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TWO-DIMENSIONAL COMPLEX BERWALD SPACES WITH (α, β)-METRICS

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ABSTRACT. In this paper we study the two-dimensional complex Finsler spaces with (α, β)-metrics by using the complex Berwald frame. A special approach is dedicated to the complex Berwald spaces with (α, β) - metrics. We establish the necessary and sufficient condition so that the complex Randers and Kropina spaces should be complex Berwald spaces, and we will illustrate the existence of these spaces in some examples.

1. INTRODUCTION

In the previous papers [16], [4] we constructed the complex Berwald frame in which the orthogonality is, with respect to the Hermitian structure, defined by the fundamental metric tensor of a 2-dimensional complex Finsler space on the holomorphic tangent manifold $T'M$. The complex Berwald frame is not only a geometrical machinery, it also satisfies important properties which contain three main real scalars which live on $T'M$: one vertical curvature scalar $\mathbf{I}$ and two horizontal curvature scalars $\mathbf{K}$ and $\mathbf{W}$. Such that, the study of the horizontal and vertical holomorphic sectional curvatures was reduced to the significance of these scalars. A first classification of the complex Finsler manifold of dimension two came from the exploration of the $v\bar{v}-$, $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors, (Theorem 2.1). An immediate interest for the 2-dimensional complex Landsberg and Berwald spaces was induced by the properties of the $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors. We found that the complex Landsberg and Berwald spaces of dimension two coincide, but also other interesting properties of these spaces, (Theorem 2.2).

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The main purpose of this paper is to characterize the complex Berwald spaces with (α,β)-metrics of dimension two. We apply some of the results obtained in [4] to the 2-dimensional complex Finsler spaces with (α,β)-metrics.

Subsequently, we make an overview of the contents of the paper.

In §2 we recall some preliminary properties of the 2-dimensional complex Finsler spaces in general and complex Landsberg and Berwald spaces in particular. In §3, we prepare the tools for our aforementioned study. After we review the construction of the complex (α,β) - metrics, we find the expression of them in terms of the complex Berwald frame. The complex Randers spaces and Kropina spaces are of particular interests. We establish the necessary and sufficient condition for these spaces to be complex Berwald spaces, (Theorems 3.1 and 3.2). We also show that $\mathbf{I} = -\frac{1}{L}$ and so, the vertical holomorphic sectional curvature in direction m is negatively, (Corollary 3.2 and Proposition 3.10). All these results are in §3.1 and §3.2. Finally, in §3.3 some examples of complex Berwald spaces with (α,β)-metrics are discussed.

2. Preliminaries

For the beginning we will make a survey of two - dimensional complex Finsler geometry and we will set the basic notions and terminology. For more, see [1, 4, 15, 16].

Let M be a 2-dimensional complex manifold, $(z^k)_{k=\overline{1,2}}$ are the complex coordinates in a local chart. Everywhere in this paper the indices $i,j,k,\ldots$ run over $\{1,2\}$.

Let M be a complex manifold, $\dim_C M = n$, with $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$, is in its turn, a complex manifold, the local coordinates in a chart will be denoted by $u = (z^k,\eta^k)$ and these are changed by the rules: $z'^k = z'^k(z)$, $\eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The complexified tangent bundle of $T'M$ is decomposed as $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, which changes according to the rules obtained with Jacobi matrix of above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly (*c.n.c.*), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*), ([1], [2], [15]).

A continuous function $F : T'M \to \mathbb{R}^+$ is called complex Finsler metric on M if it fulfills the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z,\eta)\geq 0$, the equality holds if and only if $\eta=0$;
iii) $F(z,\lambda\eta)=|\lambda|F(z,\eta)$ for $\forall\lambda\in\mathbb{C}$;
iv) the Hermitian matrix $\big(g_{i\bar{j}}(z,\eta)\big)$, with $g_{i\bar{j}}=\frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j}$, called the fundamental metric tensor, is positive definite.

The pair (M,F) is called a *complex Finsler space.* The iv)-th assumption involves the strong pseudoconvexity of the Finsler metric F on the complex indicatrix $I_{F,z}=\{\eta\in T'_zM \mid F(z,\eta)<1\}$. We notice that if $g_{i\bar{j}}=g_{i\bar{j}}(z)$ the complex Finsler metric comes from Hermitian metric on M, so-called *purely Hermitian metrics* in [15].

Let us consider the Sasaki type lift of the metric tensor $g_{i\bar{j}}$,

$$G=g_{i\bar{j}}dz^i\otimes d\overline{z}^j+g_{i\bar{j}}\delta\eta^i\otimes\delta\overline{\eta}^j. \tag{2.1}$$

A Hermitian connection of $(1,0)-$ type has a special meaning, in a complex Finsler space. Its name is the Chern-Finsler connection in [1]. In the notations from [15] it is $D\Gamma N=(L^i_{jk},0,C^i_{jk},0)$, where

$$\overset{CF}{N^i_j}=g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l,\ L^i_{jk}=g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k}=\frac{\partial N^i_k}{\partial\eta^j},\ C^i_{jk}=g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial\eta^k}.$$

We denote by $_{|}$, $|$, $_{\bar{|}}$ and $\bar{|}$, the $h-$, $v-$, $\overline{h}-$, $\overline{v}-$ covariant derivatives with respect to the Chern-Finsler connection (in brief $C-F$ connection), respectively, ([15]). The nonzero curvatures coefficients of the $C-F$ connection are denoted by

$$\begin{aligned} R^i_{j\overline{h}k}&=-\delta_{\overline{h}}L^i_{jk}-\delta_{\overline{h}}(N^l_k)C^i_{jl};\ \Xi^i_{j\overline{h}k}=-\delta_{\overline{h}}C^i_{jk}=\Xi^i_{k\overline{h}j};\\ P^i_{j\overline{h}k}&=-\dot{\partial}_{\overline{h}}L^i_{jk}-\dot{\partial}_{\overline{h}}(N^l_k)C^i_{jl};\ S^i_{j\overline{h}k}=-\dot{\partial}_{\overline{h}}C^i_{jk}=S^i_{k\overline{h}j}.\end{aligned} \tag{2.2}$$

Considering the Riemann tensor

$$\mathbf{R}(W,\overline{Z},X,\overline{Y}):=G(R(X,\overline{Y})W,\overline{Z}),$$

with

$$\mathbf{R}(W,\overline{Z},X,\overline{Y})=\overline{\mathbf{R}(Z,\overline{W},Y,\overline{X})},$$

for W,X, $\overline{Z},\overline{Y}$ horizontal or vertical vectors, it results the $h\bar{h}-$, $h\bar{v}-$, $v\bar{h}-$, $v\bar{v}-$ Riemann type tensors:

$$R_{\bar{j}i\bar{h}k}=g_{l\bar{j}}R^l_{i\bar{h}k};\ P_{\bar{j}i\bar{h}k}=g_{l\bar{j}}P^l_{i\bar{h}k};\ \Xi_{\bar{j}i\bar{h}k}=g_{l\bar{j}}\Xi^l_{i\bar{h}k};\ S_{\bar{j}i\bar{h}k}=g_{l\bar{j}}S^l_{i\bar{h}k},$$

which have properties

$$R_{i\bar{j}k\overline{h}}=R_{\bar{j}i\bar{h}k};\ \Xi_{i\bar{j}k\overline{h}}=P_{\bar{j}i\bar{h}k};\ P_{i\bar{j}k\overline{h}}=\Xi_{\bar{j}i\bar{h}k};\ S_{i\bar{j}k\overline{h}}=S_{\bar{j}i\bar{h}k}=S_{\overline{h}i\bar{j}k},$$

where $R_{i\bar{j}k\overline{h}}:=\overline{R_{\bar{\imath}j\bar{k}h}}$, etc., (see [15], p. 77). Further on, everywhere the index 0 means the contraction by η, for example $R^i_{0\overline{h}k}:=R^i_{j\overline{h}k}\eta^j$.

