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ON A CLASS OF LIE p-ALGEBRAS

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Abstract. In this paper we study the finite dimensional Lie p-algebras, $\mathcal{L}$ splitting on its abelian p-socle, the sum of its minimal abelian p-ideals. In addition, some properties of the Frattini p-subalgebra of $\mathcal{L}$ are pointed out.

1. Introduction

In this section, we recall some notions and properties necessary in the paper.

Definition 1.1. A Lie p-algebra is a Lie algebra $\mathcal{L}$ with a p-map $a \to a^p$, such that:

$$(\alpha x)^p = \alpha^p x^p, \text{ for all } \alpha \in \mathbb{K},\ x \in \mathcal{L},$$
$$x(ady^p) = x(ady)^p, \text{ for all, } x, y \in \mathcal{L},$$
$$(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x,y) \text{ for all } x, y \in \mathcal{L},$$

where $is_i(x,y)$ is the coefficient of X^{i-1} in the expansion of $x(ad(Xu+y))^{p-1}$.

A subalgebra (respectively, ideal) of $\mathcal{L}$ is p-subalgebra (respectively, p-ideal) if it is closed under the p-map.

The notions of *maximal p-subalgebra* respectively *maximal p-ideal* of $\mathcal{L}$ are defined as usual. The intersection of p-subalgebras (respectively p-ideals) is a p-subalgebra (respectively a p-ideal) of $\mathcal{L}$.

We denote by $\Phi_p(\mathcal{L})$ the p-subsubalgebra of $\mathcal{L}$ obtained by intersecting all maximal p-subalgebras of $\mathcal{L}$ and we call it *the Frattini p-subalgebra* of $\mathcal{L}$.

The largest p-ideal of $\mathcal{L}$ included into $\Phi_p(\mathcal{L})$ is called *the Frattini p-ideal* and is denoted by $\mathcal{F}_p(\mathcal{L})$.

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These are the corresponding notions to the Frattini subalgebra $\Phi(\mathcal{L})$ and the Frattini ideal $\mathcal{F}(\mathcal{L})$ for a Lie algebra.

We shall use the following notations:

- $[x, y]$ is the product of x, y in $\mathcal{L}$;
- $\mathcal{L}^{(1)}$ the derived algebra of $\mathcal{L}$;
- $\mathcal{L}^{(n)} = \left(\mathcal{L}^{(n-1)}\right)^{(1)}$, for all $n \geq 2$;
- $(\mathcal{A})$ is the subalgebra generated by the subset $\mathcal{A}$ of $\mathcal{L}$;
- $(\mathcal{A})_p = \left(\{x^{p^n} | x \in (\mathcal{A}), p \in \mathbb{N}\}\right)$, where $x^{p^n} = x^{(p^n)}$;
- $\mathcal{A}^p = (\{x^p | x \in \mathcal{A}\})$, where $\mathcal{A}$ is a subalgebra of $\mathcal{L}$;
- $\mathcal{A}^{p^n} = \left(\mathcal{A}^{p^{n-1}}\right)^p$;
- $\mathcal{L}_1 = \bigcap\limits_{i=1}^{\infty} \mathcal{L}^{p^i}$;
- $\mathcal{L}_0 = \{x \in \mathcal{L} | x^{p^n} = 0 \text{ for some } n\}$;
- $Z(\mathcal{L})$ is the center of $\mathcal{L}$;
- $\mathcal{N}(\mathcal{L})$ is the nilradical of $\mathcal{L}$.

Note that, if $\mathcal{L}$ is a p-algebra (finite dimensional), then $Z(\mathcal{L})$ is closed as p-ideal of $\mathcal{L}$.

2. Lie p-algebras which are $\mathcal{F}_p$-free

In [8], Stitzinger has proved the following

Proposition 2.1. *If $\mathcal{L}$ is a finite dimensional Lie algebra over a field $\mathbb{K}$, then*

$$\mathcal{L}^{(1)} \cap Z(\mathcal{L}) \subseteq \mathcal{F}(\mathcal{L}).$$

We may prove an analogue of this proposition for a Lie p-algebra.

