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THE BEST CONSTANT FOR CARLEMAN'S INEQUALITY OF FINITE TYPE

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Dedicated to Mr. Bi-Fan Li on the occasion of his 53rd birthday.

Abstract. In this short note, we consider the best constant for Carleman's inequality of finite type by means of weight coefficient and nonlinear algebraic equation system. The result presented here give a part of answer this problem.

1. Introduction and Main Result

The following Carleman's inequality (see [3, 9]) is well-known:

$$\sum_{k=1}^{\infty}(a_1a_2\cdots a_k)^{\frac{1}{k}} < e\sum_{k=1}^{\infty}a_k, \tag{1.1}$$

where

$$a_k \geqq 0 \quad \text{and} \quad 0 < \sum_{k=1}^{\infty}a_k < \infty.$$

For some recent investigations of Carleman's inequality, see (for example) the works by Alzer [1, 2], Yang and Debnath [15], Yan and Sun [14], Li [11], Yang [17, 16], Yuan [18], Chen [6], Duncan and McGregor [8], Chen *et al.* [4], Chen and Qi [5], Yue [19] and Liu and Zhu [12].

The finite type of (1.1) is

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{\frac{1}{k}} < e\sum_{k=1}^{n}a_k. \tag{1.2}$$

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We know that the coefficient e of (1.1) is the best possible. However, in (1.2), the coefficient e is not the best possible one. In 1963, de Brujin [7] improved on e with asymptotic methods in analysis as follows:

$$C_n = e - \frac{2\pi^2 e}{(\ln n)^2} + O\left(\frac{1}{(\ln n)^3}\right) \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

In a recent paper, Johansson *et al.* [10] improved on e to $e^{1-\frac{1}{n}}$. In this short note, we shall refine $e^{1-\frac{1}{n}}$ in our following main result.

Theorem 1. *The best constant C_n for the following inequality:*

$$\sum_{k=1}^{n} (a_1 a_2 \cdots a_k)^{\frac{1}{k}} \leqq C_n \sum_{k=1}^{n} a_k \quad (n \in \mathbb{N}) \tag{1.3}$$

is the solution of the nonlinear algebraic equation:

$$\begin{cases} 1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_{n-1}}{n-1} + \frac{x_n}{n} = C_n, \\ \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_{n-1}}{n-1} + \frac{x_n}{n} = C_n x_2^2, \\ \qquad \vdots \\ \frac{x_{n-1}}{n-1} + \frac{x_n}{n} = C_n \cdot \frac{x_{n-1}^{n-1}}{x_{n-2}^{n-2}}, \\ \frac{x_n}{n} = C_n \cdot \frac{x_n^n}{x_{n-1}^{n-1}}, \end{cases} \tag{1.4}$$

or the ratiocinate equation system:

$$\begin{cases} y_0 = C_n, \\ y_{n-1} = \frac{1}{n}, \\ y_{i-1} - \left(\frac{y_i}{C_n}\right)^{\frac{1}{i}} y_i = \frac{1}{i}, \end{cases} \tag{1.5}$$

where $1 \leqq i \leqq n - 1$.

2. Proof of Theorem 1

By applying AM–GM inequality, we can easily obtain

$$\begin{aligned} \sum_{k=1}^{n} (a_1 a_2 \cdots a_k)^{\frac{1}{k}} &= \sum_{k=1}^{n} \left(\frac{(\lambda_1 a_1)(\lambda_2 a_2) \cdots (\lambda_k a_k)}{\lambda_1 \lambda_2 \cdots \lambda_k}\right)^{\frac{1}{k}} \\ &\leqq \sum_{k=1}^{n} \frac{1}{(\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{k}}} \left(\frac{1}{k} \sum_{j=1}^{k} \lambda_j a_j\right) \\ &= \sum_{i=1}^{n} \left(\sum_{k=i}^{n} \frac{\lambda_i}{k(\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{k}}} a_i\right) \\ &= C_n \sum_{k=1}^{n} a_k. \end{aligned} \tag{2.1}$$

