Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
23 (2007), 1–6
www.emis.de/journals
ISSN 1786-0091

ON SOME FORMULAS IN THE HALL ALGEBRA OF THE KRONECKER ALGEBRA

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Abstract. In [5] Pu Zhang presents a PBW-basis of the composition subalgebra of the Hall algebra in the Kronecker case. The construction is based on formulas expressing the product of some specific elements in the Hall algebra (see Theorem 4.2, 4.3 in [5]). Using a different approach we will reprove these formulas in an entirely independent manner.

1. **Preliminaries**

Let K be the Kronecker quiver (i.e. the quiver $1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$) and k a finite field with $|k| = q$. We will consider the path algebra kK of K over k (called Kronecker algebra) and the category mod-kK of finitely generated (hence finite) right modules over kK. The category mod-kK will be identified with the category rep-kK of the finite dimensional k-representations of the Kronecker quiver. For general notions concerning the representation theory of quivers, we refer to [1] or [3].

Up to isomorphism we will have two simple objects in mod-kK corresponding to the two vertices. We shall denote them by S_1 and S_2. For a module $M \in$ mod-kK, $[M]$ will denote the isomorphism class of M. The number of automorphisms of M will be denoted by α_M and the dimension vector of M by $\underline{\dim} M = (m_{S_1}(M), m_{S_2}(M))$, where $m_{S_i}(M)$ is the number of composition factors of M isomorphic to S_i. For a module M let $tM := M \oplus \cdots \oplus M$ (t-times).

The indecomposables in mod-kK are divided into three families: the preprojectives, the regulars and the preinjectives.

The preprojective (respectively preinjective) indecomposable modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by P_n (respectively with I_n) the indecomposable preprojective module of dimension $(n+1, n)$ (respectively the indecomposable preinjective

2000 *Mathematics Subject Classification.* 16G20 (17B37).

Key words and phrases. Kronecker algebra, Hall algebra, composition algebra.

module of dimension $(n, n+1)$). So P_0, P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple) and $I_0 = S_2$, I_1 the injective indecomposable modules ($I_0 = S_2$ being simple). A preprojective (respectively a preinjective) module, i.e. a module with all its indecomposable direct summands preprojective (respectively preinjective) will be usually denoted by P (respectively by I).

Viewed as finite dimensional k-representations of the Kronecker quiver, the regular indecomposables up to isomorphism are:

$$R_1^o(t) := k[X]/(X^t) \underset{id}{\overset{X}{\leftleftarrows}} k[X]/(X^t) \ ,$$

$$R_1^\mu(t) := k[X]/((X-\mu)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/((X-\mu)^t) \ ,$$

where $t \geq 1$ and $\mu \in k$;

$$R_l^{\varphi_l}(t) := \ k[X]/(\varphi_l(X)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/(\varphi_l(X)^t) \ ,$$

where $t \geq 1$, $l \geq 2$ and $\varphi_l(X)$ is a monic irreducible polynomial of degree l over k.

Let $N(q,l) = \frac{1}{l}\sum_{d|l} \mu(\frac{l}{d})q^d$, where $l \geq 1$, and μ is the Möbius function. It is well known that $N(q,l)$ is the number of monic, irreducible polynomials of degree l over a field with q elements, for l fixed $N(q,l)$ is strictly monotonous increasing in $q \geq 1$, i.e. $N(q_1,l) < N(q_2,l)$ for $1 \leq q_1 < q_2$, $N(q,l) \geq 1$ and $N(q,l) = 1$ iff $q = l = 2$. Let $M(q,l) := N(q,l)$ when $l \geq 2$ and $M(q,1) := N(q,1) + 1 = q + 1$.

To somewhat simplify the notations we shall fix in an arbitrary way bijections $f_1 : \{\mu | \mu \in k\} \cup \{o\} \to \{1, \ldots, q+1\}$ and $f_l : \{\varphi_l | \varphi_l$ monic irreducible polynomial of degree l over $k\} \to \{1, \ldots, N(q,l)\}$ (where $l \geq 2$) and then let $R_1^o(t) = R_1^{f_1(o)}(t)$, $R_1^\mu(t) = R_1^{f_1(\mu)}(t)$, $R_l^{\varphi_l}(t) = R_l^{f_l(\varphi_l)}(t)$. So using the notations above our regular indecomposables are $R_l^a(t)$, where $l \geq 1$, $a = \overline{1, M(q,l)}$, $t \geq 1$.

