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SOME REMARKS ON BERWALD MANIFOLDS AND LANDSBERG MANIFOLDS

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Abstract. In the present paper, we shall prove new characterizations of Berwald spaces and Landsberg spaces. The main idea inthis research is the use of the so-called average Riemannian metric.

1. Introduction

Several years ago, Professor Makoto Matsumoto [6] raised the following question: are there non-Berwald Landsberg metrics at all? In his research he established a number of results which suggested the conjecture: the class of these Finsler manifolds is empty. Recently, the conjecture was affirmatively answered by Z. I. Szabó [9]. It turned out, however, that Szabó's proof contains a gap ([7],[10]), and the problem is still open. In the present paper, after reviewing these special metrics, we shall prove new characterizations of them (Theorem 4.1 and 4.3). The main idea is the use of the average Riemannian metric of the metric tensor of Finsler metric developed in [9] for characterization of Berwald manifolds.

We also need the well-known characterization of Landsberg manifolds in terms of the 'horizontal incompressibility' of its volume form (Theorem 4.2, cf. [2]). A further key observation is our theorem 4.1, which characterizes Berwald metrics by the property that the horizontal lifts of the vector fields on the base manifold are Killing fields for the natural lift of a Riemannian metric on the base manifold.

For the reader's convenience and to make the paper more readable, in sections 2 and 3 we briefly summarize our setup. Section 4 is devoted to the above mentioned characterizations of Berwald and Landsberg manifolds.

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2. Finsler functions

Let M be a smooth connected manifold of $\dim M = n$, and $\pi\colon TM \to M$ its tangent bundle. We denote by (x, y) a point in TM if $y \in \pi^{-1}(x) = T_xM$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, \dots, x^n)$. By setting $v = \sum y^i \left(\partial/\partial x^i\right)_x$ for every $v \in \pi^{-1}(U)$, we write $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n)$ on $\pi^{-1}(U)$.

Definition 2.1. A function $L\colon TM \to \mathbb{R}$ is called a *Finsler function* over M if L is a smooth assignment of a norm $\|v\|$ of every vector field v on M, that is, the conditions

(F1) $L(x, y) \geq 0$, and $L(x, y) = 0$ if and only if $y = 0$,
(F2) $L(x, \lambda y) = \lambda L(x, y)$ for all $\lambda \in \mathbb{R}^+$,
(F3) L is smooth on the punctured tangent bundle $E = TM\backslash\{0\}$

are satisfied. Then the pair (M, L) is said to be a *Finsler manifold.* The norm of a vector $v = (x, y) \in TM$ is defined by $\|v\| = L(x, y)$.

A Finsler function L is said to be *strongly convex* if its Hessian (G_{ij}) defined by

$$G_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{2.1}$$

is positive-definite on E. In the sequel, we assume the convexity of L.

Let $V = \ker \pi_*$ be the vertical bundle over E, and let π^*TM denote the pull-back of TM to E via π. Then

$$0 \to V \xrightarrow{i} TE \xrightarrow{\pi_*} \pi^*TM \to 0 \tag{2.2}$$

is a short exact sequence of bundle maps. After the identification $V_{(x,y)} = T_y(T_xM)$, the Hessian (G_{ij}) defines an inner product G on V by

$$G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = G_{ij}. \tag{2.3}$$

The multiplicative group $\mathbb{R}^+$ acts on TM by multiplication

$$m_\lambda\colon T_xM \ni v \to \lambda \cdot v \in T_xM$$

for all $\lambda \in \mathbb{R}^+$. This action generates a natural section $\mathcal{E} \in \Gamma(V)$ given by $\mathcal{E}(v) = (v, v)$. Its expression in local coordinates is

$$\mathcal{E} = \sum y^i \frac{\partial}{\partial y^i}. \tag{2.4}$$

This section is called the *tautological section* of V.