Lemma 2.2. *If $\mathcal{L}$ is a finite dimensional Lie p-algebra over a field $\mathbb{K}$, then we have*

$$(\mathcal{L}^{(1)})_p \cap Z(\mathcal{L}) \subseteq \mathcal{F}_p(\mathcal{L}).$$

Proof. Let $\mathcal{M}$ be a maximal p-subalgebra of $\mathcal{L}$ and suppose that $Z(\mathcal{L}) \nsubseteq \mathcal{M}$. Then $\mathcal{L} = \mathcal{M} + Z(\mathcal{L})$, so $\mathcal{L}^{(1)} = \mathcal{M}^{(1)} \subseteq \mathcal{M}$ and hence

$$(\mathcal{L}^{(1)})_p \subseteq (\mathcal{M})_p \subseteq \mathcal{M}.$$

□

The *abelian socle* $\mathrm{Sa}(\mathcal{L})$ is the sum of all minimal ideals of $\mathcal{L}$.

We may define the abelian p-socle of the finite dimensional Lie p-algebra $\mathcal{L}$ as being the sum of all minimal abelian p-ideals of $\mathcal{L}$ and we denote it by $\mathrm{Sap}(\mathcal{L})$.

The abelian socle (respectively, the abelian p-socle) of a finite dimensional Lie (p)-algebra is an ideal (a p-ideal) of $\mathcal{L}$, as one can show easily.

Definition 2.3. Let $\mathcal{L}$ be a finite dimensional Lie p-algebra and I be a p-ideal of $\mathcal{L}$. We say that $\mathcal{L}$ *p-splits over I* if there exists a p-subalgebra B of $\mathcal{L}$ such that $\mathcal{L} = I \dot{+} B$.

B is called a *p-complement* of the p-ideal I.

Theorem 2.4. *Let $\mathcal{L}$ be a finite dimensional Lie p-algebra such that $\mathcal{L}^{(1)} \neq 0$ and $\mathcal{L}^{(1)}$ is nilpotent. Then the following statements are equivalent:*

(i) $\mathcal{F}_p(\mathcal{L}) = 0$.

(ii) $\mathrm{Sap}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$, *and $\mathcal{L}$ p-splits over $\mathcal{N}(\mathcal{L})$.*

(iii) $\mathcal{L}^{(1)}$ is abelian, $(\mathcal{L}^{(1)})^p = 0$, $\mathcal{L}$ p-splits over $\mathcal{L}^{(1)} \oplus Z(\mathcal{L})$, and

$$\mathrm{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L}).$$

In the same hypotheses, the Cartan subalgebra of $\mathcal{L}$ are exactly those subalgebras which have $\mathcal{L}^{(1)}$ as a p-complement.

Proof. (i) $\Rightarrow$ (ii): These implications are immediate from Theorems 4.1, 4.2 of [5].

(iii)$\Rightarrow$ (i): This also follows from Theorem 4.1 of [5].

(i) $\Rightarrow$ (iii): Suppose that $\mathcal{F}_p(\mathcal{L}) = 0$. Then $\mathcal{F}(\mathcal{L}) = 0$, and $\mathcal{L}^{(1)}$ is abelian. Now $(\mathcal{L}^{(1)})^p \subseteq Z(\mathcal{L})$ by Lemma 2.1 [6], and so

$$(\mathcal{L}^{(1)})^p \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) \subseteq \mathcal{F}_p(\mathcal{L}) = 0,$$

by Lemma 2.2. Clearly $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) \subseteq \mathcal{N}(\mathcal{L}) = \mathrm{Sap}(\mathcal{L})$.

Now let $\underline{m}$ be a minimal (and hence abelian) p-ideal of $\mathcal{L}$. Then $[\mathcal{L}, \underline{m}] =\underline{m}$ is an ideal of $\mathcal{L}$ and

$$[\mathcal{L}, \underline{m}]^p \subseteq (\mathcal{L}^{(1)})^p \cap \underline{m}^p \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) = 0$$

by Lemma 2.1 of [6] and by Lemma 2.2. Hence $[\mathcal{L}, \underline{m}]$ is p-closed, therefore $[\mathcal{L}, \underline{m}] = \underline{m}$ or $[\mathcal{L}, \underline{m}] = 0$.

The former implies that $\underline{m} \subseteq \mathcal{L}^{(1)}$, and the latter that $\underline{m} \subseteq Z(\mathcal{L})$ hence $\mathrm{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L})$ and (iii) follows.

