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CLASSIFICATION OF RANDERS METRICS OF SCALAR FLAG CURVATURE

XINYUE CHENG AND ZHONGMIN SHEN

Abstract. This is a survey article about the recent developments in classifying Randers metrics of scalar flag curvature under an additional condition on the isotropic S-curvature. The authors give an outline of the proof for the classification theorem.

1. Introduction

A Randers metric on a manifold M is a Finsler metric defined in the following form:

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

Randers metrics were first introduced by physicist G. Randers in 1941 from the standpoint of general relativity. Later on, these metrics were applied to the theory of electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics.

Randers metrics also arise naturally from the navigation problem on a Riemannian space (M, h) under the influence of an external force field W [17]. It is shown that least time paths are geodesics of a Randers metric $F = \alpha + \beta$ determined by

$$h\Big(x,\ \frac{y}{F} - W_x\Big) = 1. \tag{1}$$

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Akbar-Zadeh's famous rigidity theorem says that every Finsler metric of negative constant flag curvature on a closed manifold must be a Riemannian metric. If "constant flag curvature" is changed to "scalar flag curvature", we have the following rigidity theorem, namely, every Finsler metric of negative scalar flag curvature on a closed manifold of dimension $n \geq 3$ must be a Randers metric [11]. This leads to the study of Randers metrics of scalar flag curvature.

The S-curvature plays a very important role in Finsler geometry (cf. [15, 19]). It is known that, for a Finsler metric $F = F(x, y)$ of scalar flag curvature, if the S-curvature is isotropic with $\mathbf{S} = (n+1)c(x)F$, then the flag curvature must be in the following form

$$\mathbf{K} = \frac{3\tilde{c}_{x^m} y^m}{F} + \sigma, \tag{2}$$

where $\sigma = \sigma(x)$ and $\tilde{c} = \tilde{c}(x)$ are scalar functions with $c - \tilde{c} = constant$ [5]. This leads to the study of Finsler metrics of scalar flag curvature with isotropic S-curvature. In this paper, our goal is to give an outline of the classification theorem on the Randers metrics of scalar flag curvature with isotropic S-curvature. Our main theorem is Theorem 5.3 (see section 5).

Hilbert's Fourth Problem is to characterize the distance functions on an open subset $\mathcal{U}$ in R^n such that geodesics are straight lines. A Finsler metric F is said to be *projectively flat* if it is a smooth solution of Hilbert's Fourth Problem. Projectively flat Finsler metrics on $\mathcal{U}$ can be characterized by the following equations:

$$G^i = P(x, y)y^i,$$

where $P(x, \lambda y) = \lambda P(x, y), \forall \lambda > 0$. It is easy to show that any projectively flat metric $F = F(x, y)$ is of scalar flag curvature. Moreover, the flag curvature is given by

$$\mathbf{K} = \frac{P^2 - P_{x^m} y^m}{F^2}.$$

The Beltrami theorem says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Nevertheless, examples show that this is no longer true for Finsler metrics. This leads to the study of projectively flat Finsler metrics with isotropic S-curvature.

2. Definitions and Notations

A *Finsler metric* on a manifold M is a continuous function $F \colon TM \to [0, \infty)$ satisfying the following conditions:

(1) *Regularity*: F is smooth on $TM \backslash \{0\}$.

(2) *Positive homogeneity:* $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0$.

(3) *Strong convexity:* the fundamental tensor $g_{ij}(x, y)$ is positive definite for all $(x, y) \in TM \backslash \{0\}$, where $g_{ij}(x, y) := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x, y)$.

For each vector $y \in T_xM$, we have an inner product $g_y = g_{ij} dx^i \otimes dx^j$ on T_xM.