Similarly to the preprojective (preinjective) case, a regular module, i.e. a module with all its indecomposable direct summands regular, will be usually denoted by R or by R_n if its dimension is (n,n).

Let r_n denote the sum of all isomorphism classes of regular modules of dimension (n,n), i.e. $r_n = \sum_{[R_n]}[R_n]$. By definition $r_0 := [0]$. One can easily see, that $r_1 = \sum_{a=1}^{q+1}[R_1^a(1)]$.

Consider now the Auslander-Reiten translations $\tau = D\operatorname{Ext}^1(-, kK)$, $\tau^{-1} = \operatorname{Ext}^1(D(-), kK)$ where $D = \operatorname{Hom}_k(-, k)$ (see [1] or [3]). We then have

$$\begin{aligned}\tau(P_n) =& P_{n-2}, \tau(P_0) = \tau(P_1) = 0, \tau^{-1}(P_n) = P_{n+2},\\ \tau^{-1}(I_n) =& I_{n-2}, \tau^{-1}(I_0) = \tau^{-1}(I_1) = 0, \tau(I_n) = I_{n+2},\\ \tau(R_l^a(t)) =& \tau^{-1}(R_l^a(t)) = R_l^a(t).\end{aligned}$$

The defect of $M \in$ mod-kK with dimension vector (a, b) is defined in the Kronecker case as $\partial M := b - a$. Observe that if M is a preprojective (preinjective, respectively regular) indecomposable, then $\partial M = -1$ ($\partial M = 1$, respectively $\partial M = 0$). Moreover for a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in mod-kK we have $\partial M_2 = \partial M_1 + \partial M_3$.

The following lemma summarizes facts on morphisms, automorphisms and extensions in mod-kK:

Lemma 1.1. *Using the notation above, we have:*

a) $\operatorname{Hom}(R, P) = \operatorname{Hom}(I, P) = \operatorname{Hom}(I, R) = \operatorname{Ext}^1(P, R) = \operatorname{Ext}^1(P, I) = \operatorname{Ext}^1(R, I) = 0$.
b) *If* $(a, l) \neq (a', l')$, *then* $\operatorname{Hom}(R_l^a(t), R_{l'}^{a'}(t')) = \operatorname{Ext}^1(R_l^a(t), R_{l'}^{a'}(t')) = 0$.
c) *For* $n \leq m$, *we have* $\dim_k \operatorname{Hom}(P_n, P_m) = m-n+1$ *and* $\operatorname{Ext}^1(P_n, P_m) = 0$*; otherwise* $\operatorname{Hom}(P_n, P_m) = 0$ *and* $\dim_k \operatorname{Ext}^1(P_n, P_m) = n - m - 1$. *In particular* $\operatorname{End}(P_n) \cong k$ *and* $\operatorname{Ext}^1(P_n, P_n) = 0$.
d) *For* $n \geq m$, *we have* $\dim_k \operatorname{Hom}(I_n, I_m) = n - m + 1$ *and* $\operatorname{Ext}^1(I_n, I_m) = 0$*; otherwise* $\operatorname{Hom}(I_n, I_m) = 0$ *and* $\dim_k \operatorname{Ext}^1(I_n, I_m) = m - n - 1$. *In particular* $\operatorname{End}(I_n) \cong k$ *and* $\operatorname{Ext}^1(I_n, I_n) = 0$.
e) $\dim_k \operatorname{Hom}(P_n, I_m) = n + m$ *and* $\dim_k \operatorname{Ext}^1(I_m, P_n) = m + n + 2$.
f) $\dim_k \operatorname{Hom}(P_n, R_l^a(t)) = \dim_k \operatorname{Hom}(R_l^a(t), I_n) = lt$, $\dim_k \operatorname{Ext}^1(R_l^a(t), P_n) = \dim_k \operatorname{Ext}^1(I_n, R_l^a(t)) = lt$.
g) $\dim_k \operatorname{Hom}(R_l^a(t), R_l^a(t')) = \dim_k \operatorname{Ext}^1(R_l^a(t), R_l^a(t')) = l \min(t, t')$.
h) $\alpha_{P_n} = \alpha_{I_n} = q - 1$.
i) $\alpha_{R_l^a(t)} = q^{\dim_k \operatorname{End}(R_l^a(t))} - q^{\dim_k \operatorname{rad} \operatorname{End}(R_l^a(t))} = q^{lt} - q^{l(t-1)}$.
j) *Let* $M = c_1 M_1 \oplus \cdots \oplus c_t M_t$ *such that* M_i *are pairwise nonisomorphic indecomposable modules. Then* $\alpha_M = q^m \alpha_{c_1 M_1} \dots \alpha_{c_t M_t}$, *where* $m = \sum_{i \neq j} c_i c_j \dim_k \operatorname{Hom}(M_i, M_j)$
k) *Let* $M = cN$ *with* N *indecomposable and* $\operatorname{End}(N) = k'$ *a field. Then* $\alpha_M = |\operatorname{GL}_c(k')| = \prod_{1 \leq i \leq c}(d^c - d^{i-1})$, *where* $d = |k'| = q^{[k':k]}$.
l) $\alpha_{cP_n} = \alpha_{cI_n} = |\operatorname{GL}_c(k)| = \prod_{1 \leq i \leq c}(q^c - q^{i-1})$.
m) $\alpha_{cR_l^a(t)} = q^{l(t-1)c^2} \prod_{1 \leq i \leq c}(q^{lc} - q^{l(i-1)})$.