The last part of the theorem precises that the Cartan subalgebras are exactly those subalgebras having $\mathcal{L}^{(1)}$ as a p-complement. This follows from Proposition 1 of [8], or from Theorem 4.4.1.1. of [10] and from the fact that Cartan subalgebras are p-closed. □

Corollary 2.5. *If $\mathcal{L}$ is a finite dimensional Lie p-algebra over $\mathbb{K}$ with $\mathcal{L}^{(1)}$ nilpotent, and nonzero $\mathcal{F}_p(\mathcal{L}) = 0$ and $\mathbb{K}$ is perfect, then the maximal toral subalgebras are precisely those having as p-complement $\mathcal{L}^{(1)} \oplus Z(\mathcal{L})$.*

Proof. Take $\mathcal{L} = \big(\mathcal{L}^{(1)} \oplus Z(\mathcal{L})\big) \dot{+} B$, with Bp-closed and $B^{(1)} = 0$ and let $B = B_0 \oplus B_1$ be the Fitting decomposition of B relatively to the p-map. Then $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) = \mathrm{Sap}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$ from Theorem 2.4, (ii), (iii). But $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) \dot{+} B_0$

is a nilpotent ideal of $\mathcal{L}$ and so $B_0 \subseteq \mathcal{N}(\mathcal{L}) \cap B = 0$. Hence $B = B_1$ is toral. It is clear that $B_1 + Z(\mathcal{L})_1$ is a maximal toral subalgebra of $\mathcal{L}$.

Finally, let T be any maximal torus of $\mathcal{L}$, and let $\mathcal{C} = Z_{\mathcal{L}}(T)$. Then $\mathcal{C}$ is a Cartan subalgebra of $\mathcal{L}$, (by Theorem 4.5.17 of [10]) and $\mathcal{L} = \mathcal{L}^{(1)} \dot{+} \mathcal{C}$ as above. Clearly we can write $\mathcal{C} = \mathcal{C}_0 \oplus T$. But now $\mathcal{L}^{(1)} + \mathcal{C}_0$ is a nilpotent ideal of $\mathcal{L}$, and so $\mathcal{C}_0 \subseteq \mathcal{N}(\mathcal{L}) \cap \mathcal{C} = Z(\mathcal{L})$, making T a p-complement of $\mathcal{L}^{(1)} \oplus Z(\mathcal{L})_0$. □

The condition "$\mathrm{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L})$" in (*iii*) Theorem 2.4. cannot be weakened to "$Z(\mathcal{L}) \subseteq \mathrm{Sap}(\mathcal{L})$", as the following example proves.

Example 1. We know which are the Lie algebras of dimension 2 over $\mathbb{K}$ and we take $\mathcal{L} = I + V$, where

$$I = \mathbb{K}a + \mathbb{K}b,\ V = \mathbb{K}v_1 + \mathbb{K}v_2,$$
$$v_1^p = v_2^p = b^p = 0,\ a^p = 0,$$
$$[V,V] = 0,\ [a,b] = 0,\ [a,v_1] = v_1,\ [a,v_2] = v_2,\ [b,v_1] = v_2,\ [b,v_2] = 0.$$

Then $\mathcal{L}^{(1)} = V$ is abelian, $\left(\mathcal{L}^{(1)}\right)^p = 0, Z(\mathcal{L}) = 0$. Now

$$\mathcal{N}(\mathcal{L}) = \mathbb{K}b + \mathbb{K}v_1 + \mathbb{K}v_2.$$

Also $\mathbb{K}v_2$ is a maximal p-ideal. Let J be a minimal p-ideal contained in $\mathcal{N}(\mathcal{L})$. Since $[\mathcal{N}(\mathcal{L}), \mathcal{N}(\mathcal{L})] = \mathbb{K}v_2$, either $J = \mathbb{K}v_2$ or $[\mathcal{N}(\mathcal{L}), J] = 0$. Suppose that $J \neq \mathbb{K}v_2$. Then $[b, J] = 0$ so $J \subseteq \mathbb{K}b + \mathbb{K}v_2$, and $[v_1, J] = 0$ so $J \subseteq \mathbb{K}v_1 + \mathbb{K}v_2$. Thus $J \subseteq \mathbb{K}v_2$, a contradiction. Hence $\mathcal{N}(\mathcal{L}) \neq \mathrm{Sap}(\mathcal{L})$.

