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SOME EXAMPLES OF RANDERS SPACES

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Abstract. A Riemannian almost product structure on a manifold induces on a submanifold of codimension 1 a structure generalizing the paracontact structures and containing a Riemannain metric and an one form . We show that the pair consisting of this Riemannian metric and one form defines a strongly convex Randers metric on submanifold. We establish some properties of this Randers metric and we provide some examples.

1. Randers metrics provided by induced structures

Let $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ be a Riemannian almost product manifold. This means that $\widetilde{P}$ is an almost product structure on $\widetilde{M}$ i.e. $\widetilde{P}^2 = I$ (identity) and $\widetilde{g}$ is a Riemannian structure on $\widetilde{M}$ which is compatible with $\widetilde{P}$, i.e. $\widetilde{g}(\widetilde{P}X, \widetilde{P}Y) = \widetilde{g}(X, Y)$ for any vector fields X, Y on $\widetilde{M}$.

Let M be a submanifold of codimension 1 in $\widetilde{M}$. We denote by g the Riemannian metric induced by $\widetilde{g}$ on M and by N a field of unitary vectors that are normal to M.

Then for any vector field X tangent to M, the vector field $\widetilde{P}X$ decomposes in a tangent and a normal component:

$$\widetilde{P}X = PX + b(X)N, X \in \mathcal{X}(M), \tag{1.1}$$

($\mathcal{X}(M)$ denotes the Lie algebra of vector fields on M). It is clear that b is an 1-form on M. Also, the vector field $\widetilde{P}N$ decomposes in the form

$$\widetilde{P}N = \xi + aN, \tag{1.2}$$

where ξ is a vector field tangent to M and a is a function on M.

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Based on the properties of $\widetilde{P}$, $\widetilde{g}$ and on the uniqueness in the decomposition of type (1.1) and (1.2) the following formulae can be derived. For details and arbitrary codimension see [3].

$$P^2X = X - b(X)\xi, \tag{1.3}$$
$$b(PX) = -ab(X), \quad X \in \mathcal{X}(M), \tag{1.4}$$
$$b(\xi) = \|\xi\|^2 = 1 - a^2, \quad \text{(norm is taken with respect to } g) \tag{1.5}$$
$$P\xi = -a\xi, \tag{1.6}$$
$$b(X) = g(X, \xi), \tag{1.7}$$
$$g(PX, Y) = g(X, PY), \tag{1.8}$$
$$g(PX, PY) = g(X, Y) - b(X)b(Y), \quad X, Y \in \mathcal{X}(M). \tag{1.9}$$

We say that $(\widetilde{P}, \widetilde{g})$ induces on M a (P, ξ, b, a)-structure. By (1.5) we have $a \in (-1, 1)$. If $a = 0$, this structure reduces to a paracontact structure on M. We assume in the following that $a \neq 0$ and $\xi \neq 0$.

Let (x^i), $i, j, k \ldots = 1, \ldots, n = \dim M$ be local coordinates on M and (x^i, y^i) be local coordinates on tangent bundle TM. We set $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $b_i := b(\frac{\partial}{\partial x^i})$ and consider the real functions on TM:

$$\alpha(x, y) = \sqrt{g_{ij}(x)y^iy^j}, \quad \beta(x, y) = b_i(x)y^i.$$

It is well known that the function $F(x, y) = \alpha(x, y) + \beta(x, y)$ defines a Finsler structure on M whenever $\|b\| < 1$, cf. [1], Ch. 11. Such a Finsler structure is called a Randers structure and the pair (M, F) is called a Randers space. The function F is also called a Randers metric. Here $\|b\| := \sqrt{b_ib^i}$ where $b^i = g^{ij}b_j$. By (1.7) we have $b^i = \xi^i$ if $\xi = \xi^i\frac{\partial}{\partial x^i}$. Hence $\|b\| := \sqrt{b_i\xi^i} = \sqrt{1 - a^2}$ by (1.5). Therefore, $\|b\| < 1$.

