Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
25 (2009), 175–187
www.emis.de/journals
ISSN 1786-0091

GROUPS WITH THE SAME PRIME GRAPH AS AN ALMOST SPORADIC SIMPLE GROUP

BEHROOZ KHOSRAVI

The author dedicate this paper to his parents: Professor Amir Khosravi and Mrs. Soraya Khosravi for their unending love and support.

Abstract. Let G be a finite group. We denote by $\Gamma(G)$ the prime graph of G. Let S be a sporadic simple group. M. Hagie in (Hagie, M. (2003), The prime graph of a sporadic simple group, Comm. Algebra, 31: 4405-4424) determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$. In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(A)$ where A is an almost sporadic simple group, except $\mathrm{Aut}(McL)$ and $\mathrm{Aut}(J_2)$.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then the set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of order elements of G is denoted by $\pi_e(G)$. We construct the prime graph of G as follows:

The prime graph $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G)$, $\pi_2(G)$, $\ldots$, $\pi_{t(G)}(G)$ be the connected components of $\Gamma(G)$. We use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It has been proved that for every finite group G we have $t(G) \leq 6$ [12, 22, 31] and the diameter of $\Gamma(G)$ is at most 5 [23]. In [20] and [19] finite groups with the same prime graph as a CIT simple group and $PSL(2, q)$ where $q = p^\alpha < 100$ are determined.

In [18] we introduced the following concept for finite groups:

2000 *Mathematics Subject Classification.* 20D05, 20D60, 20D08.
Key words and phrases. Almost sporadic simple groups, prime graph, order elements.
The author was supported in part by a grant from IPM (No. 85200022).

Definition 1.1. ([18]) A finite group G is called *recognizable by prime graph* (briefly, *recognizable by graph*) if $H \cong G$ for every finite group H with $\Gamma(H) = \Gamma(G)$. Also a finite simple nonabelian group P is called *quasirecognizable by prime graph*, if every finite group G with $\Gamma(G) = \Gamma(P)$ has a composition factor isomorphic to P.

It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [18, 32]. Also the authors in [21] proved that $PSL(2,p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$ is recognizable by prime graph. Hagie in [9] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In this paper, as the main result we determine finite groups G such that their prime graph is $\Gamma(A)$, where A is an almost sporadic simple group, except $\mathrm{Aut}(J_2)$ and $\mathrm{Aut}(McL)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [5], for example. We use the results of J. S. Williams [31], N. Iiyori and H. Yamaki [12] and A. S. Kondrat'ev [22] about the prime graph of simple groups and the results of M. S. Lucido [24] about the prime graph of almost simple groups. We note that the structure of the almost sporadic simple groups are described in [5].

We denote by (a,b) the greatest common divisor of positive integers a and b. Let m be a positive integer and p be a prime number. Then $|m|_p$ denotes the $p-$part of m. In other words, $|m|_p = p^k$ if $p^k \| m$ (i.e. $p^k | m$ but $p^{k+1} \nmid m$).

2. Preliminary Results

First we give an easy remark:

Remark 2.1. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

Definition 2.1. ([8]) A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.1. ([31, Theorem A]) *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or a 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group;*
(d) *an extension of a simple group by a π_1-group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.2. *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

Proof. The prime graph of G has more than one component. So let $q \in \pi_2$. Let $y \in G$ be an element of order q. Since, $H \lhd G$, y induces an automorphism $\sigma \in \mathrm{Aut}(H)$. If $\sigma(h) = h$, for some $1 \neq h \in H$, then $yh = hy$. From the assumption, H is a π_1−group and $o(y) = q$. So $(o(h), o(y)) = 1$, which implies that $o(hy) = o(h)o(y)$. Hence, $q \in \pi_1$, which is a contradiction. Therefore, σ is a fixed-point-free automorphism of order q. Thus, H is a nilpotent group, by Thompson's theorem ([7, Theorem 10.2.1]). □

The next lemma summarizes the basic structural properties of a Frobenius group [7, 25]:

Lemma 2.3. *Let G be a Frobenius group and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$. Also the following conditions hold:*

(1) *$|H|$ divides $|K| - 1$.*
(2) *K is nilpotent and if $|H|$ is even, then K is abelian.*
(3) *Sylow p-subgroups of H are cyclic for odd p and are cyclic or generalized quaternion for $p = 2$.*
(4) *If H is a non-solvable Frobenius complement, then H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$, where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$.*

Also the next lemma follows from [8] and the properties of Frobenius groups [10]:

Lemma 2.4. *Let G be a 2-Frobenius group, i.e. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

(a) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
(b) *G/K and K/H are cyclic, $|G/K| \mid (|K/H|-1)$ and $G/K \leq Aut(K/H)$;*
(c) *H is nilpotent and G is a solvable group.*

By using the above lemmas it follows that:

Lemma 2.5. *Let G be a finite group and let A be an almost sporadic simple group, i.e. there exists an sporadic simple group S such that $S \leq A \leq \mathrm{Aut}(S)$. If the prime graph of A is not connected and $\Gamma(G) = \Gamma(A)$, then one of the following holds:*