E. L. Stitzinger has shown that, for any Lie algebra $\mathcal{L}$ over the arbitrary field $\mathbb{K}$, such that $\mathcal{L}^{(1)}$ is nilpotent, $\mathcal{L}$ is $\mathcal{F}$-free (that is $\mathcal{F}(\mathcal{L}) = 0$) if and only if each subalgebra of $\mathcal{L}$ is $\mathcal{F}$-free.

The complete analogue of this result does not hold if $\mathcal{F}(\mathcal{L})$ is replaced by $\mathcal{F}_p(\mathcal{L})$, as the following example proves.

Example 2. Let $\mathcal{L} = \mathbb{K}a + \mathbb{K}b + \mathbb{K}v_1 + \mathbb{K}v_2$ with $\mathbb{K} = Z_2$,

$$a^2 = a, b^2 = a + b, [a, v_1] = v_1, [a, v_2] = v_2, [b, v_1] = v_2, [b, v_2] = v_1 + v_2,$$
$$[a, b] = [v_1, v_2] = 0, v_1^2 = v_2^2 = 0,$$

and $I = \mathbb{K}a + \mathbb{K}b$. We get $\mathcal{F}_p(\mathcal{L}) = 0$ where as $\mathcal{F}_p(I) = \mathbb{K}a$.

However some partial results can be obtained.

Theorem 2.6. *Let $\mathcal{L}$ be a finite-dimensional p-Lie algebra. Then the following statements are equivalent:*

(i) $\mathcal{L}^{(1)}$ is nilpotent and $\mathcal{F}_p(\mathcal{L}) = 0$.

(ii) $\mathcal{L} = I \dot{+} B$ where B is an abelian subalgebra, I is an abelian p-ideal, the (adjoint) action of B on I is faithful and completely reducible, $Z(\mathcal{L})$ is completely reducible under the p-map, and the p-map is trivial on $[B, I]$.

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.4, $\mathcal{L} = I \dot{+} B$, where

$$I = \mathrm{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n,$$

with I_i is a minimal p-ideal of $\mathcal{L}$, for $i = 1, 2, \ldots, n$, and B is a p-subalgebra of $\mathcal{L}$. Now $Z_B(I) = \{x \in B | [x, B] = 0\}$ is an ideal in the solvable Lie algebra $\mathcal{L}$. If $Z_B(I) \neq 0$, it follows that

$$0 \neq Z_B(I) \cap \mathrm{Sap}(\mathcal{L}) \subseteq B \cap I = 0,$$

which is a contradiction. Hence $Z_B(I) = 0$ and the action of B is faithful.

Now suppose that $I_i \nsubseteq Z(\mathcal{L})$. Then $I_i \cap Z(\mathcal{L}) \subset I_i$ and so, as $I_i \cap Z(\mathcal{L})$ is a p-ideal, $I_i \cap Z(\mathcal{L}) = 0$. If $a \in I_i$ then $(ada)^p = 0$, and so $ada^p = 0$,hence $a^p \in Z(\mathcal{L})$. Thus, $a^p \in I_i \cap Z(\mathcal{L}) = 0$, and the minimality of I_i implies that I_i is an irreducible B-module but, of course, if $I_i \subseteq Z(\mathcal{L})$ then I_i is a completely reducible B-module, so $I = I_1 \oplus \cdots \oplus I_n$ is a completely reducible B-module.

Now $\mathcal{L}^{(1)}$ is nilpotent, therefore $\mathrm{ad}\, x$ is nilpotent, for every $x \in B^{(1)}$. It follows from Engel's Theorem that $[B^{(1)}, I_i] \subset I_i$ for every $i = 1, 2, \ldots, n$. If $I_i \nsubseteq Z(\mathcal{L})$, this implies that $[B^{(1)}, I_i] = 0$, since I_i is an irreducible B-module. If $I_i \subseteq Z(\mathcal{L})$ then, clearly, $[B^{(1)}, I_i] = 0$ also. Thus $[B^{(1)}, I_i] = 0$, and so $B^{(1)} = 0$, as $Z_B(I) = 0$. Moreover, $Z(\mathcal{L}) \subseteq I$, since $Z_B(I) = 0$. If $a \in Z(\mathcal{L})$ and $a = a_1 + \cdots + a_n$, with $a_i \in I_i$, then $[x, a_1] + \cdots + [x, a_n] = 0$, for all $x \in \mathcal{L}$, so each $a_i \in Z(\mathcal{L})$. Hence $Z(\mathcal{L}) = \Sigma I_i$, where the sum is over all I_i contained in $Z(\mathcal{L})$. Since each$I_i \subseteq Z(\mathcal{L})$ is a minimal p-ideal, $Z(\mathcal{L})$ must be irreducible under the p-map.