Thus, we have

Theorem 1.1. *Any submanifold with $\xi \neq 0$, $a \neq 0$ of codimension 1 of a Riemannian almost product manifold carries a Randers structure i.e. it is a Randers space.*

2. Some properties of Randers spaces provided by induced structures

In this section assume that the almost product structure $\widetilde{P}$ is integrable. It is well known that this assumption is equivalent with the condition $\widetilde{\nabla}\widetilde{P} = 0$.

Recall that the Gauss and Weingarten formula for the immersion $M \hookrightarrow \widetilde{M}$ are respectively

$$\widetilde{\nabla}_XY = \nabla_XY + h(X, Y)N,$$
$$\widetilde{\nabla}_XN = -AX,$$

and the equality $h(X, Y) = g(AX, Y)$, holds for $X, Y \in \mathcal{X}(M)$.

Based on this condition as well as on the Gauss and Weingarten formulae, in [3], one proves that for a (P, ξ, b, a) on M the following formulae hold

$$(\nabla_X P)(Y) = h(X,Y)\xi + b(Y)AX, \tag{2.1}$$

$$(\nabla_X b)(Y) = -h(X,PY)\xi + ah(X,Y), \tag{2.2}$$

$$\nabla_X \xi = -P(AX)\xi + aAX, \tag{2.3}$$

$$X(a) = -2b(AX), \quad \text{for any } X, Y \in \mathcal{X}(M). \tag{2.4}$$

Let $\alpha + \beta$ be the Randers structure on M provided by the Theorem 1.1. It is known that a Randers structure reduces to a Berwald one if and only if $\nabla_X b = 0$, for any $X \in \mathcal{X}(M)$.

We have

Theorem 2.1. *The Randers structure on M induced by $(\widetilde{P}, \widetilde{g})$ on $\widetilde{M}$ reduces to a Berwald one if and only if*

$$PA = aA$$

holds

Indeed, by (2.2) we have

$$\begin{aligned}(\nabla_X b)(Y) &= ag(AX,Y) - g(AX,PY) = \\ &= ag(AX,Y) - g(PAX,Y) = \\ &= g(Y,(aA - PA)X), \text{ for any } X, Y \in \mathcal{X}(M)\end{aligned}$$

and

$$\nabla_X b = 0, \text{ for any } X \in \mathcal{X}(M)$$

it is obviously equivalent to $PA = aA$.

Recall that $2db(X,Y) = (\nabla_X b)(Y) - (\nabla_Y b)(X)$, for any $X, Y \in \mathcal{X}(M)$. Thus, if $\nabla b = 0$, then $db = 0$ i.e. the 1-form b is closed.

One easily check that $2db(X,Y) = -g((PA - AP)X, Y)$, for any $X, Y \in \mathcal{X}(M)$.

Thus, we have

Theorem 2.2. *The 1-form b is closed if and only if $PA{=}AP$.*

Let be again (P, g, ξ, b, a) the structure induced on M by $(\widetilde{P}, \widetilde{g})$. In [3] one proves

Theorem 2.3. *The vector field ξ is Killing if and only if*

$$PA + AP = 2aA$$

holds.

Notice that since A and P are both selfadjoint operators with respect to g, the condition $(\nabla_X b)(Y) = 0$ is also equivalent to $AP = aA$.

If one combines the Theorem 2.1 and Theorem 2.3 one gets

Theorem 2.4. *The Randers structure on M induced by $(\widetilde{P}, \widetilde{g})$ is Berwald if and only if the 1-form b is closed and ξ is Killing.*

If M is totally umbilical, i.e. $A = \lambda I$ then clearly b is closed and it follows

Corollary 2.5. *The Randers structure induced by $(\widetilde{P}, \widetilde{g})$ on a totally umbilical submanifold M, is Berwald if and only if ξ is Killing.*

This Corollary applies for spheres in E^n.