(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group with $t(K/H) \geq 2$ and $G/K \leq Out(K/H)$. Also $\pi_2(A) = \pi_i(K/H)$ for some $i \geq 2$ and $\pi_2(A) \subseteq \pi(K/H) \subseteq \pi(S)$.*

The next lemma was introduced by Crescenzo and modified by Bugeaud:

Lemma 2.6. *([6, 17]) With the exceptions of the relations* $(239)^2-2(13)^4=-1$ *and* $(3)^5-2(11)^2=1$ *every solution of the equation*

$$p^m-2q^n=\pm 1;\quad p,q \quad \textit{prime}\ ;\quad m,n>1,$$

has exponents $m=n=2$*; i.e. it comes from a unit* $p-q.2^{\frac{1}{2}}$ *of the quadratic field* $Q(2^{\frac{1}{2}})$ *for which the coefficients* p*,* q *are prime.*

Lemma 2.7. *([17]) The only solution of the equation* $p^m-q^n=1$*;* p*,* q *prime; and* $m,n>1$ *is* $3^2-2^3=1$*.*

Lemma 2.8 (Zsigmondy's Theorem [33])**.** *Let* p *be a prime and* n *be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime* p' *for* p^n-1*, that is,* $p'|(p^n-1)$ *but* $p'\nmid(p^m-1)$*, for every* $1\le m<n$*,*
(ii) $p=2$*,* $n=1$ *or 6,*
(iii) p *is a Mersenne prime and* $n=2$*.*

Definition 2.2. A group G is called a C_{pp} group if the centralizers in G of its elements of order p are $p-$groups.

Lemma 2.9. *([4]) (a) The* $C_{13,13}$*-simple groups are:* A_{13}*,* A_{14}*,* A_{15}*;* Suz*,* Fi_{22}*;* $L_2(q)$*,* $q=3^3$*,* 5^2*,* 13^n *or* $2\times 13^n-1$ *which is a prime,* $n\ge 1$*;* $L_3(3)$*,* $L_4(3)$*,* $O_7(3)$*,* $S_4(5)$*,* $S_6(3)$*,* $O_8^+(3)$*,* $G_2(q)$*,* $q=2^2$*, 3;* $F_4(2)$*,*$U_3(q)$*,* $q=2^2$*, 23;* $Sz(2^3)$*,* ${}^3D_4(2)$*,* ${}^2E_6(2)$*,* ${}^2F_4(2)'$*.*

(b) The $C_{19,19}$*-simple groups are:* A_{19}*,* A_{20}*,* A_{21}*;* J_1*,* J_3*,* $O'N$*,* Th*,* HN*;* $L_2(q)$*,* $q=19^n$*,* $2\times 19^n-1$ *which is a prime,* $(n\ge 1)$*;* $L_3(7)$*,* $U_3(2^3)$*,* $R(3^3)$*,* ${}^2E_6(2)$*.*

Definition 2.3. By using the prime graph of G, the order of G can be expressed as a product of coprime positive integers m_i, $i=1,2,\ldots,t(G)$ where $\pi(m_i)=\pi_i(G)$. These integers are called *the order components* of G. The set of order components of G will be denoted by $OC(G)$. Also we call $m_2,\ldots,m_{t(G)}$ *the odd order components* of G.

The order components of non-abelian simple groups are listed in [13, Table 1].

Lemma 2.10. *([3, Lemma 8]) Let* G *be a finite group with* $t(G)\ge 2$ *and let* N *be a normal subgroup of* G*. If* N *is a* π_i-*group for some prime graph component* π_i *of* G *and* $m_1,m_2,\ldots,m_r$ *are some order components of* G *but not* π_i*-numbers, then* $m_1m_2\cdots m_r$ *is a divisor of* $|N|-1$*.*

3. Main Results

Let A be an almost sporadic simple group, that is $S\le A\le \mathrm{Aut}(S)$ where S is a sporadic simple group. Since $|Aut(S):S|\le 2$ for sporadic simple groups S (see [5]), so $A=S$ or $A=\mathrm{Aut}(S)$. Hagie considered the case $A=S$. So in the sequel we only assume the case $A=\mathrm{Aut}(S)$.

We note that some of the sporadic simple groups have trivial outer automorphism groups. Also if S is one of the following groups: M_{12}, He, Fi_{22} or HN, then $\mathrm{Aut}(S) \neq S$. But, $\Gamma(S) = \Gamma(\mathrm{Aut}(S))$. Therefore, we consider the case $A = \mathrm{Aut}(S)$, where S is one of the following groups: M_{22}, J_3, HS, Suz, $O'N$ or Fi'_{24}.

Now, we consider the following Diophantine equations:

$$(i)\quad \frac{q^p-1}{q-1} = y^n, \qquad (ii)\quad \frac{q^p-1}{(q-1)(p,q-1)} = y^n,$$

$$(iii)\quad \frac{q^p+1}{q+1} = y^n, \qquad (iv)\quad \frac{q^p+1}{(q+1)(p,q+1)} = y^n.$$

These Diophantine equations have many applications in the theory of finite groups (for example see [16] or [17]). We note that the odd order components of some non-abelian simple groups of Lie type are of the form $(q^p \pm 1)/((q\pm 1)(p,q\pm 1))$ [13] and there exists some results about these Diophantine equations [15]. Now, we prove the following lemma about these Diophantine equations to determine some C_{pp}-simple groups.