3. Chern-Rund connection

We denote by $A^k(V)$ the space of V-valued k-forms on E. In this section, we shall define the *Chern-Rund connection* in the Riemannian vector bundle (V, G). Let $\nabla\colon A^0(V) \to A^1(V)$ be a connection on V. We suppose that a V-valued 1-form $\theta \in A^1(V)$ defined by

$$\theta = \nabla \mathcal{E} \tag{3.1}$$

satisfies the homogeneity condition $m_\lambda^* \theta = \theta$ and the splitting condition $\theta \circ i = Id$. Then $H = \ker \theta$ is a subbundle of TE complementary to V. H is called the *horizontal bundle*. If these assumption are satisfied, we call ∇ a *Finsler connection* in the vertical bundle V.

Under the identification $\pi^*TM \cong V$, we consider π_* as the natural projection $\pi_*\colon TE \to V$, and write

$$\pi_* = \sum \frac{\partial}{\partial y^i} \otimes dx^i.$$

Definition 3.1. ([3]) A Finsler connection ∇ in (V, G) is called the *Chern-Rund connection* if it satisfies the following conditions.

(1) ∇ is *symmetric*:

$$\nabla \pi_* = 0. \tag{3.2}$$

(2) ∇ is *almost G-compatible*:

$$(\nabla_{X^H} G)(Y, Z) = 0, \tag{3.3}$$

for every $X^H \in \Gamma(H)$ and for all $Y, Z \in \Gamma(V)$.

Remark 3.1. The V-valued 1-form θ defined by (3.1) is the *Cartan's* (or *Berwald's*) *non-linear connection* in [3]. For any vector field v on the base manifold M, there exists a unique vector field $v^H \in \Gamma(H)$ satisfying $\pi_*\left(v^H\right) = v$. Such a vector field v^H is called the *horizontal lift* of v with respect to θ.

We denote by $\left(\omega_j^i\right)$ the family of connection forms of ∇ with respect to the natural frame:

$$\nabla \frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i} \otimes \omega_j^i. \tag{3.4}$$

Since

$$\nabla \pi_* = \nabla \left(\sum \frac{\partial}{\partial y^i} \otimes dx^i \right) = \sum \left(\sum \frac{\partial}{\partial y^i} \otimes \omega_j^i \right) \wedge dx^j,$$

the connection forms ω_j^i satisfy $\sum \omega_j^i \wedge dx^j = 0$. Taking into account that the assumption (3.3) can be written in the form $X^H G(Y, Z) = G(\nabla_{X^H} Y, Z) + G(Y, \nabla_{X^H} Z)$, the connection forms $\omega_j^i = \sum \Gamma_{jk}^i(x, y) dx^k$ of ∇ are determined by the Christoffel symbols

$$\Gamma_{jk}^i = \frac{1}{2} \sum G^{im} \left(\frac{\delta G_{jm}}{\delta x^k} + \frac{\delta G_{mk}}{\delta x^j} - \frac{\delta G_{jk}}{\delta x^m} \right), \tag{3.5}$$

where we set

$$\frac{\delta}{\delta x^k} := \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^k} - \sum y^j \Gamma^i_{jk}\ \frac{\partial}{\partial y^i}.$$

The local functions $N^i_j = \sum y^m \Gamma^i_{mj}$ are called the *coefficients of Cartan's non-linear connection*. It is well-known that these coefficients satisfy the symmetric property

$$\frac{\partial N^i_j}{\partial y^k} = \frac{\partial N^i_k}{\partial y^j}. \tag{3.6}$$

Then θ is of form $\theta = \sum \dfrac{\partial}{\partial y^i} \otimes \theta^i$, where the 1-forms θ^i $(i = 1, \dots, n)$ are given by

$$\theta^i = dy^i + \sum N^i_j(x, y) dx^j, \tag{3.7}$$

and $\{\theta^1, \dots, \theta^n\}$ defines a local frame field on the dual bundle V^*.