3. Examples

Let be E^{2m} the Euclidean space of dimension $2m$. We denote its elements by (x^i, x'^i), $i = 1, \ldots, m$ and consider the almost product structure $\widetilde{P}$ given by $\widetilde{P}(x^i, x'^i) = (x'^i, x^i)$. This is compatible with the usual dot product $\langle , \rangle$ and so $(E^{2m}, \widetilde{P}, \langle , \rangle)$ is a Riemannian locally product manifold. Notice that $\widetilde{P}$ has m eigenvalues equal to 1 and m eigenvalues equal to -1. The tangent spaces $T_x E^{2m}$, $x \in E^{2m}$ is isomorphic to E^{2m} and we denote by (y^i, y'^i) its elements.

Now we consider the sphere of radius 1 in E^{2m}:

$$S^{2m-1} = \Big\{ (x^i, x'^i) \quad | \quad \sum_i (x^i)^2 + \sum_i (x'^i)^2 = 1 \Big\}.$$

The unitary vector field normal to S^{2m-1} is $N = (x^i, x'^i)$ and the tangent space in a point $x \in S^{2m-1}$ is

$$T_x S^{2m-1} = \Big\{ (y^i, y'^i) \quad | \quad \sum_i (x^i y^i + x'^i y'^i) = 0 \Big\}.$$

We decompose $\widetilde{P}N = (x'^i, x^i)$ into the tangent and normal parts $\widetilde{P}N = (\xi^i, \xi'^i) + a(x^i, x'^i)$ and by identification we find

$$\xi^i = x'^i - a x^i, \quad \xi'^i = x^i - a x'^i \tag{3.1}$$

and using $\sum_i (x^i \xi^i + x'^i \xi'^i) = 0$ one gets

$$a = 2 \sum_i x^i x'^i. \tag{3.2}$$

Then (3.1) yields

$$\xi = (x'^i - a x^i, x^i - a x'^i),$$

with a given by (3.2). We note that a vanishes in the points $(0, x'^i)$ and $(x^i, 0)$ of S^{2m-1}.

We insert ξ in $b(X) = \langle X, \xi \rangle$ with $X = (X^i, X'^i)$ tangent to S^{2m-1} and we find the 1-form

$$b(X) = \sum_i (x^i X'^i + x'^i X^i).$$

For S^{2m-1} the Weingarten operator is $A = \lambda I$ (where $\lambda = -1$) and the general formulae from [3] reduce to

$$\begin{aligned}(\nabla_X P)(Y) &= -g(Y,\xi)X - g(X,Y)\xi,\\ (\nabla_X b)(Y) &= g(X, PY) - ag(X,Y),\\ \nabla_X \xi &= PX - aX,\\ X(a) &= 2g(X,\xi),\ \text{for any } X, Y \in \mathcal{X}(S^{2m-1}).\end{aligned}$$

We remark that the Randers metric provided by g and b is never Berwald since $(\nabla_X b)(Y) = 0$ is equivalent with $P = aI$ (I is identity) and this contradicts the equation (1.6). In order to explicitly write the Randers function $F = \alpha + \beta$ we need a basis in the tangent space of the sphere S^{2m-1}. We do this in the case $m = 2$ that is for the sphere S^3 in E^4.

We parameterize S^3 as $(x, y, z) \to (\varepsilon, x, y, z)/\sqrt{1 + x^2 + y^2 + z^2}$, with $\varepsilon = \pm 1$.

A basis of $T_x S^3$ is as follows:

$$\begin{aligned}h_1 &= (-\varepsilon x, 1 + y^2 + z^2, -xy, -xz)/A^3,\\ h_2 &= (-\varepsilon y, -yx, 1 + x^2 + z^2, -yz)/A^3,\\ h_3 &= (-\varepsilon z, -zx, -zy, 1 + x^2 + y^2)/A^3,\end{aligned}$$

where $A := \sqrt{1 + x^2 + y^2 + z^2}$.