Lemma 3.1. *Let $p \geq 3$ and p_0 be prime numbers and $q = p_0^\alpha$.*

(a) If $y = 11$ and $p_0 \in \{2,3,5,7\}$, then $(p,q,n) = (5,3,2)$ is the only solution of (i) and (ii). Also $(p,q,n) = (5,2,1)$ is the only solution of (iii) and (iv).

(b) If $y = 29$ and $p_0 \in \{2,3,5,7,11,13,17,19,23\}$, then the Diophantine equations (i)-(iv) have no solution.

(c) If $y = 31$ and $p_0 \in \{2,3,5,7,11,19\}$, then $(p,q,n) = (5,2,1)$ and $(3,5,1)$ are the only solutions of (i) and (ii). Also (iii) and (iv) have no solution.

Proof. Let $q = p_o^\alpha$ and $(q^p-1)/(q-1) = 11^n$ or $(q^p-1)/((q-1)(p,q-1)) = 11^n$. Then $11 \mid (p_0^{\alpha p}-1)$, which implies that $p_0^{\alpha p} \equiv 1 \pmod{11}$ and hence $\beta := \mathrm{ord}_{11}(p_0)$ is a divisor of αp. Since, $p \geq 3$ and $(p_0^{\alpha p}-1)/(p_0^\alpha - 1) = 11^n$ or $(p_0^{\alpha p}-1)/(p_0^\alpha-1)(p,p_0^\alpha-1)) = 11^n$, it follows that 11 is a primitive prime for $p_0^{\alpha p}-1$. Also 11 is a primitive prime for $p_o^\beta - 1$, by the definition of $\mathrm{ord}_{11}(p_0)$. Therefore, $\beta = \alpha p$, by the definition of the primitive prime (see Lemma 2.8). Also by using the Fermat theorem we know that β is a divisor of 10. Hence, the only possibility for p is 5 and so $1 \leq \alpha \leq 2$. Now, by checking the possibilities for q it follows that $(p,q,n) = (5,3,2)$ is the only solution of the Diophantine equations (i) and (ii). Similarly consider the Diophantine equations

$$\frac{q^p+1}{q+1} = 11^n, \quad \text{and} \quad \frac{q^p+1}{(q+1)(p,q+1)} = 11^n,$$

Then 11 is a divisor of $p_o^{2\alpha p}-1$ and in a similar manner it follows that $p=5$ and $\alpha = 1$. Therefore, the only solution of these Diophantine equations is $(p,q,n) = (5,2,1)$.

The proof of (b) and (c) are similar and for convenience we omit the proof of them. □

Now, by using Lemmas 2.6, 2.7 and 3.1, we can prove the following lemma:

Lemma 3.2. *Let M be a simple group of Lie type over $GF(q)$.*

(a) *If q is a power of 2, 3, 5 or 7 and M is a $C_{11,11}$-group, then M is one of the following simple groups: $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O^+_{10}(3)$.*

(b) *If q is a power of 2, 3, 5, 7, 11, 13, 17, 19 or 23 and M is a $C_{29,29}$-group, then $M = L_2(29)$.*

(c) *If q is a power of 2, 3, 5, 7, 11 or 19 and M is a $C_{31,31}$-group, then M is $L_5(2)$, $L_3(5)$, $L_6(2)$, $L_4(5)$, $O^+_{10}(2)$, $O^+_{12}(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $Sz(32)$.*

Proof. The odd order components of finite non-abelian simple groups are listed in Table 1 in [13]. Now, by using Lemmas 2.6, 2.7, 2.8 and 3.1 we get the result. For convenience we omit the proof. □

Theorem 3.1. *Let G be a finite group satisfying $\Gamma(G) = \Gamma(A)$.*

(a) *If $A = \mathrm{Aut}(J_3)$, then $G/O_\pi(G) \cong J_3$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong J_3.2$, where $\pi \subseteq \{2, 3, 5\}$.*

(b) *If $A = \mathrm{Aut}(M_{22})$, then $G/O_2(G) \cong M_{22}$ and $O_2(G) \neq 1$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$.*

(c) *If $A = \mathrm{Aut}(HS)$, then $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong HS.2$, $U_6(2).2$ or McL, where $\pi \subseteq \{2, 3, 5\}$.*

(d) *If $A = \mathrm{Aut}(Fi'_{24})$, then $G/O_\pi(G) \cong Fi'_{24}$, where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong Fi'_{24}.2$, where $\pi \subseteq \{2, 3\}$.*

(e) *If $A = \mathrm{Aut}(O'N)$, then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong O'N.2$, where $\pi \subseteq \{2\}$.*

(f) *If $A = \mathrm{Aut}(Suz)$, then $G/O_\pi(G) \cong Suz$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_\pi(G) \cong Suz.2$, where $\pi \subseteq \{2, 3, 5\}$.*