Remark 3.2. Using the Cartan's non-linear connection θ, let

$$D_X Y := \begin{cases} \theta[X, Y] & \text{if } X \in \Gamma(H) \\ X(Y) & \text{if } X \in \Gamma(V) \end{cases}$$

for all $Y \in \Gamma(V)$. Then it is easy to see that $D\colon A^0(V) \to A^1(V)$ is also a Finsler connection, called the *Berwald connection* of (M, L). The coefficients G^i_{jk} of D are given by

$$G^i_{jk} = \frac{\partial N^i_j}{\partial y^k},$$

and the connection D is given by

$$D \frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i} \otimes \left(\sum G^i_{jk} dx^k\right). \tag{3.8}$$

It is clear from (3.6) that D has the symmetry property $D\pi_* = 0$.

We have two projections from TE to V. The first one is the natural projection π_* and second one is θ defined by (3.1).

Definition 3.2. The *torsion form* T of ∇ is defined by the V-valued 2-form given by

$$T = \nabla\theta. \tag{3.9}$$

By direct calculations, we obtain

$$
\begin{aligned}
T &= \sum \nabla \frac{\partial}{\partial y^i} \wedge \theta^i + \sum \frac{\partial}{\partial y^i} \otimes d\theta^i \\
&= \sum \frac{\partial}{\partial y^i} \otimes \left(\omega_j^i \wedge \theta^j + dN_j^i \wedge dx^j\right) \\
&= \sum \frac{\partial}{\partial y^i} \otimes \left[\sum \left(\frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}\right) dx^j \wedge dx^k + \left(\Gamma_{jk}^i - G_{jk}^i\right) dx^j \wedge \theta^k\right].
\end{aligned}
$$

In the sequel we set

$$T^{HH} = \sum \frac{\partial}{\partial y^i} \otimes \left(\sum R_{jk}^i dx^j \wedge dx^k\right)$$

and

$$T^{HV} = \sum \frac{\partial}{\partial y^i} \otimes \left(\sum P_{jk}^i dx^j \wedge \theta^k\right),$$

where we put

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}$$

and

$$P_{jk}^i = \Gamma_{jk}^i - G_{jk}^i.$$

Since

$$\theta\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = \sum R_{jk}^i \frac{\partial}{\partial y^i},$$

the horizontal part T^{HH} of T measures the integrability of H.

Definition 3.3. A Finsler manifold (M, L) is called a *Landsberg manifold* if $T^{HV} = 0$ is satisfied.

It follows at once that the Chern-Rund connection ∇ coincides with the Berwald connection D if L is Landsberg. A geometrical meaning of the mixed part T^{HV} will be explained soon.

Next we consider the curvature of ∇. The curvature form $\Omega = \left(\Omega_j^i\right)$ of ∇ is defined by

$$\nabla^2 \frac{\partial}{\partial y^j} = \sum \frac{\partial}{\partial y^i} \otimes \Omega_j^i.$$

By direct calculations, we have

$$\Omega_j^i = d\omega_j^i + \sum \omega_m^i \wedge \omega_j^m = \sum R_{jkl}^i dx^k \wedge dx^l + \sum P_{jkl}^i dx^k \wedge \theta^l,$$

where we put

$$R_{jkl}^i = \frac{\delta \Gamma_{jk}^i}{\delta x^l} - \frac{\delta \Gamma_{jl}^i}{\delta x^k} + \sum \left(\Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m\right) \tag{3.10}$$

and

$$P^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l}. \tag{3.11}$$

In the sequel, we set

$$R^{HH}\left(\frac{\partial}{\partial y^j}\right) = \sum \frac{\partial}{\partial y^i} \otimes \sum R^i_{jkl} dx^k \wedge dx^l$$

and

$$R^{HV}\left(\frac{\partial}{\partial y^j}\right) = \sum \frac{\partial}{\partial y^i} \otimes \sum P^i_{jkl} dx^k \wedge \theta^l.$$

4. Berwald and Landsberg manifolds

Definition 4.1. A Finsler manifold (M, L) is called a *Berwald manifold* if $R^{HV} = 0$ is satisfied.