(ii)$\Rightarrow$ (i). In view of Theorem 4.1. of [5], it suffices to show that $I = \mathrm{Sap}(\mathcal{L})$. Now we have $I = [B, I] \oplus Z(\mathcal{L})$, $[B, I]$ is a direct sum of irreducible B-modules (each of which is a minimal p-ideal) and $Z(\mathcal{L})$ is a direct sum of irreducible subspaces for the p-map (each of which is a minimal p-ideal). Thus, $I \subseteq \mathrm{Sap}(\mathcal{L})$. But, as B acts faithfully on $\mathcal{L}, I$ is a maximal abelian ideal. Hence $I = \mathrm{Sap}\,\mathcal{L}$, as required. □

Corollary 2.7. *Let $\mathcal{L}$ be a finite dimensional Lie p-algebra with $\mathcal{L}^{(1)}$ nilpotent and $\mathcal{F}_p(\mathcal{L}) = 0$. Let P be a p-subalgebra of $\mathcal{L}$ containing* $\mathrm{Sap}(\mathcal{L})$. *Then $\mathcal{F}_p(P) = 0$.*

Proof. Write $\mathcal{L} = I \dot{+} B$ as in Theorem 2.4 (ii). Then $P = I \dot{+} (B \cap P)$ since $I = \mathrm{Sap}(\mathcal{L}) \subseteq P$. Now B acts completely reducibly on $[B, I]$, and hence so does $B \cap P$. It follows that $B \cap P$ acts completely reducibly on $[B \cap P, I]$. Moreover, $Z(P) = Z(\mathcal{L}) \oplus Z_{[B,I]}(B \cap P)$ and the p-map is trivial on $[B, I]$, so that $Z(P)$ is completely reducible under the p-map. The result now follows from Theorem 2.4. □

Corollary 2.8. *Let $\mathcal{L}$ be a finite dimensional Lie p-algebra such that $\mathcal{L}^{(1)}$ is nilpotent and $\mathcal{F}_p(\mathcal{L}) = 0$. If J is an ideal of $\mathcal{L}$, then* $\mathrm{Sap}(J) = 0$.

Proof. It suffices to show this for maximal ideals. By Corollary 2.5, we may assume that $I_1 \nsubseteq J$, where $\mathrm{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n$, with $I_1, \ldots, I_n$ minimal abelian p-ideals. Then $\mathcal{L} = J + I_1$, since J is maximal, and $J \cap I_1 = 0$. Thus $\mathcal{L} = J \oplus I_1$, $J \cong \mathcal{L}/I_1 \cong B \dot{+} (I_2 \oplus \cdots \oplus I_n)$, and $I_1 \subseteq Z(\mathcal{L})$. Hence $Z_B(I_2 \oplus \cdots \oplus I_n) = Z_B(I) = 0$, and it is clear that all of the conditions of Theorem 2.4 (ii) hold. □

Corollary 2.9. *If $\mathcal{L}$ is an abelian finite dimensional Lie p-algebra, then $\mathcal{F}_p(\mathcal{L}) = 0$, if and only if $\mathcal{L}$ is completely reducible under the p-map.*

Proof. This statement can be proved by using Theorem 2.4 and the fact $B = 0$ and $\mathcal{L} = Z(\mathcal{L})$. □

Corollary 2.10. *Let $\mathcal{L}$ be a finite dimensional Lie p-algebra such that $\mathcal{L} = \mathrm{Sap}(\mathcal{L}) \dot{+} B$ and that the conditions of Theorem 2.4 (ii) are satisfied. Assume in addition that B is completely reducible under the p-map; that is $\mathrm{Sap}(B) = B$. Then if P is any p-subalgebra of $\mathcal{L}, P = \mathrm{Sap}\, P \dot{+} B'$, the conditions of Theorem 2.4. (ii) are satisfied and B' is completely reducible under the p-map.*

Proof. If $\mathrm{Sap}(\mathcal{L}) \subseteq P$, then $\mathrm{Sap}(P) = \mathrm{Sap}(\mathcal{L})$, and taking $B' = B \cap P$, we get the result.