By analogy with the real case, we defined in [4] the following: (M,F) is called *complex Landsberg space* iff $C_{j\bar{r}k|\bar{0}}=0$ and it is *called complex Berwald* space iff $C_{j\bar{r}k|\bar{h}}=0$. Note that, by Proposition 2.1 iii) from [4], (M,F) is a complex Landsberg space iff $\Xi_{\bar{r}j\bar{0}k}=0$ and (M,F) is a complex Berwald space iff $\Xi_{\overline{r}j\bar{h}k} = 0$. Moreover, any 2-dimensional complex Berwald space is Landsberg. Also, using the similar arguments like those used in [9], p. 65, we can prove that (M, F) is a complex Berwald space if and only if the coefficients L^i_{jk} of the complex Berwald connection (see [15]) depend only on z^k.

For the vertical section $\mathcal{L} = \eta^k \dot{\partial}_k$, called the Liouville complex field (or the vertical radial vector field in [1]), we consider its horizontal lift $\chi := \eta^k \delta_k$. According to [1], p. 108, [15], p. 81, the horizontal holomorphic curvature of the complex Finsler space (M, F) in η direction, is given by

$$K_F(z,\eta) = \frac{2}{L^2} G(\mathbf{R}(\chi,\bar{\chi})\chi,\bar{\chi}). \tag{2.3}$$

Next, we recall in brief the construction of the complex Berwald frame $\{l, m, \bar{l}, \bar{m}\}$ on $V_C T'M$. For more details see [16].

In [16] we set $l := l^i \dot{\partial}_i$ with its dual form $\omega = l_i \delta\eta^i$, where

$$l^i = \frac{1}{F}\eta^i \text{ and } l_i = \frac{1}{F} g_{i\bar{j}} \bar{\eta}^j = g_{i\bar{j}} l^{\bar{j}}. \tag{2.4}$$

As the vertical distribution $VT'M$ is a two-dimensional space, it is decomposed into $VT'M = \{l\} \oplus \{l\}^\perp$, where $\{l\}^\perp$ is spanned by the unit vector m obtained by requiring the orthogonality conditions $G(l, \bar{m}) = 0$ and $G(m, \bar{m}) = 1$. Taking $m_i := g_{i\bar{j}} m^{\bar{j}}$, these lead to the system

$$\begin{aligned} l_1 m^1 + l_2 m^2 &= 0 \\ m_1 m^1 + m_2 m^2 &= 1 \end{aligned}$$

with the solutions $m^1 = \dfrac{-l_2}{\Delta}$ and $m^2 = \dfrac{l_1}{\Delta}$, where $\Delta = l_1 m_2 - l_2 m_1$. A straightforward computation proves that $\Delta = \bar{\Delta}$ is real and if we replace these solutions in the second equation of the system, we will get that $\Delta^2 = g = \det(g_{i\bar{j}})$. Thus, we have $m = \frac{1}{\sqrt{g}}(-l_2 \frac{\partial}{\partial \eta^1} + l_1 \frac{\partial}{\partial \eta^2})$.

We note that $l_i l^i = 1$, $l_i m^i = l^i m_i = 0$, $m_i m^i = 1$ and, from the definition (2.1) of the metric structure G, the $(1,0)$ vectors are orthogonal to $(0,1)$ vectors, thus $l_i \bar{l}^i = 0$, etc. With respect to the complex Berwald frame, $\frac{\partial}{\partial \eta^k}$ and $g_{i\bar{j}}$ are given by $\frac{\partial}{\partial \eta^i} = l_i l + m_i m$ and $g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}$ and, from here we deduce that

$$C^i_{jk} = g^{\bar{m}i} \dot{\partial}_k g_{j\bar{m}} = A l^i m_k m_j + B m^i m_k m_j, \tag{2.5}$$

where we set $A := m^j m^k l_h C^h_{kj}$; $B := m_h m^k m^j C^h_{jk}$.

Therefore, the formulas from Proposition 3.2, in [16], become

$$\begin{aligned}
&l(l_i)=\frac{-1}{2F}l_i;\ \bar{l}(l_i)=\frac{1}{2F}l_i;\ l(m_i)=\frac{1}{2F}m_i;\ \bar{l}(m_i)=\frac{-1}{2F}m_i;\\
&m(l_i)=Am_i;\ \bar{m}(l_i)=\frac{1}{F}m_i;\ m(m_i)=\frac{1}{2}Bm_i-\frac{1}{F}l_i\\
&\bar{m}(m_i)=\frac{1}{2}\bar{B}m_i;\ l(l^i)=\frac{1}{2F}l^i;\ \bar{l}(l^i)=-\frac{1}{2F}l^i;\\
&l(m^i)=-\frac{1}{2F}m^i;\ \bar{l}(m^i)=\frac{1}{2F}m^i;\ m(l^i)=\frac{1}{F}m^i;\\
&\bar{m}(l^i)=0;\ m(m^i)=-\frac{1}{2}Bm^i-Al^i;\ \bar{m}(m^i)=-\frac{1}{F}l^i-\frac{1}{2}\bar{B}m^i.
\end{aligned}\tag{2.6}$$

By using the complex Berwald frame the local coefficients of the $v\bar{v}, v\bar{h}$, $h\bar{v}-$ Riemann type tensors can be written as

$$S_{\bar{r}j\bar{h}k}=\mathbf{I}m_{\bar{h}}m_{\bar{r}}m_jm_k,\ \Xi_{\bar{r}j\bar{h}k}=-A_{|\bar{h}}l_{\bar{r}}m_jm_k-B_{|\bar{h}}m_{\bar{r}}m_jm_k,$$

$$P_{\bar{r}j\bar{h}k}=-F[\bar{A}_{|k}|_j-\frac{1}{2F}\bar{A}_{|k}l_j+(B\bar{A}_{|k}+\frac{A}{F}\bar{A}_{|0}m_k+B\bar{A}_{|s}m^sm_k)m_j]m_{\bar{r}}m_{\bar{h}},$$

where $\mathbf{I}:=-B|_{\bar{s}}m^{\bar{s}}-\frac{B\bar{B}}{2}$ and it is called in [4] the *vertical curvature scalar.*

Taking into account (2.3), we defined in [4] *the vertical holomorphic sectional curvature* in direction l and m, respectively

$$K^v_{F,l}(z,\eta):=2\mathbf{R}(l,\bar{l},l,\bar{l})=0\ ;\ \ K^v_{F,m}(z,\eta):=2\mathbf{R}(m,\bar{m},m,\bar{m})=2\mathbf{I}.\tag{2.7}$$

Theorem 2.1. [4] *Let (M,F) be a complex Finsler space of dimension two. Then it is purely Hermitian, or it satisfies that $B=0$ and $A\neq 0$, or $B_{|k}=0$ and $AB\neq 0$.*

The above considerations get us the premises for some special characterizations of the 2- dimensional complex Landsberg spaces.