The geodesics are characterized by the following equations in local coordinates

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where

$$G^i = \frac{1}{4} g^{il}\big\{ \big[F^2\big]_{x^m y^l} y^m - \big[F^2\big]_{x^l} \big\}.$$

The local functions $G^i = G^i(x, y)$ are called the *geodesic coefficients.*

The *Riemann curvature* $\mathbf{R}_y := R^i{}_k dx^k \bigotimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{3}$$

For a flag $P = span\{y, u\} \subset T_xM$ with flagpole y, the *flag curvature* $\mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{g_y(u, \mathbf{R}_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}. \tag{4}$$

A Finsler metric F is said to be of *scalar flag curvature* if the flag curvature $\mathbf{K}(P, y) = \sigma(x, y)$ is a scalar function on $TM\backslash\{0\}$. It is said to be of *constant flag curvature* if $\mathbf{K}(P, y) = constant$. At every point, $\mathbf{K}(P, y) = \sigma(x, y)$ if and only if

$$R^i{}_k = \sigma F^2\{\delta^i_k - F^{-2} g_{kj} y^j y^i\}. \tag{5}$$

Let

$$\mathbf{Ric} := R^m_{\ m}. \tag{6}$$

$\mathbf{Ric}$ is a well-defined scalar function on $TM\backslash\{0\}$. We call $\mathbf{Ric}$ the *Ricci curvature.*

In Finsler geometry, there are two important non-Riemannian geometric quantities. Recall the Busemann-Hausdorff volume form $dV = \sigma_F(x)dx^1 \cdots dx^n$ which is given by

$$\sigma_F(x) := \frac{Vol(B^n)}{Vol\,\{(y^i) \in R^n | F(x, y) < 1\}}.$$

The first non-Riemannian quantity is the *distortion* defined by

$$\tau(x, y) := \ln\left[\frac{\sqrt{det(g_{ij}(x, y))}}{\sigma_F(x)}\right].$$

It is shown that F is Riemannian if and only if $\tau = 0$ [15]. Thus the distortion τ measures the non-Euclidean property of the Minkowski space (T_xM, F_x).

The second non-Riemannian quantity is he so-called S-curvature defined by

$$\mathbf{S} := \tau_{|m} y^m,$$

where "|" denotes the horizontal covariant derivative with respect to any Finsler connection (such as Berwald connection, Chern connection, etc.). In local coordinates, the S-curvature can be expressed by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m}(x,y) - y^m \frac{\partial}{\partial x^m}\big(\ln \sigma_F(x)\big). \tag{7}$$

(cf. [15, 19]). An important fact is that, for any Berwald metric, the S-curvature vanishes, $\mathbf{S} = 0$ [14, 15]. The S-curvature $\mathbf{S} = S(x,y)$ was first introduced by the second author when he studied volume comparison in Riemann-Finsler geometry [14]. He also proved that the S-curvature and the Ricci curvature determine the local behavior of the Busemann-Hausdorff measure of small metric balls around a point [16].

We say that the S-curvature is *isotropic* if there exists a scalar function $c = c(x)$ on M such that

$$\mathbf{S} = (n+1)cF. \tag{8}$$

If $c(x) = constant$, we say that F is of constant S-curvature.

3. Randers Metrics

We now discuss Randers metrics on an n-dimensional manifold M. Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric $\beta = b_iy^i$ be a 1-form on M with $\|\beta_x\|_\alpha < 1$. Then $F(x,y) := \alpha(x,y) + \beta(x,y)$ is a Finsler metric. The volume form dV_F of F is given by

$$dV_F = e^{(n+1)\rho(x)}dV_\alpha,$$

where dV_α is the volume form of α and

$$\rho(x) := \ln\sqrt{1 - \|\beta_x\|^2_\alpha}.$$

Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^{\,j},$$

where "|" denotes the covariant derivative with respect to α. Let

$$r_{ij} := \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \quad s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right), \quad s^i{}_j := a^{ih}s_{hj},$$

$$s_j := b^is_{ij}, \quad r_j := b^ir_{ij}, \quad e_{ij} := r_{ij} + b_is_j + b_js_i.$$

The S-curvature is given by

$$\mathbf{S} = (n+1)\Big\{\frac{e_{00}}{2F} - (s_0 + \rho_0)\Big\},$$

where $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$ and $\rho_0 := \rho_{x^i}(x)y^i$. See [15][19]. We have the following