The induced metric g has the matrix:

$$\frac{1}{A^4}\begin{pmatrix} 1 + y^2 + z^2 & -xy & -xz \\ -xy & 1 + x^2 + z^2 & -yz \\ -zx & -zy & 1 + x^2 + y^2 \end{pmatrix}$$

The form of ξ in the given parameterization is

$$\xi = (y\varepsilon - a, z\varepsilon - ax\varepsilon, 1 - ay\varepsilon, x\varepsilon - az\varepsilon)/A,$$

where $a = 2\dfrac{xz - \varepsilon y}{A^2}$. Thus, we have

$$\xi = \frac{1}{A^3}(\varepsilon y A^2 - 2(xz + \varepsilon y)\varepsilon, \varepsilon z A^2 - 2(xz\varepsilon + y)x,$$

$$A^2 - 2(xz\varepsilon + y)y, \varepsilon x A^2 - 2(xz\varepsilon + y)z).$$

We write $\xi = \alpha h_1 + \beta h_2 + \gamma h_3$ and by an identification we determine α, β, γ and we find that ξ in the basis $\mathbf{h} = (h_1, h_2, h_3)$ is as follows:

$$\xi = (z\varepsilon - xy)h_1 + (1 - y^2)h_2 + (x\varepsilon - yz)h_3.$$

We compute the components $b_i = b(h_i) = g(h_i, \xi)$ of b in the basis (h_1, h_2, h_3) and we find

$$b = \frac{1}{A}(\varepsilon z A^2 - 2(\varepsilon xz + y)x, A^2 - 2(xz\varepsilon + y)y, \varepsilon x A^2 - 2(\varepsilon xz + y)z).$$

We denote by (u, v, w) the components of an arbitrary tangent vector in the basis $\mathbf{h}$. The above calculations show that the Randers function $F = \alpha + \beta$, for $\varepsilon = 1$ has the form:

$$\begin{aligned} F(x, y, z; u, v, w) = &\sqrt{(A^2 - x^2)u^2 + (A^2 - y^2)v^2 + (A^2 - z^2)w^2 \\ -2xyuv - 2xzuw - 2yzvw} + \frac{(zA^2 - 2(xz + y)x)u}{A} \\ &+ \frac{(A^2 - 2(xz + y)y)v}{A} + \frac{(xA^2 - 2(xz + y)z)w}{A}. \end{aligned}$$

Recall that $A = \sqrt{1 + x^2 + y^2 + z^2}$.

Let E^{m+1} be an Euclidean space of dimension $m + 1$. Every almost product structure $\widetilde{P} : E^{m+1} \longrightarrow E^{m+1}$ has let say $k = \overline{0, m+1}$ eigenvalues equal to $+1$ and $m + 1 - k$ eigenvalues equal to -1. For a convenient choice of coordinates on E^{m+1}, the operator $\widetilde{P}$ takes the standard form $\widetilde{P}(x^1, \ldots, x^{m+1}) = (x^1, \ldots, x^k, -x^{k+1}, \ldots, -x^{m+1})$. Then $(E^{m+1}, \widetilde{P}, \langle , \rangle)$ is a locally product Riemannian manifold.

Let M be a hypersurface in E^{m+1} ($\dim M = m$). Assume it is given in an explicit form: $x^{m+1} = f(x^1, \ldots, x^m)$ with f a smooth function and denote $p_i := \dfrac{\partial f}{\partial x^i}$, $i = 1, \ldots, m$.

A natural basis in T_xM, $x \in M$ is given by $h_i = (0, \ldots, \underset{i}{1}, \ldots, p_i)$, $i = \overline{1, m}$ and an unitary normal vector field is $N = (p_1, p_2, \ldots, p_m, -1)/A$, where $A = \sqrt{1 + p_1{}^2 + \cdots + p_m{}^2}$. We have

$$\widetilde{P}(N) = (p_1, \ldots, p_k, -p_{k+1}, \ldots, -p_m, 1)/A.$$

On the other hand $\widetilde{P}(N)$ is decomposed in the form

$$\widetilde{P}(N) = \xi^1 h_1 + \cdots + \xi^m h_m + aN.$$

An identification gives:

$$\begin{aligned} \xi^1 = \frac{(1-a)p_1}{A}, \ldots, \xi^k &= \frac{(1-a)p_k}{A}, \\ \xi^{k+1} = \frac{-(1+a)p_{k+1}}{A}, \ldots, \xi^m &= \frac{-(1+a)p_m}{A}, \\ p_1\xi^1 + \ldots + p_m\xi^m &= \frac{1+a}{A} \end{aligned}$$