Theorem 2.2. [4] *Let (M,F) be a complex Finsler space of dimension two. The following statements are equivalent: i) (M,F) is a complex Landsberg space; ii) $A_{|\bar{0}}=B_{|\bar{0}}=0$; iii) $\bar{A}_{|k}=0$; iv) (M,F) is a complex Berwald space.*

An important result can be deduced, namely the class of 2-dimensional complex Landsberg spaces coincides with the Berwald class. Another remark is that $\bar{A}_{|k}=0$ implies $\bar{B}_{|k}=0$, but the converse is not true, (see [4]).

3. Complex Finsler spaces with (α,β)-metrics

Now, we consider $z\in M$, $\eta\in T'_zM$, $\eta=\eta^i\frac{\partial}{\partial z^i}$, $\tilde{a}:=a_{i\bar{j}}(z)dz^i\otimes d\bar{z}^j$ a purely Hermitian positive metric and $b=b_i(z)dz^i$ a differential $(1,0)-$ form. By these objects we have defined (for more details see [5]) the complex $(\alpha,\beta)-$ metric F on $T'M$

$$F(z,\eta):=F(\alpha(z,\eta),|\beta(z,\eta)|),\tag{3.1}$$

where

$$(3.2)\qquad \begin{aligned} &\alpha(z,\eta) := \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j};\\ &|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\overline{\beta(z,\eta)}} \text{ with } \beta(z,\eta) = b_i(z)\eta^i. \end{aligned}$$

Let us recall the coefficients of the $C-F$ connection corresponding to the purely Hermitian metric α are

$$\overset{a}{N^k_j} := a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l,\quad \overset{a}{L^i_{jk}} := a^{\bar{l}i}(\overset{a}{\delta_k}\, a_{j\bar{l}}),\quad \overset{a}{C^i_{jk}} = 0.$$

Now, we denote by $(\overset{a}{l}, \overset{a}{m}, \overset{a}{\bar{l}}, \overset{a}{\bar{m}})$ the complex Berwald frame of the purely Hermitian space (M, α). Their local coefficients are

$$(3.3)\qquad \begin{aligned} &\overset{a}{l}_i := \frac{1}{\alpha}a_{i\bar{j}}\bar{\eta}^j;\ \overset{a}{l^i} := \frac{1}{\alpha}\eta^i;\\ &\overset{a}{m}{}^1 = \frac{-\overset{a}{l}_2}{\Delta};\ \overset{a}{m}{}^2 = \frac{\overset{a}{l}_1}{\Delta};\ \overset{a}{m}_1 = -\Delta\,\overset{a}{l}{}^2;\ \overset{a}{m}_2 = \Delta\,\overset{a}{l}{}^1\\ &\Delta^2 := \det(a_{i\bar{j}}) \end{aligned}$$

On the one hand, we can decompose b_i into $\overset{a}{l}_i$ and $\overset{a}{m}_i$, this is $b_i = \varepsilon\,\overset{a}{l}_i + \tau\,\overset{a}{m}_i$. Contracting with η^i it results $\beta = \varepsilon\alpha$. Now, the contraction by $\overset{a}{m}{}^i$ gives $\tau = b_i\,\overset{a}{m}{}^i$. On the other hand, $\overset{a}{m}_i\, b^i = a_{i\bar{j}}\,\overset{a}{\bar{m}}{}^j\, b^i = b_{\bar{j}}\,\overset{a}{\bar{m}}{}^j = \bar{\tau}$. So that,

$||b||^2 := b_ib^i = \frac{\beta}{\alpha}\,\overset{a}{l}_i\, b^i + \tau\,\overset{a}{m}_i\, b^i = \frac{|\beta|^2}{\alpha^2} + |\tau|^2$. From here immediately results

$$(3.4)\qquad b_i = \frac{\beta}{\alpha}\,\overset{a}{l}_i + \tau\,\overset{a}{m}_i;\quad b^i = \frac{\bar{\beta}}{\alpha}\,\overset{a}{l}{}^i + \bar{\tau}\,\overset{a}{m}{}^i,$$

where $|\tau|^2 = \frac{\alpha^2||b||^2 - |\beta|^2}{\alpha^2}$.

Using (2.6) it is easy to show that

$$(3.5)\qquad \begin{aligned} &\overset{a}{l}(\alpha) = \frac{1}{2};\ \overset{a}{l}(\beta) = \frac{\beta}{\alpha};\ \overset{a}{l}(\bar{\beta}) = 0;\ \overset{a}{l}(|\beta|) = \frac{|\beta|}{2\alpha};\\ &\overset{a}{l}(\tau) = -\frac{\tau}{2\alpha};\ \overset{a}{l}(\bar{\tau}) = \frac{\bar{\tau}}{2\alpha};\ \overset{a}{l}(|\tau|) = 0;\\ &\overset{a}{m}(\alpha) = 0;\ \overset{a}{m}(\beta) = \tau;\ \overset{a}{m}(\bar{\beta}) = 0;\ \overset{a}{m}(|\beta|) = \frac{\bar{\beta}\tau}{2|\beta|};\\ &\overset{a}{m}(\tau) = 0;\ \overset{a}{m}(\bar{\tau}) = -\frac{\bar{\beta}}{\alpha^2};\ \overset{a}{m}(|\tau|) = -\frac{\bar{\beta}\tau}{2|\tau|\alpha^2}. \end{aligned}$$

Let $(l, m, \bar{l}, \bar{m})$ be the complex Berwald frame of the complex Finsler space with $(\alpha,\beta)-$ metric F, $(M, F(\alpha(z,\eta), |\beta(z,\eta)|))$. The link between these frames is $\frac{\partial}{\partial\eta^i} = l_il + m_im = \overset{a}{l}_i\overset{a}{l} + \overset{a}{m}_i\overset{a}{m}$.