Lemma 3.1 ([6])**.** *Let $F = \alpha + \beta$ be a Randers metric on a manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent:*

(a) *F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$;*

(b) *α and β satisfy that* $e_{00} = 2c(\alpha^2 - \beta^2)$, *i.e.*

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j).$$

Every Randers metric $F = \alpha + \beta$ with $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ can be described as a solution to the following equation:

$$h\Big(x,\ \frac{y}{F} - W_x\Big) = 1, \tag{9}$$

where $h(x,y) = \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric and $W = W^i(x)\frac{\partial}{\partial x^i}$ is a vector field with $\|W_x\|_h = h(x, W_x) < 1$. The relationship between (α, β) and (h, W) are given below.

$$a_{ij} = \frac{(1 - \|W\|^2)h_{ij} + W_i W_j}{(1 - \|W\|^2)^2}, \qquad b_i = -\frac{W_i}{1 - \|W\|^2}.$$

$$h_{ij} = (1 - \|\beta\|^2)(a_{ij} - b_i b_j), \quad W^i = -\frac{b^i}{1 - \|\beta\|^2},$$

where $W_i := h_{ij}W^j$ and $b^i := a^{ij}b_j$. Moreover,

$$\|W_x\|_h^2 := h_{ij}W^i W^j = a^{ij}b_i b_j =: \|\beta_x\|_\alpha^2,$$

Zermelo's navigation problem is to determine shortest time paths on a Riemannian manifold (M, h) with external force W. It turns out that the shortest paths are the geodesics of the Randers metric $F = \alpha + \beta$ determined by (9) ([4, 17]). We call (h, W) the navigation represention of F. One can study the geometry of a Randers metric $F = \alpha+\beta$ via its navigation representation (h, W).

Let

$$\mathcal{R}_{ij} := \frac{1}{2}\left(W_{i;j} + W_{j;i}\right), \quad \mathcal{S}_{ij} = \frac{1}{2}\left(W_{i;j} - W_{j;i}\right), \quad \mathcal{S}^i{}_j := h^{ir}\mathcal{S}_{rj},$$

$$\mathcal{R}_j := W^i\mathcal{R}_{ij}, \quad \mathcal{S}_j := W_i\mathcal{S}^i{}_j = W^i\mathcal{S}_{ij}, \quad \mathcal{R} := \mathcal{R}_j W^j,$$

where ";" denotes the covariant derivative with respect to h.

Lemma 3.2 ([7]). *Let $F = \alpha + \beta$ be a Randers metric on a manifold M, which is expressed in terms of a Riemannian metric h and a vector field W by (9). Then*

$$\mathbf{S} = \frac{n+1}{2F}\big\{2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}\big\}.$$

From Lemma 3.2, we can prove the following

Lemma 3.3 ([7, 22]). *Let $F = \alpha + \beta$ be a Randers metric on a manifold M, which is expressed in terms of a Riemannian metric h and a vector field W by (9). Then $\mathbf{S} = (n+1)cF$ if and only if $\mathcal{R}_{00} = -2ch^2$. In this case,*

$$G^i = \bar{G}^i - F\mathcal{S}^i{}_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i, \tag{10}$$

where $\bar{G}^i$ denote the geodesic coefficients of h.

Now, we are ready to study and characterize Randers metrics of scalar flag curvature with isotropic S-curvature.

4. Projectively Flat Randers Metrics with Isotropic S-Curvature

First recall a classification theorem.

Theorem 4.1 ([18]). *Let $F = \alpha + \beta$ be an n-dimensional Randers metric of constant Ricci curvature $\mathbf{Ric} = (n-1)\sigma F^2$ with $\beta \not\equiv 0$. Suppose that F is locally projectively flat. Then $\sigma \leq 0$. Further, if $\sigma = 0$, F is locally Minkowskian. If $\sigma = -1/4$, F can be expressed in the following form*

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2} \pm \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \ y \in T_xR^n, \tag{11}$$

where $a \in R^n$ is a constant vector with $|a| < 1$. The Randers metric in (11) has the following properties:

(a) $\mathbf{K} = -1/4$;
(b) $\mathbf{S} = \pm\frac{1}{2}(n+1)F$;
(c) *all geodesics of F are straight lines.*

Later on, D. Bao and C. Robles proved the following result: if a Randers metric F is Einstein with $\mathbf{Ric} = (n-1)\sigma(x)F^2$, then F is of constant S-curvature [2]. This leads to the study of projectively flat Randers metrics with isotropic S-curvature.