It follows from the last two expressions of (2.1) that

$$
\begin{cases}
1+\frac{\lambda_1}{2(\lambda_1\lambda_2)^{\frac{1}{2}}}+\frac{\lambda_1}{3(\lambda_1\lambda_2\lambda_3)^{\frac{1}{3}}}+\cdots \\
\qquad +\frac{\lambda_1}{(n-1)(\lambda_1\lambda_2\cdots\lambda_{n-1})^{\frac{1}{n-1}}}+\frac{\lambda_1}{n(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}}}=C_n, \\
\frac{\lambda_2}{2(\lambda_1\lambda_2)^{\frac{1}{2}}}+\frac{\lambda_2}{3(\lambda_1\lambda_2\lambda_3)^{\frac{1}{3}}}+\cdots \\
\qquad +\frac{\lambda_2}{(n-1)(\lambda_1\lambda_2\cdots\lambda_{n-1})^{\frac{1}{n-1}}}+\frac{\lambda_2}{n(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}}}=C_n, \\
\qquad \vdots \\
\frac{\lambda_{n-1}}{(n-1)(\lambda_1\lambda_2\cdots\lambda_{n-1})^{\frac{1}{n-1}}}+\frac{\lambda_{n-1}}{n(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}}}=C_n, \\
\frac{\lambda_n}{n(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}}}=C_n.
\end{cases}
\tag{2.2}
$$

Now, we set

$$\frac{1}{\lambda_1}=x_1, \quad \frac{1}{(\lambda_1\lambda_2)^{\frac{1}{2}}}=x_2, \quad \cdots, \quad \frac{1}{(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}}}=x_n,$$

then

$$\frac{1}{\lambda_1}=x_1, \quad \frac{1}{\lambda_2}=\frac{x_2^2}{x_1}, \quad \cdots, \quad \frac{1}{\lambda_n}=\frac{x_n^n}{x_{n-1}^{n-1}}.$$

We also know that (2.2) can be rewritten as the following nonlinear algebraic equation system:

$$
\begin{cases}
x_1+\frac{x_2}{2}+\frac{x_3}{3}+\cdots+\frac{x_{n-1}}{n-1}+\frac{x_n}{n}=C_nx_1, \\
\frac{x_2}{2}+\frac{x_3}{3}+\cdots+\frac{x_{n-1}}{n-1}+\frac{x_n}{n}=C_n\cdot\frac{x_2^2}{x_1}, \\
\qquad \vdots \\
\frac{x_{n-1}}{n-1}+\frac{x_n}{n}=C_n\cdot\frac{x_{n-1}^{n-1}}{x_{n-2}^{n-2}}, \\
\frac{x_n}{n}=C_n\cdot\frac{x_n^n}{x_{n-1}^{n-1}}.
\end{cases}
\tag{2.3}
$$

Since the nonlinear algebraic equation system (2.3) is homogeneous for x_i ($1 \leqq i \leqq n$), we can set $x_1 = 1$, hence, (2.3) reduces to (1.4).

If we set

$$y_{i-1}=C_n\left(\frac{x_i}{x_{i-1}}\right)^{i-1} \quad \text{and} \quad y_0=C_n,$$

we know that the nonlinear algebraic equation system (2.3) can be written as (1.5).

3. Remarks and Observations

Remark 2. When $n = 2$, the nonlinear algebraic equation system (1.4) reduces to

$$
\begin{cases}
1+\frac{x_2}{2}=C_2, \\
\frac{x_2}{2}=C_2x_2^2.
\end{cases}
$$

It's easy to get

$$C_2 = \frac{\sqrt{2}+1}{2}.$$

Remark 3. When $n = 3$, the nonlinear algebraic equation system (1.4) becomes

$$\begin{cases} 1 + \frac{x_2}{2} + \frac{x_3}{3} = C_3, \\ \frac{x_2}{2} + \frac{x_3}{3} = C_3 x_2^2, \\ \frac{x_3}{3} = C_3 \frac{x_3^3}{x_2^2}. \end{cases} \tag{3.1}$$

For

$$x_2 > 0 \quad \text{and} \quad x_3 > 0,$$

we know that (3.1) can be written as follows:

$$\begin{cases} 1 + \frac{x_2}{2} + \frac{x_3}{3} - C_3 = 0, \\ \frac{x_2}{2} + \frac{x_3}{3} - C_3 x_2^2 = 0, \\ \frac{x_2^2}{3} - C_3 x_3^2 = 0. \end{cases} \tag{3.2}$$

