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PROJECTIVE RANDERS CHANGES OF SPECIAL FINSLER SPACES

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Abstract. A change of Finsler metric $L(x,y) \to \bar{L}(x,y)$, is called a Randers change of L if $\bar{L}(x,y) = L(x,y) + b_\alpha(x)y^\alpha$. The purpose of this paper is to study the conditions for a Finsler space of weakly Berwald/Landsberg type which could be transformed by a Randers change to a Finsler space of the same type.

1. Introduction

Randers's well-known method for giving examples of Finsler spaces has the form

$$L(x,y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

where a_{ij} is a Riemannian metric and $\beta(y^i) = b_i y^i$ is a one form with the condition $\|b\| = \sqrt{a^{ij}b_i b_j} < 1$ (a^{ij} is the inverse of a_{ij}). If we change $\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ to a given Finsler metric, this method may lead to another Finsler metric.

Definition ([5])**.** A change of Finsler metric $L(x,y) \to \bar{L}(x,y)$, is called a *Randers change* of L if

$$\bar{L}(x,y) = L(x,y) + b_i(x)y^i \tag{1}$$

where $\beta(x,y) = b_i(x)y^i$ is a one form on a smooth manifold M^n.

Thorough this paper we always suppose the regularity, positive homogeneity and strong convexity for the Finsler structure ([3]), thus we assume *a priory* that $\bar{L}$ satisfies the ordinary conditions as fundamental function.

Another important change of Finsler metrics is the so called projective change. A change of Finsler metric $L(x,y) \to \bar{L}(x,y)$, is called a *projective change* of L

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if geodesic curves are preserved. It is a well-known fact that $L(x,y) \to \bar{L}(x,y)$ is projective if and only if there exists a scalar field $p(x,y)$ which is positive homogeneous of order one, called the projective factor, satisfying $\bar{G}^i = G^i + p(x,y)y^i$ where G^i are the geodesic spray coefficients.

Projective Randers changes are characterised by the following theorem:

Theorem ([4])**.** *A Randers change is projective if and only if b is a gradient vector field.*

Randers changes of special Finsler spaces were studied e.g. in the papers [1], [7]. In [7] Park and Lee gave conditions for Finsler spaces changed by a Randers change to be of Douglas type.

Theorem ([7])**.** *Let $F^n(M^n, L) \to \bar{F}^n(M^n, \bar{L})$ a projective Randers change. If F^n is a Douglas space, then $\bar{F}^n$ is also a Douglas space, and vice versa.*

The terminology and notations are referred basically to monograph [6]. Let M^n be an n-dimensional ($n > 2$) differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x,y)$ on M^n. A short review of the basic notations:

- the Finsler metric tensor: $g_{ij} = \dot{\partial}_i\dot{\partial}_j L^2/2$ where $\dot{\partial}_i$ refers to the partial derivation with respect to y^i. g^{ij} is the inverse of g_{ij}
- the distinguished section: $\ell^i = y^i/L$, $\ell_i = y_i/L$
- the angular metric tensor: $h_{ij} = g_{ij} - \ell_i\ell_j$
- the geodesic spray coefficients and successive y-derivatives:

$$4G_j = (\dot{\partial}_j\partial_i L^2)y^i - \partial_j L^2, \quad G^i = g^{i\alpha}G_\alpha,$$
$$G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jk},$$
$$g_{\alpha l}G^\alpha_{ijk} = G_{lijk}$$

- the hv-torsion

$$-2P_{ijk} = y_\alpha G^\alpha_{ijk}. \tag{2}$$

Throughout the paper we shall use the notation $L_i = \dot{\partial}_i L$, $L_{ij} = \dot{\partial}_j\dot{\partial}_i L$ etc. We use the following properties of the angular metric tensor freely:

- $h_{ij} = LL_{ij}$
- $h_{ij}\ell^j = 0$
- $g^{ij}h_{ik} = \delta^j_k - \ell^j\ell_k$
- $g^{ij}h_{ij} = n - 1.$

In the projective geometry of Finsler manifolds, there is an important projective invariant quantity, the *Douglas* tensor defined by

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}\left(G_{ijk}y^h + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij}\right). \tag{3}$$

2. Projective Randers changes

Lemma 1. *For a Randers change we have* $\frac{1}{L}\cdot h_{ij} = \frac{1}{\bar{L}}\cdot \bar{h}_{ij}$.