Let $F = \alpha + \beta$ be a locally projectively flat Randers metric. Then α is locally projectively flat and β is closed. According to the Beltrami theorem in Riemann geometry, α is locally projectively flat if and only if it is of constant sectional curvature. Thus we may assume that α of constant sectional curvature μ. It is locally isometric to the following standard metric α_μ on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$ or the whole R^n for $\mu = -1, 0, +1$:

$$\alpha_{-1}(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \quad y \in T_x\mathrm{B}^n \cong \mathrm{R}^n, \tag{12}$$

$$\alpha_0(x, y) = |y|, \quad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n, \tag{13}$$

$$\alpha_{+1}(x, y) = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + |x|^2}, \quad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{14}$$

Then we can determine β if $\mu + 4c(x)^2 \neq 0$,

$$\beta = -\frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2}.$$

On the other hand, we have

$$c_{i|j} = -c(\mu + 4c^2)a_{ij} + \frac{12cc_ic_j}{\mu + 4c^2}.$$

Now, we can solve the above equations for c and determine β and the flag curvature $\mathbf{K}$.

Theorem 4.2 ([5])**.** *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M and μ denote the constant sectional curvature of α. Suppose that the S-curvature is isotropic, $\mathbf{S} = (n+1)c(x)F$. Then F can be classified as follows.*

(A) *If $\mu + 4c(x)^2 \equiv 0$, then $c(x) = constant$ and $\mathbf{K} = -c^2 \leq 0$.*

(A1) *if $c = 0$, then F is locally Minkowskian with flag curvature $\mathbf{K} = 0$;*

(A2) *if $c \neq 0$, then after a normalization, F is locally isometric to the following Randers metric on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$,*

$$F(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} \pm \langle x,y\rangle}{1-|x|^2} \pm \frac{\langle a,y\rangle}{1+\langle a,x\rangle}, \tag{15}$$

where $a \in \mathrm{R}^n$ with $|a| < 1$, and the flag curvature of F is negative constant, $\mathbf{K} = -\frac{1}{4}$.

(B) *If $\mu + 4c(x)^2 \neq 0$, then F is given by*

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2} \tag{16}$$

and the flag curvature of F is given by

$$\mathbf{K} = \frac{3c_{x^k}(x)y^k}{F(x,y)} + 3c(x)^2 + \mu. \tag{17}$$

(B1) *when $\mu = -1$, $\alpha = \alpha_{-1}$ can be expressed in the form (12) on B^n. In this case,*

$$c(x) = \frac{\lambda + \langle a,x\rangle}{2\sqrt{(\lambda + \langle a,x\rangle)^2 \pm (1-|x|^2)}}, \tag{18}$$

where $\lambda \in \mathrm{R}$ and $a \in \mathrm{R}^n$ with $|a|^2 < \lambda^2 \pm 1$.

(B2) *when $\mu = 0$, $\alpha = \alpha_0$ can be expressed in the form (13) on R^n. In this case,*

$$c(x) = \frac{\pm 1}{2\sqrt{\kappa + \langle a,x\rangle + |x|^2}}, \tag{19}$$

where $\kappa > 0$ and $a \in \mathrm{R}^n$ with $|a|^2 < \kappa$.

(B3) *when $\mu = 1$, $\alpha = \alpha_{+1}$ can be expressed in the form (14) on R^n. In this case,*

$$c(x) = \frac{\epsilon + \langle a,x\rangle}{2\sqrt{1+|x|^2 - (\epsilon + \langle a,x\rangle)^2}}, \tag{20}$$

where $\epsilon \in \mathrm{R}$ and $a \in \mathrm{R}^n$ with $|\epsilon|^2 + |a|^2 < 1$.