By inserting (ξ^i) in the very last equation, one obtains

$$a = \frac{p_1^2 + \cdots + p_k^2 - p_{k+1}^2 - \cdots - p_m^2 - 1}{A^2}$$

Moving a in the form of ξ^i we find

$$\xi = \frac{2}{A^3}(1 + p_{k+1}^2 + \cdots + p_m^2)(p_1, \ldots, p_k, 0, \ldots, 0) \\ - \frac{2}{A^3}(p_1^2 + \cdots + p_k^2)(0, \ldots, 0, p_{k+1}, \ldots, p_m).$$

As to the induced metric g, we have

$$g_{ij} = \begin{cases} 1 + p_i^2, & i \neq j \\ p_i p_j, & i = j. \end{cases}$$

We compute $b_i = g_{ih}\xi^h$ and obtain:

$$b_i = \frac{2p_i}{A}, \ i = 1, 2, \ldots, k, \\ b_i = 0, \quad i = k+1, \ldots, m.$$

Let (y^i) be the components of an arbitrary element of $T_x M$. The Randers metric derived from (g_{ij}) and (b_i) has the form:

$$F(x,y) = \sqrt{\sum_i (1 + p_i^2)(y^i)^2 + \sum_{i,j} p_i p_j y^i y^j} + \frac{2}{A}\sum_{h=1}^{k} p_h y^h, \text{ or}$$

$$F(x,y) = \sqrt{\left(\sum_{i=1}^{m} p_i y^i\right)^2 + \sum_{i=1}^{m} (y^i)^2} + \frac{2}{A}\sum_{h=1}^{k} p_h y^h.$$

Recall that $p_i = \dfrac{\partial f}{\partial x^i}$, $A = \sqrt{1 + {p_1}^2 + \cdots + {p_m}^2}$. With this procedure we generically find a set of m Randers metrics on M. Notice that for a hyperplane $x^{m+1} = a_1 x^1 + \cdots + a_m x^m$ all these Randers metrics are locally Minkowski.

We have to separately treat the cases $a = \pm 1$ and $a = 0$. The case $a = 1$ is equivalent to $1 + p_{k+1}^2 + \cdots + p_m^2 = 0$, that never holds. The equality $a = -1$ holds in the points of M, where $p_1 = p_2 = \cdots = p_k = 0$. In this case $\xi = 0$ and we cannot construct F. Thus we have to delete from M the points $\{(x^1, \ldots, x^m) | p_1 = p_2 = \cdots = p_k = 0\}$. Let denote by M_0 the new hypersurface obtained in such a way.

On M_0 all functions F from above are Randers metrics but only those obtained for $a \neq 0$ are strongly convex. Thus in order to obtain only strongly convex Randers metrics we have to delete from M_0 the points $\{(x^1, \ldots, x^m) | p_1^2 + \cdots + p_k^2 - p_{k+1}^2 - \cdots - p_m^2 - 1 = 0\}$. Let M_{00} be the hypersurface obtained after this elimination. On M_{00} all Randers metrics from above are strongly convex. We note that at the same time these Randers metrics are y-global (cf. [1], pg. 304).

Let now confine ourselves to the case $m = 2$. We have two different types of product structures obtained for $k = 1$ and $k = 2$. The corresponding Randers metrics are as follows:

$$k=1: F(x,y;u,v)=\sqrt{(pu+qv)^2+u^2+v^2}+\frac{2pu}{\sqrt{1+p^2+q^2}},$$
$$k=2: F(x,y;u,v)=\sqrt{(pu+qv)^2+u^2+v^2}+\frac{2(pu+qv)}{\sqrt{1+p^2+q^2}},$$

where $p=\dfrac{\partial f}{\partial x}$, $q=\dfrac{\partial f}{\partial y}$ and (u,v) are the components of a tangent vector.