Proof. It follows from (1) that $\bar{L}_i = L_i + b_i$, $\bar{L}_{ij} = L_{ij}$. The angular metric tensor satisfies $h_{ij} = LL_{ij}$, thus $\frac{\bar{h}_{ij}}{\bar{L}} = \bar{L}_{ij} = L_{ij} = \frac{h_{ij}}{L}$. □

Lemma 2. *If* $\bar{L}(x,y) = L(x,y) + \beta(x,y)$ *is a projective Randers change, then*

$$\frac{1}{L}G_{lijk} + \frac{2}{L^2}\ell_l P_{ijk} - \frac{1}{(n+1)L}\left(h_{il}G_{jk} + h_{jl}G_{ik} + h_{kl}G_{ij}\right) = \frac{1}{\bar{L}}\bar{G}_{lijk} + \frac{2}{\bar{L}^2}\bar{\ell}_l\bar{P}_{ijk} - \frac{1}{(n+1)\bar{L}}\left(\bar{h}_{il}\bar{G}_{jk} + \bar{h}_{jl}\bar{G}_{ik} + \bar{h}_{kl}\bar{G}_{ij}\right). \tag{4}$$

Proof. From (3) one obtains

$$\frac{1}{L}h_{\alpha l}D^{\alpha}_{ijk} = \frac{1}{L}\left(g_{\alpha l} - \ell_\alpha \ell_l\right)\cdot G^{\alpha}_{ijk} - \frac{1}{(n+1)L}\left(G_{ijk}h_{\alpha l}y^\alpha + h_{il}G_{jk} + h_{jl}G_{ik} + h_{kl}G_{ij}\right).$$

From the property $h_{\alpha l}y^\alpha = 0$ it follows that

$$\frac{1}{L}h_{\alpha l}D^{\alpha}_{ijk} = \frac{1}{L}\left(g_{\alpha l} - \ell_\alpha \ell_l\right)\cdot G^{\alpha}_{ijk} - \frac{1}{(n+1)L}\left(h_{il}G_{jk} + h_{jl}G_{ik} + h_{kl}G_{ij}\right).$$

From the definition of the hv-torsion (see (2)) we conclude that

$$\begin{aligned}\frac{1}{L}h_{\alpha l}D^{\alpha}_{ijk} &= \frac{1}{L}\left(g_{\alpha l} - \frac{y_\alpha}{L}\ell_l\right)\cdot G^{\alpha}_{ijk} \\ &\quad - \frac{1}{(n+1)L}\left(h_{il}G_{jk} + h_{jl}G_{ik} + h_{kl}G_{ij}\right) \\ &= \frac{1}{L}G_{lijk} + \frac{2}{L^2}\ell_l P_{ijk} \\ &\quad - \frac{1}{(n+1)L}\left(h_{il}G_{jk} + h_{jl}G_{ik} + h_{kl}G_{ij}\right).\end{aligned}$$

The Douglas tensor D_{ijk} is projective invariant. Moreover, by Lemma 1 we have $\frac{1}{L}h_{\alpha l}D^{\alpha}_{ijk} = \frac{1}{\bar{L}}\bar{h}_{\alpha l}\bar{D}^{\alpha}_{ijk}$ and this fact completes the proof. □

In the next two sections we give two consequences of the relation (4).

3. Projective Randers change between Landsberg spaces

Definition. If a Finsler space satisfies the condition $P_{ijk} = 0$, we call it a *Landsberg* space.

Theorem 1. *Let F_n and $\bar{F}_n$ be Landsberg spaces and let $\bar{L}(x,y) = L(x,y) + \beta(x,y)$ be a projective Randers change between them. Then*

$$G_{lk} - \bar{G}_{lk} = \frac{1}{n-1} h_{kl} \lambda(x,y).$$

where $\lambda(x,y)$ is a scalar field.

Proof. Let F_n and $\bar{F}_n$ be Landsberg spaces, i.e. $P_{ijk} = \bar{P}_{ijk} = 0$. Then (4) becomes

$$\frac{1}{L} G_{lijk} - \frac{1}{(n+1)L} \left(h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij} \right) = \frac{1}{\bar{L}} \bar{G}_{lijk} - \frac{1}{(n+1)\bar{L}} \left(\bar{h}_{il} \bar{G}_{jk} + \bar{h}_{jl} \bar{G}_{ik} + \bar{h}_{kl} \bar{G}_{ij} \right).$$

Moreover, for Landsberg spaces we have $G_{lijk} - G_{iljk} = 0$, $\bar{G}_{lijk} - \bar{G}_{iljk} = 0$. These properties lead to

$$\frac{1}{(n+1)L} \left(h_{ij} G_{lk} + h_{ik} G_{lj} - h_{jl} G_{ik} - h_{kl} G_{ij} \right) = \frac{1}{(n+1)L} \left(h_{ij} \bar{G}_{lk} + h_{ik} \bar{G}_{lj} - h_{jl} \bar{G}_{ik} - h_{kl} \bar{G}_{ij} \right).$$