Theorem 4.2 (A) follows from the classification theorem in [18] after we prove that the flag curvature is constant in this case. From Theorem 4.2 we obtain some interesting projectively flat Randers metrics with isotropic S-curvature.

Example 4.1. Let

$$F_{-}(x,y) = \frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}\sqrt{(1-|x|^2)+\lambda^2}+\lambda\langle x,y\rangle}{(1-|x|^2)\sqrt{(1-|x|^2)+\lambda^2}}, \; y \in T_x\mathrm{B}^n, \tag{21}$$

where $\lambda \in \mathrm{R}$ is an arbitrary constant. The geodesics of F_{-} are straight lines in B^n. One can easily verify that F_{-} is complete in the sense that every unit speed geodesic of F_{-} is defined on $(-\infty,\infty)$. Moreover F_{-} has strictly negative flag curvature $\mathbf{K} \le -\frac{1}{4}$.

Example 4.2. Let

$$F_0(x,y) = \frac{|y|\sqrt{1+|x|^2}+\langle x,y\rangle}{\sqrt{1+|x|^2}}, \qquad y \in T_x\mathrm{R}^n. \tag{22}$$

The geodesics of F_0 are straight lines in R^n. One can easily verify that F_0 is positively complete in the sense that every unit speed geodesic of F_0 is defined on $(-a,\infty)$. Moreover F_0 has positive flag curvature $\mathbf{K} > 0$.

Theorem 4.2 is a local classification theorem. If we assume that the manifold is closed (compact without boundary), then the scalar function $c(x)$ takes much more special values [5]. In particular, we have the following

Theorem 4.3 ([5])**.** *Let $S^n = (M,\alpha)$ is the standard unit sphere and $F = \alpha+\beta$ be a projectively flat Randers metric on S^n. Suppose that S-curvature is isotropic, $\mathbf{S} = (n+1)c(x)F$. Then*

$$c(x) = \frac{f(x)}{2\sqrt{1-f(x)^2}}$$

and

$$F(x,y) = \alpha(x,y) - \frac{f_{x^k}(x)y^k}{\sqrt{1-f(x)^2}},$$

where $f(x)$ is an eigenfunction of S^n corresponding to the first eigenvalue. Moreover,

(a) *$\delta := \sqrt{|\nabla f|^2_\alpha(x)+f(x)^2} < 1$ is a constant and we have the following estimates for flag curvature*

$$\frac{2-\delta}{2(1+\delta)} \le \mathbf{K} \le \frac{2+\delta}{2(1-\delta)}.$$

(b) *The geodesics of F are the great circles on S^n with F-length 2π.*

5. Randers Metrics of Scalar Flag Curvature with Isotropic S-curvature

In this section we are going to discuss Randers metrics of scalar flag curvature with isotropic S-curvature.

Using (10), we get the following

Lemma 5.1 ([7]). *Let $F=\alpha+\beta$ be a Randers metric expressed by (9). Suppose that it has* isotropic S-curvature, $\mathbf{S}=(n+1)cF$. *Then for any scalar function $\mu=\mu(x)$ on M,*

$$R^i{}_k-\Big(\frac{3c_{x^m}y^m}{F}+\mu-c^2-2c_{x^m}W^m\Big)\Big\{F^2\delta^i_k-FF_{y^k}y^i\Big\}$$
$$=\tilde{R}^i{}_k-\mu\Big(\tilde{h}^2\delta^i_k-\xi_k\xi^i\Big)-\frac{\xi_k}{\tilde{h}+\tilde{W}_0}\Big\{\tilde{R}^i{}_p-\mu\Big(\tilde{h}^2\delta^i_p-\xi_p\xi^i\Big)\Big\}W^p, \tag{23}$$

where

$$\xi^i:=y^i-F(x,y)W^i,\qquad \xi_k:=h_{ik}\xi^i,$$
$$\tilde{W}_0:=W_i\xi^i,\qquad \tilde{h}=h_{ij}\xi^i\xi^j$$

and

$$\tilde{R}^i{}_k:=\bar{R}_p{}^i{}_{kq}\xi^p\xi^q.$$

Here $\bar{R}_p{}^i{}_{kq}$ denote the Riemann curvature tensor of h.