Proof. **(a)** Let $\Gamma(G) = \Gamma(\mathrm{Aut}(J_3))$. First, let G be a solvable group. Then G has a Hall $\{5, 17, 19\}$-subgroup H. Since, G is solvable, it follows that H is solvable. Hence, $t(H) \leq 2$, which is a contradiction, since there exists no edge between 5, 17 and 19 in $\Gamma(G)$. Thus, G is not solvable, and so G is not a 2-Frobenius group, by Lemma 2.4. If G is a non-solvable Frobenius group and H and K be the Frobenius complement and the Frobenius kernel of G, respectively, then by using Lemma 2.3 it follows that H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$ where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$. We know that $3 \nsim 17$ and $3 \nsim 19$ in $\Gamma(G)$. Therefore, $Z = 1$. Hence, $\{17, 19\} \subseteq \pi(K)$. This is a contradiction, since K is nilpotent and $17 \nsim 19$ in $\Gamma(G)$. Hence, G is neither a Frobenius group nor a 2-Frobenius group. So by using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{19,19}$ simple group. By using Lemma 2.9, K/H is A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN, $L_3(7)$, $U_3(8)$, $R(27)$, ${}^2E_6(2)$, $L_2(q)$, where $q = 19^n$ or $L_2(q)$, where $q = 2 \times 19^n - 1$ $(n \geq 1)$ is a prime number. But, $\pi(K/H) \subseteq \pi(J_3)$ and $\pi(J_3) \cap \{7, 11, 13, 31\} = \emptyset$. Also $q = 2 \times 19^n - 1 > 19$.

Hence, the only possibilities for K/H are J_3 and $L_2(19^n)$, where $n \geq 1$. The orders of maximal tori of $A_m(q) = PSL(m+1, q)$ are

$$\frac{\prod_{i=1}^{k}(q^{r_i} - 1)}{(q-1)(m+1, q-1)}; \quad (r_1, \ldots, r_k) \in Par(m+1).$$

Therefore, every element of $\pi_e(PSL(2, q))$ is a divisor of q, $(q+1)/d$ or $(q-1)/d$, where $d = (2, q-1)$. If $q = 19^n$, then $3 \mid (19^n - 1)/2$ and since $3 \sim 5$ and $3 \nsim 17$ in $\Gamma(G)$, it follows that if 5 divides $|G|$, then $5 \mid (19^n - 1)$ and if 17 is a divisor of $|G|$, then $17 \mid (19^n + 1)$. Note that $\pi(19 - 1) = \{2, 3\}$, $\pi(19^2 - 1) = \{2, 3, 5\}$ and $17 \mid (19^4 + 1)$. Now by using the Zsigmondy's Theorem, Lemmas 2.6 and 2.7 it follows that the only possibility is $n = 1$.

Now, we consider these possibilities for K/H, separately.

Case 1. Let $K/H \cong J_3$.

We note that $Out(J_3) \cong \mathbb{Z}_2$ and hence G/H is isomorphic to J_3 or $J_3.2$. Also H is a nilpotent π_1-group. Hence, $\pi(H) \subseteq \{2, 3, 5, 17\}$. If $17 \in \pi(H)$, then let T be a $\{3, 17, 19\}$ subgroup of G, since J_3 has a $19 : 9$ subgroup. Obviously, T is solvable and hence $t(T) \leq 2$, which is a contradiction. Therefore, $\pi = \pi(H) \subseteq \{2, 3, 5\}$ and $G/O_\pi(G) \cong J_3$ or $G/O_\pi(G) \cong J_3.2$. If $G/O_\pi(G) \cong J_3$, then $O_\pi(G) \neq 1$ and $2 \in \pi$, since $2 \nsim 17$ in $\Gamma(J_3)$.

Case 2. Let $K/H \cong L_2(19)$.

Since $Out(L_2(19)) \cong \mathbb{Z}_2$, it follows that $G/H \cong L_2(19)$ or $L_2(19).2$. But, in this case $\pi(K/H) = \{2, 3, 5, 19\}$ and so $17 \mid |H|$. We know that $L_2(19)$ contains a $19 : 9$ subgroup and hence G has a $\{3, 17, 19\}$-subgroup T which is solvable and so $t(T) \leq 2$. But, this is a contradiction, since $t(T) = 3$. Therefore, $K/H \ncong L_2(19)$.

(b) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(M_{22}))$.

If G is a solvable group, then let T be a Hall $\{3, 5, 7\}$-subgroup of G. Obviously T is solvable and hence $t(T) \leq 2$, which is a contradiction. If G is a non-solvable Frobenius group, then G has a Frobenius kernel K and a Frobenius complement H. By using Lemma 2.3, it follows that H has a normal subgroup $H_0 = SL(2, 5) \times Z$, where $|H : H_0| \leq 2$ and $(|Z|, 30) = 1$. Since, $5 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$, it follows that $Z = 1$ and so $\pi(K) = \{7, 11\}$, which is a contradiction since K is nilpotent and $7 \nsim 11$ in $\Gamma(G)$. Therefore, G is not a Frobenius group or a 2-Frobenius group. By using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{11,11}$-simple group. If K/H is an alternating group or a sporadic simple group which is a $C_{11,11}$-group, then K/H is: A_{11}, A_{12}, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, HS, Sz, $O'N$, Co_2 or J_1. Also $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$, by Remark 2.1. Therefore, $3 \nsim 5$ in $\Gamma(K/H)$ and $\pi(K/H) \subseteq \{2, 3, 5, 7, 11\}$, which implies that the only possibilities for K/H are $L_2(11)$, M_{11}, M_{12} and M_{22}. If $K/H \cong M_{11}$, M_{12} or $L_2(11)$, then K/H has a $11 : 5$ subgroup by [5]. Also in these cases $7 \notin \pi(K/H)$ and hence $7 \in \pi(H)$. Now, consider the $\{5, 7, 11\}$ subgroup T of G which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore, $K/H \cong M_{22}$ and since $Out(M_{22}) \cong \mathbb{Z}_2$ it follows that $G/H \cong M_{22}$ or $M_{22}.2$. Also H is a nilpotent

