^2, \tag{4.4}$$

where $r = \frac{1}{F\left(\frac{dX}{d\sigma}\right)}$, σ is Euclidean length parameter of the curve L.

For Euclidean space it known Fenchel theorem: the total curvature of closed curve in Euclidean space E^{n+1} satisfies the inequality:

$$\int k_e d\sigma \geq 2\pi \tag{4.5}$$

For Euclidean space it true Fary-Milnor theorem: If the total curvature of closed curve in Euclidean space E^{n+1}

$$\int k_e d\sigma < 4\pi \tag{4.6}$$

then the curve is unknotted. The analog of the these theorems is true for Minkowski space.

Theorem 7 ([9], A. A. Borisenko, K. Tenenblat, 2009)**.** *The total curvatures of the closed curve in Minkowski and Euclidean spaces satisfy the inequalities*

$$\left(\frac{k_1}{k_2}\right) \int k_e d\sigma \leq \int k_M ds \leq \left(\frac{k_2}{k_1}\right)^2 \int k_e d\sigma$$

For the closed convex curve in Minkowski plane

$$\omega_M = \int k_M ds = S_M(F_0),$$

where $S_M(F_0)$ is Minkowski length of the F_0. It is known that $6 \leq S_M(F_0) \leq 8$

With the curvature F_0 $k_2 \geq k \geq k_1 > 0$ the length of the indicatrix F_0 satisfies the inequalities:

$$2\pi\left(\frac{k_1}{k_2}\right) \leq S_M(F_0) \leq 2\pi\left(\frac{k_2}{k_1}\right)^2$$

Theorem 8 ([9], A. A. Borisenko, K. Tenenblat, 2009)**.** *If the total curvature of the closed curve in Minkowski space satisfies the inequality:*

$$\omega_M = \int k_M ds < 4\pi\left(\frac{k_1}{k_2}\right)$$

then the curve is unknotted.

From

$$\left(\frac{k_1}{k_2}\right)\omega_e = \left(\frac{k_1}{k_2}\right)\int k_e d\sigma \leq \omega_M = \int k_M ds \leq \left(\frac{k_2}{k_1}\right)^2 \int k_e d\sigma = \left(\frac{k_2}{k_1}\right)^2 \omega_e$$

we have $\omega_e < 4\pi$ and apply Fary–Milnor theorem.

The length of closed curve in Euclidean space which belongs to the ball of radius R satisfies the following inequality: $\sigma(L) \leq R\omega_e$

Theorem 9 ([9], A. A. Borisenko, K. Tenenblat, 2009)**.** *If the closed curve L in Minkowski space lies in ball of radius R then the Minkowski length of the curve L satisfies the inequality*

$$S_M(L) \leq \left(\frac{k_2}{k_1}\right)^4 R\omega_M(L) \tag{4.7}$$

Let M^3 be a Minkowski space with the symmetric Minkowski norm $F(y^1, y^2, y^3)$. The equation of indicatrix F_0 is $F(y) = 1$; In Minkowski space we take auxiliary Euclidean metric and some rectangular Cartesian coordinates (x^1, x^2, x^3). The normal curvatures of the indicatrix as surface in Euclidean space E^3 satisfies the inequality:

$$0 < k_1 \leq k \leq k_2$$

Let F be a surface in Minkowski space M^3 and simultaneously a surface in Euclidean space E^3. In explicit form $x^3 = f(x^1, x^2)$ is the equation of the surface. The question is how connected the normal curvatures k_e, k_M of the surface F as surfaces in Euclidean and Minkowski spaces.

The normal curvature of the surface in Finsler space it is possible define in the following way [19]: Let P be a point on the submanifold F in Finsler space, $y \in T_pF$ be an unit tangent vector to F, $c = c(s)$ be a unique geodesic in F such that the tangent vector $\dot{c}(0) = y$. The vector of normal curvature is called the vector

$$A(y) = -\nabla_{\dot{c}}\dot{c}(0),$$

where
$$\nabla_{\dot{c}}\dot{c}(0) = \left\{ \frac{d^2c^i}{ds^2}(0) + 2G^i(y) \right\} \left. \frac{\partial}{\partial x^i} \right|_p ,$$
(x^i) is a local coordinate in Finsler ambient space. And the absolute value of normal curvature
$$k_M(y) = \|\nabla_{\dot{c}}\dot{c}(0)\|$$

The normal curvature in the direction y with respect the unit Minkowski normal n is equal
$$k_n(y) = g_{ij}(n)n^i \frac{d^2x^j}{ds^2}$$

It is true the inequalities

$$k_e(y)r\left(\frac{k_1}{k_2}\right) \le k_M(y) \le k_e(y)r\left(\frac{k_2}{k_1}\right)\sqrt{2\frac{\left(1+\left(\frac{k_2}{k_1}\right)^2\right)}{\left(1+\frac{k_1}{k_2}\right)^2}} \tag{4.8}$$

If $k_e(y) > 0$ then

$$k_e(y)r\left(\frac{k_1}{k_2}\right) \le k_n(y) \le k_e(y)r\left(\frac{k_2}{k_1}\right)^2 \tag{4.9}$$

The sign of Euclidean normal curvature coincides with the sign of Minkowski normal curvature with respect the unit Minkowski normal.

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