Lemma 3.1. *Let $(M, F(\alpha(z,\eta), |\beta(z,\eta)|))$ be a complex Finsler space with (α,β)-metric of dimension two. Then $\alpha_{|k} = 0$ if and only if $(N^i_j - \overset{a}{N^i}_j)\overset{a}{l_i} = 0$.*

Proof. $0 = \alpha_{|k} = \delta_k \alpha = \frac{\partial a_{i\bar{j}}}{\partial z^k}\eta^i \bar{\eta}^j - \alpha N^r_k \overset{a}{l_r} = \alpha(\overset{a}{N^r}_k - N^r_k)\overset{a}{l_r}$. □

By $0 = F_{|k} = L_\alpha\, \alpha_{|k} + L_{|\beta|}|\beta|_{|k}$ and by the expression of $|\tau|$ it results the following

Lemma 3.2. *Let $(M, F(\alpha(z,\eta), |\beta(z,\eta)|))$ be a complex Finsler space with (α,β)-metric of dimension two. If $\alpha_{|k} = 0$ then $|\beta|_{|k} = 0$. Moreover, if $||b||^2$ is a constant on M then $|\tau|_{|k} = 0$.*

Further on, we focus on the two classes of complex $(\alpha,\beta)-$ metrics.

3.1. **Complex Randers metric $F := \alpha + |\beta|$.** For the complex Randers metric $F := \alpha + |\beta|$ we have, ([6])

$$\begin{aligned}
& g_{i\bar{j}} = \frac{F}{\alpha}a_{i\bar{j}} - \frac{F}{2\alpha}\overset{a}{l_i}\overset{a}{l_{\bar{j}}} + \frac{F}{2|\beta|}b_i b_{\bar{j}} + \frac{1}{2}l_i l_{\bar{j}};\\
& g^{\bar{j}i} = \frac{\alpha}{F}a^{\bar{j}i} + \frac{|\beta|(\alpha||b||^2 + |\beta|)}{\gamma}l^i l^{\bar{j}} - \frac{\alpha^3}{F\gamma}b^i \bar{b}^j - \frac{\alpha}{\gamma}(\bar{\beta}l^i\bar{b}^j + \beta b^i l^{\bar{j}});\\
(3.6)\quad & l_i := \frac{\eta_i}{F} = \frac{1}{F}\frac{\partial L}{\partial \eta^i} = \frac{L_\alpha}{F}\frac{\partial \alpha}{\partial \eta^i} + \frac{L_{|\beta|}}{F}\frac{\partial|\beta|}{\partial \eta^i} = \overset{a}{l_i} + \frac{\bar{\beta}}{|\beta|}b_i;\\
& l^i := \frac{1}{F}\eta^i = \frac{\alpha}{F}\overset{a}{l^i};\ g := \det(g_{i\bar{j}}) = \frac{\gamma F^2}{2\alpha^3|\beta|}\Delta^2\ ,
\end{aligned}$$

where $\gamma := L + \alpha^2(||b||^2 - 1)$. One can check that $\overset{a}{l}(F) = \frac{F}{2\alpha}$, $\overset{a}{l}(\gamma) = \frac{\gamma}{\alpha}$, $\overset{a}{m}(F) = \frac{\bar{\beta}\tau}{2|\beta|}$, $\overset{a}{m}(\gamma) = \frac{\bar{\beta}\tau F}{|\beta|}$. Next we compute

$$\begin{aligned}
m_1 &= -\sqrt{g}l^2 = -\sqrt{\frac{2\alpha|\beta|}{\gamma}}\Delta\,\overset{a}{l^2} = \sqrt{\frac{\gamma}{2\alpha|\beta|}}\,\overset{a}{m}_1;\\
m_2 &= \sqrt{g}l^1 = \sqrt{\frac{2\alpha|\beta|}{\gamma}}\Delta\,\overset{a}{l^1} = \sqrt{\frac{\gamma}{2\alpha|\beta|}}\,\overset{a}{m}_2;\\
m^1 &= -\frac{l_2}{\sqrt{g}} = \frac{\alpha}{F}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\overset{a}{m^1} - \frac{\bar{\beta}}{\Delta|\beta|}b_2)\\
&= \frac{\alpha}{F}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\overset{a}{m^1} - \frac{\bar{\beta}}{\Delta|\beta|}\frac{\beta}{\alpha}\overset{a}{l}_2 - \frac{\bar{\beta}}{\Delta|\beta|}\tau\,\overset{a}{m}_2) = \frac{\alpha}{F}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\overset{a}{m^1} - \frac{\bar{\beta}\tau}{|\beta|}\overset{a}{l^1}).
\end{aligned}$$

By analogy we have,

$$m^2 = \frac{l_1}{\sqrt{g}} = \frac{\alpha}{F}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\overset{a}{m^2} + \frac{\bar{\beta}}{\Delta|\beta|}b_1) = \ldots = \frac{\alpha}{F}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\frac{F}{\alpha}\overset{a}{m^2} - \frac{\bar{\beta}B}{|\beta|}\overset{a}{l^2}).$$

So, we have proved

Proposition 3.1. *Let $(M, F := \alpha + |\beta|)$ be a complex Randers space of dimension two. The coefficients of the complex Berwald frame $(l, m, \bar{l}, \bar{m})$ are*

$$l_i = \frac{F}{\alpha} \overset{a}{l}_i + \frac{\bar{\beta}\tau}{|\beta|} \overset{a}{m}_i; \; l^i = \frac{\alpha}{F} \overset{a}{l^i}; \tag{3.7}$$
$$m_i = \sqrt{\frac{\gamma}{2\alpha|\beta|}} \overset{a}{m}_i; \; m^i = \sqrt{\frac{2\alpha|\beta|}{\gamma}} (\overset{a}{m^i} - \frac{\alpha\bar{\beta}B}{|\beta|F} \overset{a}{l^i}).$$

In the theory of two-dimensional complex Berwald spaces, an important role is played by the scalars A and B. Therefore, our next goal is to determine the scalars A and B for a complex Randers space.

Proposition 3.2. *Let $(M, F := \alpha + |\beta|)$ be a complex Randers space of dimension two. Then*

i) $l = \frac{\alpha}{F} \overset{a}{l}$; $m = \sqrt{\frac{2\alpha|\beta|}{\gamma}} (\overset{a}{m} - \frac{\alpha\bar{\beta}\tau}{|\beta|F} \overset{a}{l})$;

ii) $g_{i\bar{j}} = \frac{F^2}{\alpha^2} \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{\alpha|\beta|} (\beta\bar{\tau} \overset{a}{l}_i \overset{a}{m}_{\bar{j}} + \bar{\beta}\tau \overset{a}{m}_i \overset{a}{l}_{\bar{j}}) + (|\tau|^2 + \frac{\gamma}{2\alpha|\beta|}) \overset{a}{m}_i \overset{a}{m}_{\bar{j}}$;

iii) $C_{j\bar{h}k} := \frac{\partial g_{j\bar{h}}}{\partial \eta^k} = -\frac{\bar{\beta}^2\tau^2}{2|\beta|^3} \overset{a}{m}_j \overset{a}{m}_k \overset{a}{l}_{\bar{h}} + \frac{\bar{\beta}\tau(4|\beta|^2 - \alpha^2|\tau|^2)}{4\alpha|\beta|^3} \overset{a}{m}_j \overset{a}{m}_k \overset{a}{m}_{\bar{h}}$.