π_1-group and so $\pi(H) \subseteq \{2, 3, 5, 7\}$. By using [5] we know that M_{22} has a $11 : 5$ subgroup. If $3 \in \pi(H)$, then let T be a $\{3, 5, 11\}$ subgroup of G which is solvable and hence $t(T) \leq 2$, which is a contradiction, since there exists any edge between 3, 5 and 11 in $\Gamma(G)$. Therefore, $3 \notin \pi(H)$. Similarly, it follows that $7 \notin \pi(H)$. Let $5 \in \pi(H)$ and $Q \in Syl_5(H)$. Also let $P \in Syl_3(K)$. We know that H is nilpotent and hence Q *char* H. Since $H \lhd K$ it follows that $Q \lhd K$. Therefore P acts by conjugation on Q and since $3 \nsim 5$ in $\Gamma(G)$ it follows that P acts fixed point freely on Q. Hence, QP is a Frobenius group with Frobenius kernel Q and Frobenius complement P. Now by using Lemma 2.3 it follows that P is a cyclic group which implies that a Sylow 3-subgroup of M_{22} is cyclic. But, this is a contradiction since a 3-Sylow subgroup of M_{22} are elementary abelian by [5]. Therefore, H is a 2-group. Then $G/O_2(G) \cong M_{22}$, where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$.

(C) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(HS))$.

If G is solvable, then G has a Hall $\{3, 7, 11\}$-subgroup T. Hence, T is solvable and so $t(T) \leq 2$, which is a contradiction. Hence, G is not a 2-Frobenius group. If G is a non-solvable Frobenius group, then by using Lemma 2.3, H, the Frobenius complement of G, has a normal subgroup $H_0 = SL(2, 5) \times Z$, where $(|Z|, 30) = 1$ and $|H : H_0| \leq 2$. Since, $5 \nsim 7$ and $5 \nsim 11$ in $\Gamma(G)$, it follows that $Z = 1$ and hence 77 is a divisor of $|K|$, where K is the Frobenius kernel of G. But, this is a contradiction. Since, $7 \nsim 11$ in $\Gamma(G)$ and K is nilpotent.

Now, similar to (b), G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is one of the following groups: M_{11}, M_{12}, M_{22}, McL, HS, $U_5(2)$, $U_6(2)$ and $L_2(11)$.

Case 1. Let $K/H \cong M_{11}$, M_{12}, $U_5(2)$ or $L_2(11)$.

By using [5] we know that $|Out(K/H)|$ is a divisor of 2. Therefore, $7 \notin \pi(G/H)$, and hence $7 \in \pi(H)$. Since in each case, K/H has a $11 : 5$ subgroup it follows that G has a $\{5, 7, 11\}$ subgroup T, which is solvable and hence $t(T) \leq 2$. But, this is a contradiction and so this case is impossible.

Case 2. Let $K/H \cong M_{22}$.

We note that $out(M_{22}) \cong \mathbb{Z}_2$. Hence, $G/H \cong M_{22}$ or $M_{22}.2$. First let $G/H \cong M_{22}$, where H is a π_1-group and $\pi_1 = \{2, 3, 5, 7\}$. We know that M_{22} has a $11 : 5$ subgroup (see [5]). If $2 \in \pi(H)$, then G has a $\{2, 5, 11\}$ subgroup T which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore, $2 \notin \pi(H)$. If $3 \in \pi(H)$ or $7 \in \pi(H)$, then let T be a $\{3, 5, 11\}$ or $\{5, 7, 11\}$ subgroup of G, respectively. Then $t(T) \leq 2$, which is a contradiction. If $5 \in \pi(H)$, then let P be a Sylow 5-subgroup of H. If $Q \in Syl_3(G)$, then Q acts fixed point freely on P, since $3 \nsim 5$ in $\Gamma(G)$. Therefore, PQ is a Frobenius group which implies that Q be a cyclic group and it is a contradiction. Hence $H = 1$ and so $G = M_{22}$. But, $\Gamma(M_{22}) \neq \Gamma(\mathrm{Aut}(HS))$, since $2 \nsim 5$ in $\Gamma(M_{22})$. Therefore, this case is impossible.

Now, let $G/H \cong M_{22}.2$. By using [5], M_{22} has a $11 : 5$ subgroup. Similar to the above discussion we conclude that $\{3, 5, 7\} \cap \pi(H) = \emptyset$, and hence H is a 2-group. But, in this case 3 and 5 are not joined which is a contradiction. Therefore, Case 2 is impossible, too.