By applying Wu's method (see [13]), we find that the solutions of (3.2) are the union of the solutions of the following nonlinear algebraic equation systems:

$$\begin{cases} x_2 = 0, \\ x_3 = 0, \\ C_3 - 1 = 0, \end{cases} \qquad \begin{cases} 2x_2 - 1 = 0, \\ 4x_3 - 1 = 0, \\ 3C_3 - 4 = 0, \end{cases}$$

and

$$\begin{cases} 108C_3^3 - 108C_3^2 - 108C_3^2 x_2 + 27C_3 - 4 = 0, \\ 108C_3^3 - 108C_3^2 - 72C_3^2 x_3 - 27C_3 + 4 = 0, \\ 3888C_3^5 - 2592C_3^4 - 1512C_3^3 - 360C_3^2 + 51C_3 - 4 = 0. \end{cases}$$

It's not difficult to find that only the following equation system satisfies our restricted conditions on x_k $(k = 2, 3)$ and C_3,

$$\begin{cases} 2x_2 - 1 = 0, \\ 4x_3 - 1 = 0, \\ 3C_3 - 4 = 0. \end{cases}$$

Thus, we get

$$C_3 = \frac{4}{3}.$$

Remark 4. When $n = 4$, the nonlinear algebraic equation system (1.4) can be written as follows:

$$\begin{cases} 1 + \frac{x_2}{2} + \frac{x_3}{3} + \frac{x_4}{4} = C_4, \\ \frac{x_2}{2} + \frac{x_3}{3} + \frac{x_4}{4} = C_4 x_2^2, \\ \frac{x_3}{3} + \frac{x_4}{4} = C_4 \frac{x_3^3}{x_2^2}, \\ \frac{x_4}{4} = C_4 \frac{x_4^4}{x_3^3}. \end{cases} \tag{3.3}$$

By similarly applying the method of Remark 3 and using (1.5), we know that C_4 in (3.3) is the largest positive root of the following equation:

$$
\begin{aligned}
&109049173118505959030784\, C_4^{24} - 654295038711035754184704\, C_4^{23} \\
&\quad + 1472163837099830446915584\, C_4^{22} - 1387347813563214701002752\, C_4^{21} \\
&\quad + 220843507713085418766336\, C_4^{20} + 361130725214496730644480\, C_4^{19} \\
&\quad + 187384441880508884919296\, C_4^{18} - 149735761790067869220864\, C_4^{17} \\
&\quad - 20033038006659651207168\, C_4^{16} + 14417509185682352898048\, C_4^{15} \\
&\quad + 16905530303693690241024\, C_4^{14} - 2098418839125516877824\, C_4^{13} \\
&\quad - 198705178996352483328\, C_4^{12} + 427447433656163893248\, C_4^{11} \\
&\quad + 41447678188009291776\, C_4^{10} - 2629784260986273792\, C_4^{9} \\
&\quad + 660475521813381120\, C_4^{8} + 342213608420278272\, C_4^{7} \\
&\quad + 42624005978423296\, C_4^{6} - 201976270848000\, C_4^{5} \\
&\quad + 274965186525696\, C_4^{4} + 12841816536576\, C_4^{3} \\
&\quad + 373658292864\, C_4^{2} + 22039921152\, C_4 + 387420489 = 0.
\end{aligned} \tag{3.4}
$$

Therefore, we find from (3.4) that

$$C_4 \approx 1.420844385.$$

Remark 5. When $n \geqq 5$, we fail to obtain C_n is one of the roots of certain algebraic equation. But in view of (1.5) and the numerical method of equation (with the function `fsolve()` in mathematical software Maple 10), we can get the following approximate results for $5 \leqq n \leqq 12$:

n	C_n	n	C_n
5	1.486353229	9	1.645509523
6	1.537937557	10	1.671759812
7	1.580037211	11	1.694891445
8	1.615322400	12	1.715500223

With the aid of the numerical method of equation, we also can get the approximate results of C_n ($n \geqq 13$). Since the computations are too complex, we here choose to omit the details.

Remark 6. Clearly, our results of C_n ($2 \leqq n \leqq 12$) are improvements of the corresponding results obtained by Johansson *et al.* [10].

Finally, by virtue of the results obtained by de Brujin [7], we know that C_n ($n \in \mathbb{N}$) in Theorem 1 are monotonous and bounded for $n \in \mathbb{N}$. Here, we pose the following problem.

Problem 7. What are the best $f(n)$ and $g(n)$ in the following inequality?

$$f(n) \leqq C_{n+1} - C_n \leqq g(n) \qquad (n \in \mathbb{N}),$$

where C_n and C_{n+1} are given by Theorem 1.

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