Hence we see that

$$h_{ji} \left(G_{lk} - \bar{G}_{lk} \right) + h_{ki} \left(G_{lj} - \bar{G}_{lj} \right) - h_{jl} \left(G_{ik} - \bar{G}_{ik} \right) - h_{kl} \left(G_{ij} - \bar{G}_{ij} \right) = 0.$$

Contraction with g^{ij} gives

$$(n-1) \left(G_{lk} - \bar{G}_{lk} \right) + h_k^\alpha \left(G_{l\alpha} - \bar{G}_{l\alpha} \right) - h_l^\alpha \left(G_{\alpha k} - \bar{G}_{\alpha k} \right) - h_{kl} g^{ji} \left(G_{ij} - \bar{G}_{ij} \right) = 0.$$

Denoting $g^{ji} \left(G_{ij} - \bar{G}_{ij} \right)$ by $\lambda(x,y)$ we find that

$$(n-1) \left(G_{lk} - \bar{G}_{lk} \right) + \left(G_{lk} - \bar{G}_{lk} \right) - \left(G_{lk} - \bar{G}_{lk} \right) - h_{kl} \lambda(x,y) = 0.$$

□

4. Projective Randers change between weakly-Berwald spaces

Definition ([2])**.** If a Finsler space satisfies the condition $G_{ij} = 0$, we call it a *weakly-Berwald* space.

Theorem 2. *Let F_n and $\bar{F}_n$ be two weakly Berwald Finsler spaces which are related by a projective Randers change $L \to \bar{L}$. Let $p(x,y)$ denote the projective factor of the change, that is $\bar{G}^i = G^i + p(x,y) y^i$. Then $\dot{\partial}_i p(x,y)$ does not depend on y.*

Proof. The equation (4) for a weakly-Berwald space becomes:

$$\frac{1}{L}G_{lijk}+\frac{2}{L^2}\ell_l P_{ijk}=\frac{1}{\bar L}\bar G_{lijk}+\frac{2}{\bar L^2}\ell_l \bar P_{ijk}.$$

Because of

$$\frac{1}{L}G_{lijk}+\frac{2}{L^2}\ell_l P_{ijk}=\frac{1}{L}g_{\alpha l}G^\alpha_{ijk}-\frac{\ell_l}{L^2}y_\alpha G^\alpha_{ijk}=\frac{1}{L}\left[\left(g_{\alpha l}-\ell_l\ell_\alpha\right)G^\alpha_{ijk}\right]$$

we have

$$\frac{1}{L}h_{l\alpha}G^\alpha_{ijk}=\frac{1}{\bar L}\bar h_{l\alpha}\bar G^\alpha_{ijk}.$$

Then it follows from $h_{l\alpha}/L=\bar h_{l\alpha}/\bar L$ that

$$\frac{1}{L}h_{l\alpha}G^\alpha_{ijk}=\frac{1}{L}h_{l\alpha}\bar G^\alpha_{ijk},$$

that is

$$0=h_{l\alpha}\left(\bar G^\alpha_{ijk}-G^\alpha_{ijk}\right). \tag{5}$$

After successive derivations we have:

$$\begin{aligned}
\bar G^i &= G^i+p(x,y)y^i\\
\bar G^j_i &= G^i_j+p_j y^i+p\delta^i_j\\
\bar G^i_{jk} &= G^i_{jk}+p_{jk}y^i+p_j\delta^i_k+p_k\delta^i_j\\
\bar G^\alpha_{ijk} &= G^\alpha_{ijk}+p_{ijk}y^\alpha+p_{jk}\delta^\alpha_i+p_{ik}\delta^\alpha_j+p_{ij}\delta^\alpha_k.
\end{aligned}$$

Substituting the last formula into (5) we have

$$\begin{aligned}
0 &= h_{l\alpha}\left(p_{ijk}y^\alpha+p_{jk}\delta^\alpha_i+p_{ik}\delta^\alpha_j+p_{ij}\delta^\alpha_k\right)\\
0 &= h_{li}p_{jk}+h_{lj}p_{ik}+h_{lk}p_{ij}.
\end{aligned}$$

By contracting with g^{li} we obtain $0=(n-1)p_{jk}+p_{jk}+p_{jk}$. This shows that $(n+1)p_{jk}=0$ therefore $p_j(x,y)$ does not depend on y. ☐

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