Using the Ricci identity $\nabla^2 \mathcal{E} = R(\mathcal{E})$, we obtain the identity $R^{HV}(\mathcal{E}) = T^{HV}$ which implies that all Berwald manifolds are Landsberg manifolds.

Remark 4.1. By this definition and (3.11), if (M, L) is a Berwald manifold, then the coefficients Γ^i_{jk} are independent of the fibre coordinate $(y^1, \ldots, y^n)$. Therefore ∇ may be considered as a linear connection on the base manifold M, and we obtain the relation

$$G^i_{jk} = \frac{\partial}{\partial y^k}\left[\sum y^m \Gamma^i_{mj}(x)\right] = \Gamma^i_{jk} \circ \pi.$$

Hence, roughly speaking, a Finsler manifold (M, L) is a Berwald manifold if and only if its Berwald connection D is a linear connection on M.

A Riemannian manifold is an obvious example of Berwald manifolds. The following theorem due to [8] played an important role in understanding Berwald manifolds.

Theorem 4.1. *([8]) If (M, L) is a Berwald manifold, then there exists a Riemannian metric g on M such that the $\nabla = \pi^* \nabla^g$ for the Levi-Civita connection ∇^g of (M, g).*

Let $v = \sum v^i \frac{\partial}{\partial x^i}$ be a vector field on the base manifold M, and $\{\varphi_t\}$ be the 1-parameter group of local transformations φ_t $(|t| < \varepsilon)$ on M generated by v. If we denote by $\widetilde{\varphi}_t$ the horizontal lift of φ_t, then $\pi \circ \widetilde{\varphi}_t = \varphi_t$ and

$$\left.\frac{d\widetilde{\varphi}_t}{dt}\right|_{t=0} = v^H, \tag{4.1}$$

where v^H is the horizontal lift of v with respect to the Cartan's non-linear connection θ. For each t, we denote by $\widetilde{\Phi}_t$ the automorphism acting on tensor fields. Then the Lie derivation $\mathcal{L}_{v^H} G$ with respect to v^H is defined by

$$\mathcal{L}_{v^H} G = \left.\frac{d}{dt}\right|_{t=0} \left[\widetilde{\Phi}_t(G_t)\right] \tag{4.2}$$

By Theorem 3.1, if L is a Berwald function, then θ is given by

$$\theta = \sum \frac{\partial}{\partial y^i} \otimes \left(dy^i + \sum y^j \left(\gamma^i_{jk} \circ \pi\right) dx^k\right),$$

where γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of the associated to g. Let $\widetilde{g}$ be the lifted metric of g:

$$\widetilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij} \circ \pi.$$

Then we obtain

$$\begin{aligned}(\mathcal{L}_{v^H}\widetilde{g})\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= v^H \widetilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) - \widetilde{g}\left(\left[v^H, \frac{\partial}{\partial y^i}\right], \frac{\partial}{\partial y^j}\right) \\ &\quad - \widetilde{g}\left(\frac{\partial}{\partial y^i}, \left[v^H, \frac{\partial}{\partial y^j}\right]\right) \\ &= \sum \left(v^k \circ \pi\right) \left(\frac{\partial g_{ij}}{\partial x^k} - \sum g_{rj}\gamma^r_{ki} - \sum g_{ir}\gamma^r_{kj}\right) = 0.\end{aligned}$$

It is extremely useful that the converse of this fact is also true.

Theorem 4.2. *A Finsler function L is Berwald if and only if there exists a Riemannian metric g on M satisfying*

$$\mathcal{L}_{v^H}\widetilde{g} = 0 \tag{4.3}$$

for the horizontal lift v^H of every vector field v on M.