From (23), we can easily prove the following

Theorem 5.2 ([7]). *Let F be a Randers metric on n-dimensional manifold M defined by (9). Suppose that $\mathbf{S}=(n+1)c(x)F$. Then F is of scalar flag curvature if and only if h is of sectional curvature $\bar{\mathbf{K}}=\mu$, where $\mu=\mu(x)$ is a scalar function (=constant if $n\geq 3$). In this case, the flag curvature of F is given by*

$$\mathbf{K}=\frac{3\tilde{c}_{x^m}y^m}{F}+\sigma,$$

where $\sigma:=\mu-c^2-2c_{x^m}W^m$ and $\tilde{c}-c=$ constant.

In dimension $n\geq 3$, if $\bar{\mathbf{K}}=\mu(x)$, then $\mu(x)=\mu$ is a constant. At any point, there is a local coordinate system in which h is given by

$$h=\frac{\sqrt{|y|^2+\mu(|x|^2|y|^2-\langle x,y\rangle^2)}}{1+\mu|x|^2}, \tag{24}$$

Suppose that $\mathbf{S}=(n+1)c(x)F$, namely, W satisfies

$$W_{i;j}+W_{j;i}=-4ch_{ij}. \tag{25}$$

One can solve (25) and obtain

$$c=\frac{\delta+\langle a,x\rangle}{\sqrt{1+\mu|x|^2}}, \tag{26}$$

(27) $$W = -2\Big\{\Big(\delta\sqrt{1+\mu|x|^2}+\langle a,x\rangle\Big)x-\frac{|x|^2a}{\sqrt{1+\mu|x|^2}+1}\Big\}+xQ+b+\mu\langle b,x\rangle x,$$

where δ is a constant, $Q=(q_j{}^i)$ is an anti-symmetric matrix and $a,b\in \mathrm{R}^n$ are constant vectors. See [20] for more details. We obtain the following classification theorem.

Theorem 5.3 ([7]). *Let $F=\alpha+\beta$ be a Randers metric on a manifold M of dimension $n\geq 3$, which is expressed in terms of a Riemannian metric h and a vector field W by (9). Then F is of scalar flag curvature $\mathbf{K}=K(x,y)$ and of isotropic S-curvature $\mathbf{S}=(n+1)c(x)F$ if and only if at any point, there is a local coordinate system in which h is given by (24) and c and W are given by (26) and (27) respectively. In this case, the flag curvature is given by*

(28) $$\mathbf{K}=\frac{3c_{x^m}y^m}{F}+\sigma,$$

where $\sigma=\mu-c^2-2c_{x^m}W^m$.

Proof. By assumption, the dimension of M is not less than 3. First we assume that $F=\alpha+\beta$ is of isotropic S-curvature and of scalar flag curvature. By Theorem 5.2, the flag curvature of F is given by (28) and h has constant sectional curvature $\bar{\mathbf{K}}=\mu$. At any point, there is a local coordinate system in which h is given by (24). By the Theorem 1.2 in [20], if $\mathbf{S}=(n+1)cF$, then c and W are given by (26) and (27) respectively in the same local coordinate system.

Conversely, assume that there is a local coordinate system in which h, c and W are given by (24), (26) and (27) respectively, then by Theorem 1.2 in [20], $\mathbf{S}=(n+1)cF$. Since h has constant sectional curvature $\bar{\mathbf{K}}=\mu$, by Theorem 5.2, F is of scalar curvature with flag curvature given by (28). □

Let us take a look at a special example.