It suffices to prove the Corollary for maximal p-subalgebras. So assume that P is maximal and that $I_1 \nsubseteq P$, where $\mathrm{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n$, with $I_1, \ldots, I_n$ minimal abelian p-ideals. Then $\mathcal{L} = I_1 + P$, with $P \cap I_1 = 0$. Hence $P \cong B \dot{+} (I_2 \oplus \cdots \oplus I_n)$. As B is completely reducible under the p-map, we have

$$B = B' \oplus Z_B(I_2 \oplus \cdots \oplus I_n).$$

Then $\mathrm{Sap}(P) = Z_B(I_2 \oplus \cdots \oplus I_n) \oplus I_2 \oplus \cdots \oplus I_n, P = \mathrm{Sap}(P) \dot{+} B^{'}$, the conditions of Theorem 2.4 (ii) are satisfied and $B^{'}$ is completely reducible under the p-map. □

Definition 2.11. A finite dimensional Lie p-algebra $\mathcal{L}$ is called *p-elementary*, if $\mathcal{F}_p(P) = 0$ for every p-subalgebra P of $\mathcal{L}$.

Corollary 2.12. *Assume $\mathcal{L}^{(1)}$ is a finite dimensional Lie p-algebra with nilpotent $\mathcal{L}^{(1)}$ and $\mathcal{F}_p(\mathcal{L}) = 0$. Let $\mathcal{L} = \mathrm{Sap}(\mathcal{L}) \dot{+} B$ as in Theorem 2.4 (ii). Then $\mathcal{L}$ is p-elementary, if and only if $B = \mathrm{Sap}(B)$*

Proof. As $\mathcal{F}_p(\mathcal{L}) = 0$ and $\mathcal{L} = \mathrm{Sap}(\mathcal{L}) \dot{+} B$ (Theorem 2.4. (ii)), then B has a faithful completely reducible representation on $\mathrm{Sap}(\mathcal{L})$. This is equivalent to the fact that B has a non-zero nilideals as in [7]. Since B is abelian, this is equivalent to the injectivity of the p-map. Since $\mathbb{K}$ is algebraically closed, this is equivalent to $\mathrm{Sap}(B) = B$ as in [4]. It follows from Corollary 2.14 that $\mathcal{L}$ is p-elementary. The converse is immediate from the definition. □

The result above cannot be extended to the case when $\mathbb{K}$ is a perfect field. Let us see the following example.

Example 3. Let $\mathcal{L}$ be any abelian Lie p-algebra for which the p-map is injective but $\mathcal{L}$ is not completely reducible under the p-map. Then $\mathcal{L}$ has a faithful completely reducible module B. Make B into an abelian Lie p-algebra with trivial p-map. Then $\mathcal{F}_p(B \dot{+} \mathcal{L}) = 0$, but $\mathcal{F}_p(\mathcal{L}) \neq 0$.

Now, if $\mathbb{K}$ is not perfect, let $\lambda \in \mathbb{K} \backslash \mathbb{K}^{\,p}$and take $\mathcal{L} = \mathbb{K}a + \mathbb{K}b$, with $a^p = a, b^p = \lambda a$. If $\lambda \in \mathbb{K}$ and $\mu^p - \mu + \lambda = 0$ has no solution in $\mathbb{K}$, take $\mathcal{L} = \mathbb{K}a + \mathbb{K}b$ with $a^p = a, b^p = b + \lambda a$. Here we may take B to be$p$-dimensional with a represented by the identity matrix and b represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -\lambda & 1 & 0 & \dots & 0 \end{pmatrix}$$

(the companion matrix of $\mu^p - \mu + \lambda$). If $\mathbb{K} = Z_p$ we may take $\lambda = -1$.

Putting $p = 2$, we get the example 2.7.

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