Proof. Suppose that there exists a Riemannian metric g on M satisfying (4.3). Then, similarly to the computation above, we obtain

$$(\mathcal{L}_{v^H}\widetilde{g})\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \sum \left(v^k \circ \pi\right) \left(\frac{\partial g_{ij}}{\partial x^k} - \sum g_{rj}G^r_{ki} - \sum g_{ir}G^r_{kj}\right).$$

Since G^i_{jk} have the symmetry property $G^i_{jk} = G^i_{kj}$, we obtain

$$G^i_{jk} = \frac{1}{2}\sum g^{ir}\left(\frac{\partial g_{jr}}{\partial x^k} + \frac{\partial g_{rk}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^r}\right) = \gamma^i_{jk} \circ \pi.$$

Therefore (M, L) is a Berwald manifold. □

If (M, L) is Berwald, a Riemannian metric satisfying (4.3) is constructed as follows. Since the metric G on vertical subbundle V is given by $G = \sum G_{ij}\theta^i \otimes \theta^j$, we may define an n-form $d\mu$ on V by

$$d\mu := \sqrt{\det (G_{ij})}\, \theta^1 \wedge \ldots \wedge \theta^n. \tag{4.4}$$

The restriction of $d\mu$ to each fibre T_xM is the volume form of the Riemannian space (T_xM, G_x). For the unit ball $B_x = \{y \in T_xM \mid L(x, y) \leq 1\}$ in T_xM, Szabó[9] introduced an average Riemannian metric g on M by the integral

$$g(z, w) = \int_{B_x} G(z, w) d\mu, \tag{4.5}$$

for all vector field z, w on M, where, at the right-hand side, we consider z, w as sections of V.

Proposition 4.1. *We suppose that (M, L) be Berwald. Then the Riemannian metric g defined by* (4.5) *satisfies* (4.3).

Proof. By the assumption, the coefficients G^i_{jk} are independent of the fibre coordinates $(y^1, \ldots, y^n)$. Therefore we have

$$\begin{aligned}
&\frac{\partial g_{ij}}{\partial x^k} - \sum g_{rj} G^r_{ki} - \sum g_{ir} G^r_{kj} \\
&\quad = \frac{\partial}{\partial x^k} \int_{B_x} G_{ij} d\mu - \sum \left(\int_{B_x} G_{rj} d\mu \right) G^r_{ki} - \sum \left(\int_{B_x} G_{ir} d\mu \right) G^r_{jk} \\
&\quad = \int_{B_x} \left(\frac{\partial G_{ij}}{\partial x^k} - \sum G_{rj} G^r_{ki} - \sum G_{ir} G^r_{kj} \right) d\mu = 0.
\end{aligned}$$

Consequently we obtain (4.3). □

On the other hand, we have the following characterization of Landsberg manifolds.

Theorem 4.3. *A Finsler function L is Landsberg if and only if*

$$\mathcal{L}_{v^H} G = 0 \tag{4.6}$$

for the horizontal lift v^H of every vector field v on M.

Proof. By direct computations, we have

$$\begin{aligned}
(\mathcal{L}_{v^H} G) \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) &= v^H G_{ij} - G \left(\left[v^H, \frac{\partial}{\partial y^i} \right], \frac{\partial}{\partial y^j} \right) - G \left(\frac{\partial}{\partial y^i}, \left[v^H, \frac{\partial}{\partial y^j} \right] \right) \\
&= \sum \left(v^k \circ \pi \right) \left(\frac{\delta G_{ij}}{\delta x^k} - \sum G_{rj} G^r_{ki} - \sum G_{ir} G^r_{kj} \right) \\
&= \sum \left(v^k \circ \pi \right) \left[\frac{\delta G_{ij}}{\delta x^k} - \sum G_{rj} \left(\Gamma^r_{ik} - P^r_{ik} \right) - \sum G_{ir} \left(\Gamma^r_{jk} - P^r_{jk} \right) \right] \\
&= \sum \left(v^k \circ \pi \right) \left(G_{rj} P^r_{ik} + G_{ir} P^r_{jk} \right) = 2 \sum \left(v^k \circ \pi \right) P_{ijk},
\end{aligned}$$

where we put $P_{ijk} = \sum G_{ir} P^r_{jk}$ and use the well-known fact that P_{ijk} are symmetric in the indices $i,\ j,\ k$. Therefore L is Landsberg if and only if (4.6) is satisfied. $\square$