Example 5.1. In (24)-(27), let $\mu=0,\delta=0,Q=0$ and $b=0$. We get

$$h=|y|,\qquad c=\langle a,x\rangle,\qquad W=-2\langle a,x\rangle x+|x|^2a.$$

The Randers metric $F=\alpha+\beta$ is given by

$$F=\frac{\sqrt{(1-|a|^2|x|^4)|y|^2+(|x|^2\langle a,y\rangle-2\langle a,x\rangle\langle x,y\rangle)^2}}{1-|a|^2|x|^4}-\frac{|x|^2\langle a,y\rangle-2\langle a,x\rangle\langle x,y\rangle}{1-|a|^2|x|^4}.$$

The above defined Randers metric F is of isotropic S-curvature and scalar flag curvature, i.e.,

$$\mathbf{S}=(n+1)\langle a,x\rangle\,F,\qquad \mathbf{K}=\frac{3\langle a,y\rangle}{F}+3\langle a,x\rangle^2-2|a|^2|x|^2.$$

6. Randers metrics with almost isotropic flag curvature

From the discussion above, it is natural to consider a Randers metric $F = \alpha(x, y) + \beta(x, y)$ of scalar flag curvature with

$$\mathbf{K} = \frac{3\tilde{c}_{x^m}(x)y^m}{F(x, y)} + \sigma(x), \tag{29}$$

where $\tilde{c} = \tilde{c}(x)$ and $\sigma = \sigma(x)$ are scalar functions on the manifold. Randers metrics with such property are said to be of *almost isotropic flag curvature.*

Note that for a Randers metric satisfying (29), the Ricci curvature is given by

$$\mathbf{Ric} = (n-1)\Big\{\frac{3\tilde{c}_{x^m}(x)y^m}{F(x, y)} + \sigma(x)\Big\}F(x, y)^2. \tag{30}$$

We have the following

Lemma 6.1 ([21])**.** *If a Randers metric $F = \alpha + \beta$ satisfies (30), then it has isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$ with $\tilde{c} - c = constant$.*

By Theorem 5.3 and Lemma 6.1, we obtain a local classification theorem of Randers metrics with (29).

Theorem 6.2 ([21])**.** *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$, which is expressed in terms of a Riemannian metric h and a vector field W by (9). Then F is of scalar flag curvature with*

$$\mathbf{K} = \frac{3\tilde{c}_{x^m}(x)y^m}{F(x, y)} + \sigma(x)$$

if and only if at any point, there is a local coordinate system in which h and W are given by

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{31}$$

$$W = -2\Big\{\Big(\delta\sqrt{1 + \mu|x|^2} + \langle a, x\rangle\Big)x - \frac{|x|^2 a}{\sqrt{1 + \mu|x|^2} + 1}\Big\} + xQ + b + \mu\langle b, x\rangle x, \tag{32}$$

where δ, μ are constants, $Q = (q_j{}^i)$ is an anti-symmetric matrix and $a, b \in \mathrm{R}^n$ are constant vectors. Moreover, $\tilde{c} - c = constant$ and $\sigma = \mu - c^2 - 2c_{x^m}W^m$, where

$$c = \frac{\delta + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}} \tag{33}$$

Suppose that $\mathbf{K} = \sigma = constant$. Then $\tilde{c} = constant$ and $c = constant$. By (33), we see that if $\mu = 0$, then $c = \delta$ and $\sigma = \mu - c^2$. W is given by

$$W = -2\delta x + xQ + b + \mu\langle b, x\rangle x. \tag{34}$$

If $\mu \neq 0$, then $c = 0$ and $\sigma = \mu$, W is given by

$$W = xQ + b + \mu\langle b, x\rangle x, \tag{35}$$

Corollary 6.3 ([3]). *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$, which is expressed in terms of a Riemannian metric h and a vector field W by (9). Then F is of constant flag curvature $\mathbf{K} = \sigma$ if and only if at any point, there is a local coordinate system in which h and W are given by (31) and W is given by (34) or (35) depending on the value of μ.*

Corollary 6.3 is the classification theorem due to D. Bao, C. Robles and Z. Shen [4].

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School of Mathematics and Physics,
Chongqing Institute of Technology,
Chongqing 400050, P.R. China
E-mail address: chengxy@cqit.edu.cn

Department of Mathematical Sciences,
Indiana University Purdue University Indianapolis (IUPUI),
402 N. Blackford Street,
Indianapolis, IN 46202-3216, USA
E-mail address: zshen@math.iupui.edu