Case 3. Let $K/H \cong U_6(2)$.

By using [5], it follows that $Out(K/H) \cong S_3$. We know that $U_6(2).3$ has an element of order 21. Therefore, $G/H \cong U_6(2)$ or $U_6(2).2$. Also $7 \notin \pi(H)$, since $U_6(2)$ has a $11 : 5$ subgroup. Therefore, if $G/H \cong U_6(2)$, then $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $G/O_\pi(G) \cong U_6(2)$, where $O_\pi(G) \neq 1$. Similarly, if $G/H \cong U_6(2).2$, then $G/O_\pi(G) \cong U_6(2).2$, where $\pi \subseteq \{2, 3, 5\}$.

Case 4. Let $K/H \cong McL$.

Note that $Out(McL) = 2$. But, $G/H \ncong McL.2$, since $McL.2$ has an element of order 22. Similar to the above proof it follows that $G/O_\pi(G) \cong McL$ and $\pi \subseteq \{2, 3, 5\}$, since McL has a $11 : 5$ subgroup.

Case 5. Let $K/H \cong HS$.

There exists a $11 : 5$ subgroup in HS. Similar to Case 3, it follows that $G/O_\pi(G) \cong HS$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$, or $G/O_\pi(G) \cong HS.2$, where $\pi \subseteq \{2, 3, 5\}$.

(d) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(Fi'_{24}))$.

We claim that G is not solvable, otherwise let T be a Hall $\{7, 17, 23\}$-subgroup of G, which is solvable but $t(T) = 3$, a contradiction. If G is a non-solvable Frobenius group, then $\{11, 13, 17, 23, 29\} \subseteq \pi(K)$, where K is the Frobenius kernel of G. But, this is a contradiction since $11 \nsim 13$ and K is nilpotent. Hence, by using Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{29,29}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore, K/H is $L_2(29)$, Ru or Fi'_{24}. If $K/H \cong L_2(29)$ or Ru, then $\{17, 23\} \subseteq \pi(H)$, which is a contradiction. Since, H is nilpotent and $17 \nsim 23$ in $\Gamma(G)$. Therefore, $K/H \cong Fi'_{24}$ and so $G/H \cong Fi'_{24}$ or $Fi'_{24}.2$. By using [5], we know that Fi'_{24} has a $23 : 11$ subgroup. Therefore, $\pi(H) \cap \{5, 7, 13, 17\} = \emptyset$. Also Fi'_{24} has a $29 : 7$ subgroup, and hence $\pi(H) \cap \{11, 13\} = \emptyset$. Therefore, $\pi(H) \subseteq \{2, 3\}$ and so $G/O_\pi(G) \cong Fi'_{24}$, where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong Fi'_{24}.2$, where $\pi \subseteq \{2, 3\}$.

(e) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(O'N))$.

If G is solvable, then G has a Hall $\{3, 11, 31\}$-subgroup T, which has three components and this is a contradiction. If G is a non-solvable Frobenius group, then the Frobenius kernel of G has elements of order 7 and 11. But, $77 \notin \pi_e(G)$, which is a contradiction. Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{31,31}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Hence, K/H is $L_3(5)$, $L_5(2)$, $L_6(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $O'N$. If $K/H \cong L_2(5)$, $L_6(2)$, $L_2(31)$ or $G_2(5)$, then $11, 19 \in \pi(H)$, which is a contradiction. Since, $209 \notin \pi_e(G)$ and H is nilpotent. If $K/H \cong L_3(5)$ or $L_2(32)$, then $\{7, 19\} \subseteq \pi(H)$, which is a contradiction. Since, $7 \nsim 19$ in $\Gamma(G)$. Therefore, $K/H \cong O'N$ and $Out(O'N) = 2$, which implies that $G/H \cong O'N$ or $O'N.2$. We know that $O'N$ has a $11 : 5$ subgroup by [5] and if we consider $\{5, 11, p\}$-subgroup of G, where $p \in \{7, 19, 31\}$, it follows that $\pi(H) \cap \{7, 19, 31\} = \emptyset$. Therefore, $\pi(H) \subseteq \{2, 3, 5, 11\}$. Also $O'N$ has a $19 : 3$ subgroup, which implies that $\pi(H) \cap \{11\} = \emptyset$. Let $p \in \{3, 5\}$. If $p \in \pi(H)$, then let P be the p-Sylow subgroup of H. If $Q \in Syl_7(G)$, then Q acts fixed point freely on P, since $7 \nsim 3$ and $7 \nsim 5$ in $\Gamma(G)$. Therefore, PQ is a Frobenius group and hence Q is a cyclic group. But, this is a contradiction. Since, Sylow 7-subgroups of $O'N$ are elementary abelian by [5]. Therefore, $\pi(H) \cap \{3, 5\} = \emptyset$. Hence, $\pi(H)$ is a 2-group. Then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$; or $G/O_\pi(G) \cong O'N.2$ where $\pi \subseteq \{2\}$.

(f) Let $\Gamma(G) = \Gamma(\mathrm{Aut}(Suz))$.