Here are some particular cases.

1. For a hyperbolic paraboloid of equation $z=xy$ we get:

$$k=1: F_{11}(x,y;u,v)=\sqrt{(yu+xv)^2+u^2+v^2}+\frac{2yu}{\sqrt{1+x^2+y^2}}$$

$$k=2: F_{12}(x,y;u,v)=\sqrt{(yu+xv)^2+u^2+v^2}+\frac{2(yu+xv)}{\sqrt{1+x^2+y^2}}$$

2. For the hemisphere S^2 in E^3 given by the equation

$$z=\sqrt{1-x^2-y^2},\qquad x^2+y^2<1$$

we get:

$$k=1: F_{21}(x,y;u,v)=\sqrt{(1-y^2)u^2+2xyuv+(1-x^2)v^2}-\frac{2xu}{1+x^2+y^2}$$

$$k=2: F_{22}(x,y;u,v)=\sqrt{(1-y^2)u^2+2xyuv+(1-x^2)v^2}-\frac{2(xu+yv)}{1+x^2+y^2}$$

3. For the cylinder $z=\sqrt{1-x^2}$, $|x|<1$ we get:

$$k=1: F_{31}(x,y;u,v)=\sqrt{\frac{u^2}{1-x^2}+v^2}+2xu$$

$$k=2: F_{32}(x,y;u,v)=\sqrt{\frac{u^2}{1-x^2}+v^2}-2xu$$

4. If we consider the hemisphere S^2 in E^3 parameterized by

$$\left(\frac{x}{\sqrt{1+x^2+y^2}},\frac{y}{\sqrt{1+x^2+y^2}},\frac{1}{\sqrt{1+x^2+y^2}}\right),$$

we get

$$k=1: F_{41}(x,y;u,v)=\frac{\sqrt{(xv-yu)^2+u^2+v^2}}{1+x^2+y^2}+\frac{2(xu+yv)}{(1+x^2+y^2)^2}$$

$$k=2: F_{42}(x,y;u,v)=\frac{\sqrt{(xv-yu)^2+u^2+v^2}}{1+x^2+y^2}+\frac{2((1+y^2)u-2xyv)}{(1+x^2+y^2)^2}$$

Notice that the Randers metrics F_{41} and F_{42} coincide with F_{21} and F_{22} respectively, but are written in different parameterizations.

In [2] one proves that any strongly convex Randers metric $F = \sqrt{g_{ij}y^iy^j}+b_iy^i$, $\|b\| < 1$ solves a Zermelo's navigation problem on the Riemannian manifold (M, h) with $h_{ij} = \epsilon(g_{ij} - b_ib_j)$ with the vector field ("wind") $W^i = -\dfrac{b^i}{\epsilon}$, for $\epsilon = 1-\|b\|^2$. The Randers metric induced by $(\widetilde{P}, \widetilde{g})$ solves the following Zermelo's navigation problem:

$$h_{ij} = a^2(g_{ij} - b_ib_j), \quad W^i = -\frac{\xi^i}{a^2}$$

since in our case $\epsilon = 1 - (1 - a^2) = a^2$.

It is also proved in [2] that the Randers metric $F = \sqrt{g_{ij}y^iy^j} + b_iy^i$ is of constant flag curvature K if and only if

(i) (M, h) is of constant sectional curvature $K + \dfrac{1}{16}\sigma^2$ for some constant σ,

(ii) $\mathcal{L}_W h = -\sigma h$, where $\mathcal{L}_W$ denotes the Lie derivative with respect to h.

It is easy to see that the condition (ii) is satisfied if ξ is Killing and a is a constant function.

As we have seen, ξ is Killing if and only if $PA + AP = 2aA$ and a is not a constant function. Thus it will be hard to find Randers metrics of constant flag curvature among our examples. Before this theoretical analysis was done, some checking using Maple showed the same conclusion.

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