Proof. By Proposition 3.1 and (3.7) it results i).

ii) Using (3.6) and (3.7) we compute

$$\begin{aligned}
g_{i\bar{j}} &= \frac{F}{\alpha} a_{i\bar{j}} - \frac{F}{2\alpha} \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{2|\beta|} b_i b_{\bar{j}} + \frac{1}{2} l_i l_{\bar{j}} \\
&= \frac{F}{\alpha} \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{\alpha} \overset{a}{m}_i \overset{a}{m}_{\bar{j}} - \frac{F}{2\alpha} \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{2|\beta|} (\frac{\beta}{\alpha} \overset{a}{l}_i + \tau \overset{a}{m}_i)(\frac{\bar{\beta}}{\alpha} \overset{a}{l}_{\bar{j}} + \bar{\tau} \overset{a}{m}_{\bar{j}}) \\
&\quad + \frac{1}{2} (\frac{F}{\alpha} \overset{a}{l}_i + \frac{\bar{\beta}\tau}{|\beta|} \overset{a}{m}_i)(\frac{F}{\alpha} \overset{a}{l}_{\bar{j}} + \frac{\beta\bar{\tau}}{|\beta|} \overset{a}{m}_{\bar{j}}) \\
&= (\frac{F}{\alpha} - \frac{F}{2\alpha} + \frac{F|\beta|}{2\alpha^2} + \frac{F^2}{2\alpha^2}) \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{\alpha|\beta|} (\beta\bar{\tau} \overset{a}{l}_i \overset{a}{m}_{\bar{j}} + \bar{\beta}\tau \overset{a}{m}_i \overset{a}{l}_{\bar{j}}) \\
&\quad + (\frac{F}{\alpha} + \frac{F}{2|\beta|} |\tau|^2 + \frac{|\tau|^2}{2}) \overset{a}{m}_i \overset{a}{m}_{\bar{j}} \\
&= \frac{F^2}{\alpha^2} \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + \frac{F}{\alpha|\beta|} (\beta\bar{\tau} \overset{a}{l}_i \overset{a}{m}_{\bar{j}} + \bar{\beta}\tau \overset{a}{m}_i \overset{a}{l}_{\bar{j}}) + (|\tau|^2 + \frac{\gamma}{2\alpha|\beta|}) \overset{a}{m}_i \overset{a}{m}_{\bar{j}} \; .
\end{aligned}$$

We can write $C_{j\bar{h}k} = \frac{\partial g_{j\bar{h}}}{\partial \eta^k} = (\overset{a}{l}_k \overset{a}{l} + \overset{a}{m}_k \overset{a}{m}) g_{j\bar{h}}$.

Taking into account (2.6) and (3.5) we obtain $\overset{a}{l} g_{j\bar{h}} = 0$ and

$$\overset{a}{m} g_{j\bar{h}} = -\frac{\bar{\beta}^2\tau^2}{2|\beta|^3} \overset{a}{m}_j \overset{a}{l}_{\bar{h}} + \frac{\bar{\beta}\tau(4|\beta|^2 - \alpha^2|\tau|^2)}{4\alpha|\beta|^3} \overset{a}{m}_j \overset{a}{m}_{\bar{h}},$$

which lead to iii). □

Using now (3.7) and Proposition 3.2 iii) we obtain

$$C_{j\bar{h}k} = -\frac{\alpha^2\bar{\beta}^2\tau^2}{\gamma F|\beta|^2}m_j m_k l_{\bar{h}} + \sqrt{\frac{2\alpha|\beta|}{\gamma}}\frac{\bar{\beta}\tau}{2|\beta|^2}(\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma})m_j m_k m_{\bar{h}}. \tag{3.8}$$

Moreover, we find

Proposition 3.3. *Let $(M, F := \alpha + |\beta|)$ be a complex Randers space of dimension two. Then*

$$A = -\frac{\alpha^2\bar{\beta}^2\tau^2}{\gamma F|\beta|^2}\ ;\ \ B = \sqrt{\frac{2\alpha|\beta|}{\gamma}}\frac{\bar{\beta}\tau}{2|\beta|^2}(\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma}). \tag{3.9}$$

Further on, our aim is to disclose the conditions in which a complex Randers space of dimension two is a complex Berwald space. As it has already been obtained in Theorem 2.1, we can talk about only three classes of 2 - dimensional complex Finsler spaces: i) the purely Hermitian class ($A = 0$), ii) the class with $B = 0$ and $A \neq 0$ and iii) the class with $B_{|k} = 0$ and $AB \neq 0$. In order to solve the stated problem we use (3.9). On the one hand, we note that $A = 0$ iff $\tau = 0$. Indeed, $\tau = 0$ is equivalent to $\alpha^2||b||^2 = |\beta|^2$ and so this last condition is equivalent to $F = \alpha(1 + ||b||)$, namely it is purely Hermitian. On the other hand, if $B = 0$ and $A \neq 0$ imply $\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma} = 0$. Taking $||b||^2 = 1$ into $\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma} = 0$, it results $\alpha = 3|\beta|$. This means that the metric is purely Hermitian, too. So, it is interesting for us to discuss about the class of two-dimensional complex Randers spaces with $B_{|k} = 0$ and $AB \neq 0$.

Firstly, we compute

$$\begin{aligned} B_{|k} &= [\frac{1}{2|\beta|^2}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma})]_{|k}\bar{\beta}\tau \\ &+ \frac{1}{2|\beta|^2}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma})(\bar{\beta}_{|k}\tau + \bar{\beta}\tau_{|k}). \end{aligned} \tag{3.10}$$

In addition, if $||b||^2$ is a constant on M and using that $\alpha_{|k} = -|\beta|_{|k}$ then $\gamma_{|k} = 2\alpha\alpha_{|k}(||b||^2 - 1)$. Thus the term $[\frac{1}{2|\beta|^2}\sqrt{\frac{2\alpha|\beta|}{\gamma}}(\frac{|\beta|-\alpha}{F} + \frac{2|\beta|F}{\gamma})]_{|k}$ is proportional to $\alpha_{|k}$.

Proposition 3.4. *Let $(M, F := \alpha + |\beta|)$ be a complex Randers space of dimension two with $AB \neq 0$. If $\alpha_{|k} = 0$ and $||b||^2$ is a constant on M then $(M, F := \alpha + |\beta|)$ is a complex Berwald space.*

Proof. Because $AB \neq 0, \alpha_{|k} = 0$ and $||b||^2$ is a constant on M, by (3.10) it results that $\bar{\beta}_{|k}\tau + \bar{\beta}\tau_{|k} = 0$. On the other hand, by Lemma 3.1, $\beta_{|\bar{k}}\bar{\beta} + \beta\bar{\beta}_{|\bar{k}} = 0$ and $\tau_{|\bar{k}}\bar{\tau} + \tau\bar{\tau}_{|\bar{k}} = 0$. Multiplying the first with $\frac{\tau}{\beta}$ and the second with $\frac{\bar{\beta}}{\bar{\tau}}$ and, by adding them we obtain

$$\beta_{|\bar{k}}\frac{\tau\bar{\beta}}{\beta}+\bar{\tau}_{|\bar{k}}\frac{\tau\bar{\beta}}{\bar{\tau}}+\bar{\beta}_{|\bar{k}}\tau+\bar{\beta}\tau_{|\bar{k}}=0.$$

But, $\beta_{|\bar{k}}\frac{\tau\bar{\beta}}{\beta}+\bar{\tau}_{|\bar{k}}\frac{\tau\bar{\beta}}{\bar{\tau}}=0$ because $\beta_{|\bar{k}}\bar{\tau}=-\beta\bar{\tau}_{|\bar{k}}$. Hence $\bar{\beta}_{|\bar{k}}\tau+\bar{\beta}\tau_{|\bar{k}}=0$.