Since, $7 \nsim 11$, $11 \nsim 13$ and $7 \nsim 13$, it follows that G is not a solvable group. If G is a 2-Frobenius group, then $\{11, 13\} \subseteq \pi(K)$, where K is the Frobenius kernel of G. Then $11 \sim 13$, since K is nilpotent. But, this is a contradiction. Therefore, G is neither a Frobenius group nor a 2-Frobenius group. Hence, there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a $C_{13,13}$ simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore, K/H is $Sz(8)$, $U_3(4)$, ${}^3D_4(2)$, Suz, Fi_{22}, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $S_6(3)$, $G_2(4)$, $S_4(5)$ or $G_2(3)$.

If $K/H \cong {}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, $L_4(3)$, $S_4(5)$ or $G_2(3)$, then $\{7, 11\} \subseteq \pi(H)$, which implies that $7 \sim 11$, since H is nilpotent. But, this is a contradiction. If $K/H \cong {}^3D_4(2)$, $L_2(27)$, $L_2(13)$ or $L_3(3)$, then $\{5, 11\} \subseteq \pi(H)$ and we get a contradiction similarly. Since, $5 \nsim 11$.

If $K/H \cong G_2(4)$, $S_6(3)$, $O_7(3)$ or $O_8^+(3)$, then $11 \in \pi(H)$ and K/H has a $13 : 3$ subgroup by [5]. Let T be a $\{3, 11, 13\}$-subgroup of G. It follows that $t(T) = 3$, which is a contradiction. Since, T is solvable.

If $K/H \cong Fi_{22}$, then $G/H \cong Fi_{22}$ or $Fi_{22}.2$, where $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since, Fi_{22} has $11 : 5$ and $13 : 3$ subgroups it follows that $\{7, 11\} \cap \pi(H) = \emptyset$. Therefore, $G/O_\pi(G) \cong Fi_{22}$ or $Fi_{22}.2$, where $\pi \subseteq \{2, 3, 5\}$.

Let $K/H \cong Sz(8)$. It is known that $Out(Sz(8)) \cong \mathbb{Z}_3$ and so $G/H \cong Sz(8)$ or $Sz(8).3$. If $G/H \cong Sz(8)$, then $\{3, 11\} \subseteq \pi(H)$ which is a contradiction. Since, $3 \nsim 11$. If $G/H \cong Sz(8).3$, then let T be $\{3, 7, 11\}$-subgroup of G. Since, $Sz(8)$ has a $7 : 6$ subgroup. Then $t(T) = 3$, which is a contradiction.

If $K/H \cong Suz$, then $G/H \cong Suz$ or $Suz.2$. If $G/K \cong Suz$, then $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since, Suz has a $11 : 5$ and $13 : 3$ subgroups it follows that $7, 11 \notin \pi(H)$. Therefore, $G/O_\pi(G) \cong Suz$, where $2 \in \pi$ and $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$. If $G/H \cong Suz.2$, then it follows that $G/O_\pi(G) \cong Suz.2$, where $\pi \subseteq \{2, 3, 5\}$. □

Remark 3.1. W. Shi and J. Bi in [29] put forward the following conjecture:

Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$, and, (ii) $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for sporadic simple groups [27], alternating groups and some simple groups of Lie type [28, 26, 29]. As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.2. *Let G be a finite group and A be an almost sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$, then $G \cong A$.*

We note that Theorem 3.2 was proved in [14] by using the characterization of almost sporadic simple groups with their order components. Now, we give a new proof for this theorem. In fact we prove the following result which is a generalization of the Shi-Bi Conjecture and so Theorem 3.2 is an immediate consequence of Theorem 3.3. Also note that Theorem 3.3 is a generalization of a result in [1].

Theorem 3.3. *Let A be an almost sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If G is a finite group satisfying $|G| = |A|$ and $\Gamma(G) = \Gamma(A)$, then $G \cong A$.*

Proof. First, let $A = \mathrm{Aut}(M_{22})$. By using Theorem 3.1, it follows that $G/O_2(G) \cong M_{22}$ or $G/O_\pi(G) \cong M_{22}.2$, where $\pi \subseteq \{2\}$. If $G/O_2(G) \cong M_{22}$, then $|O_2(G)| = 2$. Hence, $O_2(G) \subseteq Z(G)$ which is a contradiction. Since, G has more than one component and hence $Z(G) = 1$. Therefore, $G/O_\pi(G) \cong M_{22}.2$, where $2 \in \pi$, which implies that $O_\pi(G) = 1$ and hence $G \cong M_{22}.2$

Let $A = \mathrm{Aut}(HS)$. By using Theorem 3.1, it follows that $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong U_6(2).2$, McL or $HS.2$, where $\pi \subseteq \{2, 3, 5\}$.

By using [5], it follows that 3^6 divides the orders of $U_6(2)$, $U_6(2).2$ and McL, but $3^6 \nmid |G|$.

Therefore, $G/O_\pi(G) \cong HS$ or $HS.2$. Now, we get the result similarly to the last case.

For convenience we omit the details of the proof of other cases. □

Acknowledgements

The author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, IRAN for the financial support.

References

[1] G. Chen. A new characterization of sporadic simple groups. *Algebra Colloq.*, 3(1):49–58, 1996.

[2] G. Chen. On Thompson's conjecture. *J. Algebra*, 185(1):184–193, 1996.