As a consequence of this theorem, the metric G is invariant by each transformation $\widetilde{\varphi}_t$ defined in (4.1), that is, $\widetilde{\varphi}_t^* G = G$. This implies that each transformation $\widetilde{\varphi}_t$ is an isometry from T_xM to $T_{\varphi_t(x)}M$.

We also calculate the Lie derivative of $d\mu$ with respect to the horizontal lift v^H of a vector field v on M. Using the definition of $\mathcal{L}_{v^H}$, we obtain

$$
\begin{aligned}
&(\mathcal{L}_{v^H} d\mu)\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) = v^H\left(d\mu\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right)\right)\\
&\quad - \sum_{j=1}^n d\mu\left(\frac{\partial}{\partial y^1}, \dots, \left[v^H, \frac{\partial}{\partial y^j}\right], \dots, \frac{\partial}{\partial y^n}\right)\\
&= \sum \left(v^k \circ \pi\right)\left[\frac{\delta}{\delta x_k}\left(\sqrt{\det G}\right) - \sum_{j=1}^n d\mu\left(\frac{\partial}{\partial y^1}, \dots, \sum G^m_{kj}\frac{\partial}{\partial y^m}, \dots, \frac{\partial}{\partial y^n}\right)\right]\\
&= \sum \left(v^k \circ \pi\right)\left[\frac{\delta}{\delta x_k}\left(\sqrt{\det G}\right) - \sum_{m=1}^n G^m_{km}\sqrt{\det G}\right]\\
&= \sum \left(v^k \circ \pi\right)\left[\frac{1}{2}\left(\det G\right)^{-1/2}\cdot\frac{\delta}{\delta x_k}\left(\det G\right) - \left(\sum G^m_{km}\right)\sqrt{\det G}\right].
\end{aligned}
$$

Since $\dfrac{\delta}{\delta x_k}(\det G) = 2\det G \sum \mathit{\Gamma}^m_{km}$, we find

$$
\begin{aligned}
(\mathcal{L}_{v^H} d\mu)\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) &= \sqrt{\det G}\ \sum \left(v^k \circ \pi\right)\left[\sum_{m=1}^n \left(\mathit{\Gamma}^m_{km} - G^m_{km}\right)\right]\\
&= \sqrt{\det G}\ \sum \left(v^k \circ \pi\right)\left[\sum_{m=1}^n P^m_{km}\right].
\end{aligned}
$$

If we set $(\operatorname{tr}.T^{HV})(X) := \operatorname{trace}\{Y \to T^{HV}(X,Y)\}$, then we see that $\mathcal{L}_{v^H} d\mu = 0$ if and only if $\operatorname{tr}.T^{HV} = 0$. Therefore, if L is Landsberg, we have

$$\mathcal{L}_{v^H} d\mu = 0, \tag{4.7}$$

that is, the volume form $d\mu$ is invariant by the automorphism $\widetilde{\mathit{\Phi}}_t$:

$$\widetilde{\mathit{\Phi}}_t(d\mu_t) = d\mu. \tag{4.8}$$

A Finsler manifold (M, L) satisfying $\operatorname{tr}.T^{HV} = 0$ is called a *weak Landsberg space.* The condition $\operatorname{tr}.T^{HV} = 0$ means that each fibre T_xM is is minimal submanifold in TM with some Sasakian-type metric (cf. [1]).

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