Now, in our assumptions, by (3.9) $A_{|\bar{k}}=-\frac{2\alpha^2\bar{\beta}\tau}{\gamma F|\beta|^2}(\bar{\beta}_{|\bar{k}}\tau+\bar{\beta}\tau_{|\bar{k}})=0$, i.e. the space is Berwald. □

Proposition 3.5. *Let $(M, F:=\alpha+|\beta|)$ be a complex Randers space of dimension two with $AB\neq 0$. If $\bar{\beta}_{|k}\tau+\bar{\beta}\tau_{|k}=0$ and $||b||^2$ is a constant on M then $(M, F:=\alpha+|\beta|)$ is a complex Berwald space.*

Proof. Because $AB\neq 0$, $\bar{\beta}_{|k}\tau+\bar{\beta}\tau_{|k}=0$ and $||b||^2$ is a constant on M, by (3.10) it results that $\alpha_{|k}=0$. Applying Proposition 3.4, the claim is proved. □

Corollary 3.1. *Let $(M, F:=\alpha+|\beta|)$ be a complex Randers space of dimension two with $AB\neq 0$. If $N^i_j=\overset{a}{N^i}_j$ and $||b||^2$ is a constant on M then $(M, F:=\alpha+|\beta|)$ is a complex Berwald space.*

Proof. Immediately results by Lemma 3.1 and Proposition 3.4. □

Theorem 3.1. *Let $(M, F:=\alpha+|\beta|)$ be a complex Randers space of dimension two with $AB\neq 0$ and $||b||^2=1$. Then $(M, F:=\alpha+|\beta|)$ is a complex Berwald space if and only if $\alpha_{|k}=0$.*

Proof. We suppose that $(M, F:=\alpha+|\beta|)$ is Berwald, i.e. $A_{|\bar{k}}=B_{|\bar{k}}=0$. By (3.9) these conditions lead to the system $\alpha|\beta|(\bar{\beta}\tau)_{|\bar{k}}+F\bar{\beta}\tau\alpha_{|\bar{k}}=0$ and $2\alpha|\beta|(3|\beta|-\alpha)(\bar{\beta}\tau)_{|\bar{k}}-3F(\alpha-|\beta|)\bar{\beta}\tau\alpha_{|\bar{k}}=0$ with the solution $\alpha_{|\bar{k}}=(\bar{\beta}\tau)_{|\bar{k}}=0$. So, $\alpha_{|k}=0$. The converse results from Proposition 3.4. □

Further on, we aim to find other features of the complex Randers spaces. Namely, we determine the vertical curvature scalar $\mathbf{I}$ of a complex Randers space.

Proposition 3.6. *Let $(M, F:=\alpha+|\beta|)$ be a complex Randers space of dimension two. Then*

$$\mathbf{I}=\frac{\alpha|\beta|(1-||b||^2)}{\gamma}\left(\frac{1}{2L}-\frac{4|\beta|F}{\gamma^2}\right)-\frac{1}{\gamma}. \tag{3.11}$$

Proof. it results with $\mathbf{I}=-B|_{\bar{s}}m^{\bar{s}}-\frac{B\bar{B}}{2}=\bar{m}(B)-\frac{B\bar{B}}{2}$ and the relations (3.5), (3.9) and Proposition 3.2 i). □

For $||b||^2=1$ in (3.11) we obtain

Corollary 3.2. *Let $(M, F:=\alpha+|\beta|)$ be a complex Randers space of dimension two with $||b||^2=1$. Then $\mathbf{I}=-\frac{1}{L}$ and the vertical holomorphic sectional curvature in direction m is $K^v_{F,m}(z,\eta)=-\frac{2}{L}<0$.*

3.2. **Complex Kropina metric** $F := \frac{\alpha^2}{|\beta|}$, $|\beta| \neq 0$**.** A similar approach, we make to the complex Kropina metric $F := \frac{\alpha^2}{|\beta|}$, $|\beta| \neq 0$. From [3] we recall that

$$
\begin{aligned}
&g_{i\bar{j}} = 2q^2 a_{i\bar{j}} - 2q^2 \overset{a}{l}_i \overset{a}{l}_{\bar{j}} + l_i l_{\bar{j}}, \text{ where } q = \frac{\alpha}{|\beta|};\\
&g^{\bar{j}i} = \frac{1}{2q^2} a^{\bar{j}i} - \frac{2 - q^2||b||^2}{2} l^i l^{\bar{j}} + \frac{1}{2|\beta|}(\beta b^i l^{\bar{j}} + \bar{\beta} l^i \bar{b}^j);\\
&g := \det(g_{i\bar{j}}) = 2q^4\Delta^2.
\end{aligned}
\tag{3.12}
$$

Proposition 3.7. *Let* $(M, F := \frac{\alpha^2}{|\beta|})$ *be a complex Kropina space of dimension two. Then*

i) $\overset{a}{l}(F) = \frac{q}{2}$; $\overset{a}{l}(q) = 0$; $\overset{a}{m}(F) = -\frac{q^2\bar{\beta}\tau}{2|\beta|}$; $\overset{a}{m}(q) = -\frac{q\bar{\beta}\tau}{2|\beta|^2}$;

ii) $l_i = q\overset{a}{l}_i - \frac{q^2\bar{\beta}\tau}{|\beta|}\overset{a}{m}_i$; $l^i = \frac{1}{q}\overset{a}{l}^i$; $m_i = q\sqrt{2}\,\overset{a}{m}_i$; $m^i = \frac{1}{q\sqrt{2}}(\overset{a}{m}^i + \frac{q\bar{\beta}\tau}{|\beta|}\overset{a}{l}^i)$;

iii) $l = \frac{1}{q}\overset{a}{l}$; $m = \frac{1}{q\sqrt{2}}(\overset{a}{m} + \frac{q\bar{\beta}\tau}{|\beta|}\overset{a}{l})$;

iv) $g_{i\bar{j}} = q^2\overset{a}{l}_i\overset{a}{l}_{\bar{j}} - \frac{q^3}{|\beta|}(\beta\bar{\tau}\,\overset{a}{l}_i\overset{a}{m}_{\bar{j}} + \bar{\beta}\tau\,\overset{a}{m}_i\overset{a}{l}_{\bar{j}}) + q^2(q^2|\tau|^2 + 2)\,\overset{a}{m}_i\overset{a}{m}_{\bar{j}}$.

v) $C_{j\bar{h}k} = \frac{2q^3\bar{\beta}^2\tau^2}{|\beta|^3}\overset{a}{m}_j\overset{a}{m}_k\overset{a}{l}_{\bar{h}} - \frac{2\bar{\beta}\tau q^2(q^2||b||^2+1)}{|\beta|^2}\overset{a}{m}_j\overset{a}{m}_k\overset{a}{m}_{\bar{h}}$.