[3] G. Chen. Further reflections on Thompson's conjecture. *J. Algebra*, 218(1):276–285, 1999.

[4] Z. M. Chen and W. J. Shi. On simple c_{pp} groups. *J. Southwest Teach. Univ.*, 18(3):249–256, 1993.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups.* Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.
[6] P. Crescenzo. A Diophantine equation which arises in the theory of finite groups. *Advances in Math.*, 17(1):25–29, 1975.
[7] D. Gorenstein. *Finite groups.* Harper & Row Publishers, New York, 1968.
[8] K. W. Gruenberg and K. W. Roggenkamp. Decomposition of the augmentation ideal and of the relation modules of a finite group. *Proc. London Math. Soc. (3)*, 31(2):149–166, 1975.
[9] M. Hagie. The prime graph of a sporadic simple group. *Comm. Algebra*, 31(9):4405–4424, 2003.
[10] B. Huppert. *Endliche Gruppen. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, 1967.
[11] B. Huppert and N. Blackburn. *Finite groups. III*, volume 243 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1982.
[12] N. Iiyori and H. Yamaki. Prime graph components of the simple groups of Lie type over the field of even characteristic. *J. Algebra*, 155(2):335–343, 1993.
[13] A. Iranmanesh, S. H. Alavi, and B. Khosravi. A characterization of $\mathrm{PSL}(3,q)$ where q is an odd prime power. *J. Pure Appl. Algebra*, 170(2-3):243–254, 2002.
[14] A. Khosravi and B. Khosravi. A new characterization of almost sporadic groups. *J. Algebra Appl.*, 1(3):267–279, 2002.
[15] A. Khosravi and B. Khosravi. On the Diophantine equation $\frac{q^n-1}{q-1} = y$. *Comment. Math. Univ. Carolin.*, 44(1):1–7, 2003.
[16] A. Khosravi and B. Khosravi. A new characterization of $\mathrm{PSL}(p,q)$. *Comm. Algebra*, 32(6):2325–2339, 2004.
[17] A. Khosravi and B. Khosravi. A new characterization of some alternating and symmetric groups. II. *Houston J. Math.*, 30(4):953–967 (electronic), 2004.
[18] A. Khosravi and B. Khosravi. Quasirecognition of the simple group ${}^2G_2(q)$ by a prime graph. *Sibirsk. Mat. Zh.*, 48(3):707–716, 2007.
[19] B. Khosravi and S. S. S. Amiri. On the prime graph of $L_2(q)$ where $q = p^\alpha < 100$. *Quasigroups Related Systems*, 14(2):179–190, 2006.
[20] B. Khosravi, B. Khosravi, and B. Khosravi. Groups with the same prime graph as a CIT simple group. *Houston J. Math.*, 33(4):967–977 (electronic), 2007.
[21] B. Khosravi, B. Khosravi, and B. Khosravi. On the prime graph of $\mathrm{PSL}(2,p)$ where $p > 3$ is a prime number. *Acta Math. Hungar.*, 116(4):295–307, 2007.
[22] A. S. Kondrat′ev. On prime graph components of finite simple groups. *Mat. Sb.*, 180(6):787–797, 864, 1989.
[23] M. S. Lucido. The diameter of the prime graph of a finite group. *J. Group Theory*, 2(2):157–172, 1999.
[24] M. S. Lucido. Prime graph components of finite almost simple groups. *Rend. Sem. Mat. Univ. Padova*, 102:1–22, 1999.
[25] D. Passman. *Permutation groups.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.
[26] W. J. Shi. A new characterization of some simple groups of Lie type. In *Classical groups and related topics (Beijing, 1987)*, volume 82 of *Contemp. Math.*, pages 171–180. Amer. Math. Soc., Providence, RI, 1989.
[27] W. J. Shi. A new characterization of the sporadic simple groups. In *Group theory (Singapore, 1987)*, pages 531–540. de Gruyter, Berlin, 1989.
[28] W.-J. Shi. Pure quantitative characterization of finite simple groups. I. *Progr. Natur. Sci. (English Ed.)*, 4(3):316–326, 1994.

[29] W. J. Shi and J. X. Bi. A characteristic property for each finite projective special linear group. In *Groups—Canberra 1989*, volume 1456 of *Lecture Notes in Math.*, pages 171–180. Springer, Berlin, 1990.

[30] W. J. Shi and J. X. Bi. A new characterization of the alternating groups. *Southeast Asian Bull. Math.*, 16(1):81–90, 1992.

[31] J. S. Williams. Prime graph components of finite groups. *J. Algebra*, 69(2):487–513, 1981.

[32] A. V. Zavarnitsin. Recognition of the simple groups $U_3(q)$ by element orders. *Algebra Logika*, 45(2):185–202, 252–253, 2006.

[33] K. Zsigmondy. Zur Theorie der Potenzreste. *Monatsh. Math. Phys.*, 3:265–285, 1892.

Received on August 10, 2007; accepted on January 18, 2009

Department of Pure Mathematics,
Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
424, Hafez Ave,
Tehran 15914, IRAN
and
School of Mathematics,
Institute for Research in Fundamental Sciences (IPM)
E-mail address: khosravibbb@yahoo.com