Proof. i) follows by (3.5).

ii) $l_i := \frac{\eta_i}{F} = \frac{1}{F}\frac{\partial L}{\partial \eta^i} = \frac{L_\alpha}{F}\frac{\partial \alpha}{\partial \eta^i} + \frac{L_{|\beta|}}{F}\frac{\partial|\beta|}{\partial\eta^i} = 2q\overset{a}{l}_i - q^2\frac{\bar{\beta}}{|\beta|}b_i = q\overset{a}{l}_i - \frac{q^2\bar{\beta}\tau}{|\beta|}\overset{a}{m}_i$ by (3.4). $l^i := \frac{1}{F}\eta^i = \frac{1}{q}\overset{a}{l}^i$.

$$
m^1 := -\frac{l_2}{\sqrt{g}} = -\frac{1}{q^2\Delta\sqrt{2}}(q\overset{a}{l}_2 - \frac{q^2\bar{\beta}\tau}{|\beta|}\overset{a}{m}_2) = \frac{1}{q\sqrt{2}}(\overset{a}{m}^1 + \frac{q\bar{\beta}\tau}{|\beta|}\overset{a}{l}^1).
$$

Analogue, for m^2.

iii) is a consequence of ii). (3.12) with ii) gives iv).

Again, we write $C_{j\bar{h}k} = \frac{\partial g_{j\bar{h}}}{\partial \eta^k} = (\overset{a}{l}_k\overset{a}{l} + \overset{a}{m}_k\overset{a}{m})g_{j\bar{h}}$. Using (2.6), (3.5) and i) we obtain $\overset{a}{l}\,g_{j\bar{h}} = 0$ and

$$
\overset{a}{m}\,g_{j\bar{h}} = \frac{2q^3\bar{\beta}^2\tau^2}{|\beta|^3}\overset{a}{m}_j\overset{a}{l}_{\bar{h}} - \frac{2\bar{\beta}\tau q^2(q^2||b||^2+1)}{|\beta|^2}\overset{a}{m}_j\overset{a}{m}_{\bar{h}},
$$

which lead to v). □

Now, taking into account i) and v) of above Proposition we obtain

$$
C_{j\bar{h}k} = \frac{\bar{\beta}^2\tau^2}{|\beta|^3}m_j m_k l_{\bar{h}} - \frac{\bar{\beta}\tau\sqrt{2}}{|\beta|^2 q}m_j m_k m_{\bar{h}}.
\tag{3.13}
$$

So, we have proved

Proposition 3.8. *Let $(M, F := \frac{\alpha^2}{|\beta|})$ be a complex Kropina space of dimension two. Then*

$$A = \frac{\bar{\beta}^2\tau^2}{|\beta|^3} \ ; \ B = -\frac{\bar{\beta}\tau\sqrt{2}}{|\beta|^2 q}. \tag{3.14}$$

Having the expressions of the scalars A and B, we can deduce the conditions in which a 2-dimensional complex Kropina space is a complex Berwald space. Taking into account Theorem 2.1 and (3.14), we obtain only two cases:

1. $A = 0$ iff $\tau = 0$. Indeed, $\tau = 0$ is equivalent to $\alpha^2||b||^2 = |\beta|^2$. This leads to $F = \frac{\alpha}{||b||}$ and so, the metric is purely Hermitian.
2. $B_{|k} = 0$ and $AB \neq 0$. This case is developed follow up.

Firstly, $F_{|k} = 0$ implies $|\beta|_{|k} = \frac{2}{q}\alpha_{|k}$. Secondly, a direct computation gives

$$B_{|k} = \frac{3\sqrt{2}}{q^2|\beta|^3}\bar{\beta}\tau\alpha_{|k} - \frac{\sqrt{2}}{|\beta|^2 q}(\bar{\beta}_{|k}\tau + \bar{\beta}\tau_{|k}). \tag{3.15}$$

So that, $B_{|k} = 0$ is equivalent to $\bar{\beta}_{|k}\tau + \bar{\beta}\tau_{|k} = \frac{3\bar{\beta}\tau}{\alpha}\alpha_{|k}$. Moreover, $\alpha_{|k} = 0$ iff $\bar{\beta}_{|k}\tau + \bar{\beta}\tau_{|k} = 0$.

Proposition 3.9. *Let $(M, F := \frac{\alpha^2}{|\beta|})$ be a complex Kropina space of dimension two with $AB \neq 0$. If $\alpha_{|k} = 0$ then $(M, F := \frac{\alpha^2}{|\beta|})$ is a complex Berwald space.*

Proof. By means of Lemma 3.1, $\beta_{|\bar{k}}\bar{\beta} + \beta\bar{\beta}_{|\bar{k}} = 0$ and $\tau_{|\bar{k}}\bar{\tau} + \tau\bar{\tau}_{|\bar{k}} = 0$. Multiplying the first with $\frac{\tau}{\beta}$ and the second with $\frac{\bar{\beta}}{\bar{\tau}}$ and, by adding them, we obtain $\beta_{|\bar{k}}\frac{\tau\bar{\beta}}{\beta} + \bar{\tau}_{|\bar{k}}\frac{\tau\bar{\beta}}{\bar{\tau}} + \bar{\beta}_{|\bar{k}}\tau + \bar{\beta}\tau_{|\bar{k}} = 0$. But, $\beta_{|\bar{k}}\frac{\tau\bar{\beta}}{\beta} + \bar{\tau}_{|\bar{k}}\frac{\tau\bar{\beta}}{\bar{\tau}} = 0$ because $\beta_{|\bar{k}}\bar{\tau} = -\beta\bar{\tau}_{|\bar{k}}$. Hence $\bar{\beta}_{|\bar{k}}\tau + \bar{\beta}\tau_{|\bar{k}} = 0$.

With our hypothesis and by (3.14), $A_{|\bar{k}} = \frac{2\bar{\beta}\tau}{|\beta|^3}(\bar{\beta}_{|\bar{k}}\tau + \bar{\beta}\tau_{|\bar{k}}) = 0$. So, the space is Berwald. □

Corollary 3.3. *Let $(M, F := \frac{\alpha^2}{|\beta|})$ be a complex Kropina space of dimension two with $AB \neq 0$. If $N^i_j = \overset{a}{N}{}^i{}_j$ then $(M, F := \frac{\alpha^2}{|\beta|})$ is a complex Berwald space.*

Proof. It results from Lemma 3.1 and Proposition 3.9. □

Theorem 3.2. *Let $(M, F := \frac{\alpha^2}{|\beta|})$ be a complex Kropina space of dimension two with $AB \neq 0$ and $||b||^2$ a nonzero constant on M. $(M, F := \frac{\alpha^2}{|\beta|})$ is a complex Berwald space if and only if $\alpha_{|k} = 0$.*

Proof. If $(M, F := \frac{\alpha^2}{|\beta|})$ is Berwald then $A_{|\bar{k}} = 0$. But, by (3.14), $A_{|\bar{k}} = \frac{2\bar{\beta}\tau}{|\beta|^3}(\bar{\beta}\tau)_{|\bar{k}} - \frac{6\bar{\beta}^2\tau^2}{|\beta|^4 q}\alpha_{|k}$. So that, $(\bar{\beta}\tau)_{|\bar{k}} = \frac{3\bar{\beta}\tau}{\alpha}\alpha_{|\bar{k}}$. On the one hand, $\bar{B}_{|\bar{k}} = 0$ means that $(\beta\bar{\tau})_{|\bar{k}} = \frac{3\beta\bar{\tau}}{\alpha}\alpha_{|\bar{k}}$. From the last two equations we obtain

$$\beta\bar{\tau}(\bar{\beta}\tau)_{|\bar{k}} + \bar{\beta}\tau(\beta\bar{\tau})_{|\bar{k}} = \frac{6}{\alpha}|\beta|^2|\tau|^2\alpha_{|\bar{k}},$$

equivalent $|\beta|^2_{|\bar{k}}|\tau|^2 + |\beta|^2|\tau|^2_{|\bar{k}} = \frac{6}{\alpha}|\beta|^2|\tau|^2\alpha_{|\bar{k}}$.

On the other hand, $|\tau|^2 = -\frac{2\alpha_{|\bar{k}}}{q^3|\beta|}$. Therefore, $\frac{2|\beta|||b||^2}{q}\alpha_{|\bar{k}} = 0$, which leads to $\alpha_{|\bar{k}} = 0$. By conjugation, $\alpha_{|k} = 0$. The converse results from Proposition 3.9. □

Proposition 3.10. *Let $(M, F := \frac{\alpha^2}{|\beta|})$ be a complex Kropina space of dimension two. The vertical curvature scalar* $\mathbf{I}$ *and the vertical holomorphic sectional curvature in direction m are*

$$\mathbf{I} = -\frac{1}{L}\ ;\ K^v_{F,m}(z,\eta) = -\frac{2}{L} < 0.$$

Proof. $\mathbf{I} : = -B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2} = \bar{m}(B) - \frac{B\bar{B}}{2} = \frac{1}{q\sqrt{2}}[\overset{a}{\bar{m}}(B) + \frac{q\beta\bar{\tau}}{|\beta|}\overset{a}{\bar{l}}(B)]$. Using (3.5) and (3.14) we have $\overset{a}{\bar{m}}(B) = \sqrt{2}(\frac{1}{q\alpha^2} - \frac{|\tau|^2}{2\alpha|\beta|})$ and $\overset{a}{\bar{l}}(B) = -\frac{\bar{\beta}\tau\sqrt{2}}{2\alpha^2|\beta|}$. From here, a quick computation leads to $\mathbf{I} = -\frac{1}{L}$ and $K^v_{F,m}(z,\eta) = -\frac{2}{L}$. □

3.3. **Some examples.** In order to reduce clutter, let us relabel the local coordinates z^1, z^2, η^1, η^2 as z, w, η, θ, respectively. Let $\Delta = \{(z,w) \in \mathbf{C}^2,\ |w| < |z| < 1\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log \frac{1}{(1-|z|^2)(|z|^2-|w|^2)});\ \alpha^2(z,w;\eta,\theta) = a_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{3.16}$$

where $|z^i|^2 := z^i\bar{z}^i$, $z^i \in \{z,w\}$, $\eta^i \in \{\eta,\theta\}$. We choose

$$b_z = \frac{w}{|z|^2-|w|^2};\ b_w = -\frac{z}{|z|^2-|w|^2}. \tag{3.17}$$

With these tools we construct $\alpha(z,w,\eta,\theta) := \sqrt{a_{i\bar{j}}(z,w)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta) = b_i(z,w)\eta^i$ and from here the complex Randers metric $F = \alpha + |\beta|$ and the complex Kropina metric $F := \frac{\alpha^2}{|\beta|}$. By a direct computation, we deduce

$$\begin{aligned}
&a_{z\bar{z}} = \frac{1}{(1-|z|^2)^2} + b_z b_{\bar{z}};\ a_{z\bar{w}} = b_z b_{\bar{w}};\ a_{w\bar{w}} = b_w b_{\bar{w}};\\
&a^{\bar{z}z} = (1-|z|^2)^2;\ a^{\bar{w}z} = \frac{\bar{w}z(1-|z|^2)^2}{|z|^2};\\
&a^{\bar{w}w} = \frac{(|z|^2-|w|^2)^2}{|z|^2} + \frac{|w|^2(1-|z|^2)^2}{|z|^2};\\
&b^z = 0;\ b^w = -\frac{|z|^2-|w|^2}{z};\ ||b||^2 = 1;\ \alpha^2 - |\beta|^2 = \frac{|\eta|^2}{(1-|z|^2)^2}
\end{aligned} \tag{3.18}$$

and the coefficients of the Chern-Finsler ($c.n.c.$) are

$$\begin{aligned}
&\overset{CF}{N_z^z} = \overset{a}{N_z^z} = \frac{2\overline{z}\eta}{1-|z|^2};\ \overset{CF}{N_w^z} = \overset{a}{N_w^z} = 0;\\
&\overset{CF}{N_z^w} = \overset{a}{N_z^w} = \frac{2\overline{z}w}{z}\left(\frac{1}{1-|z|^2}+\frac{1}{|z|^2-|w|^2}\right)\eta - \frac{|z|^2+|w|^2}{z\left(|z|^2-|w|^2\right)}\theta;\\
&\overset{CF}{N_w^w} = \overset{a}{N_w^w} = -\frac{|z|^2+|w|^2}{z\left(|z|^2-|w|^2\right)}\eta + \frac{2\overline{w}\theta}{|z|^2-|w|^2}.
\end{aligned} \tag{3.19}$$

Thus, these complex Finsler spaces with $(\alpha,\beta)-$ metric fulfill the assumptions of the Corollaries 3.1 and 3.3. So, they give us some examples of complex Berwald spaces with $(\alpha,\beta)-$ metrics.

Conclusions

By means of the complex Berwald frame, we made an approach to the geometry of the two-dimensional complex Finsler spaces with (α,β)-metrics. A trivial class of complex Berwald spaces is represented by the purely Hermitian complex Finsler spaces. But this paper moots a new class of complex Berwald manifolds. It is desirable that the complex Randers and Kropina spaces will manage to enrich the complex Berwald geometry, leading to new interesting issues.

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