We end this paragraph presenting some facts on Hall algebras.

The Hall algebra $\mathcal{H}(kK)$ associated to the Kronecker algebra kK, is the $\mathbb{Q}$-space having as basis the isomorphism classes in mod-kK together with a multiplication (the so called Hall product) defined by:

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M].$$

The structure constants $F^M_{N_1N_2} = |\{M \supseteq U | U \cong N_2, M/U \cong N_1\}|$ are called Hall numbers.

It is easy to see that the Hall algebra is a well-defined, associative, usually noncommutative algebra with unit element the isomorphism class of the zero module. Also the Hall numbers have the following important property:

Lemma 1.2. a) *If both M and N have no projective indecomposable direct summands, then $F^{\tau L}_{\tau M \tau N} = F^L_{MN}$.*

b) *If both M and N have no injective indecomposable direct summands, then $F^{\tau^{-1}L}_{\tau^{-1}M\tau^{-1}N} = F^L_{MN}$.*

The unital subalgebra of $\mathcal{H}(kK)$ generated by the two simple isomorphism classes $[S_1], [S_2]$ is called the composition algebra of the Kronecker algebra and it is denoted by $\mathcal{C}(kK)$.

Hall algebras and their composition subalgebras were used by C. M. Ringel and J. A. Green to connect the representation theory of finite dimensional algebras with the theory of quantum groups (see [4], [2]). More precisely it turned out that in some cases (including the Kronecker case) a twisted generic version of the composition algebra is isomorphic with the positive part of a corresponding Drinfeld-Jimbo quantized enveloping algebra. So a PBW-basis in the composition algebra will give us a PBW-basis in the corresponding quantized algebra.

In [5] Pu Zhang constructed a PBW-basis in the composition algebra of the Kronecker algebra. His construction is based on some formulas expressing the Hall product of some specific elements in $\mathcal{C}(kK)$ like $[P_n], [I_n]$ and r_n.

In the next paragraph we will present a new proof for the following formulas obtained by Zhang:

Theorem 1.3 ([5], Theorem 4.2., 4.3)**.** *We have:*

a) $r_n[P_m] = \sum_{0 \le i \le n} \frac{q^{n+1}-q^i}{q-1}[P_{m+n-i}]r_i$.

b) $[I_m]r_n = \sum_{0 \le i \le n} \frac{q^{n+1}-q^i}{q-1} r_i[I_{m+n-i}]$.

We mention that our approach is entirely independent from the proof given by Zhang.

2. The proof

Observe that it is enough to prove a), b) being dual to a).

Consider an exact sequence of the form

$$0 \longrightarrow P_m \xrightarrow{u} X \xrightarrow{v} R \longrightarrow 0\ ,$$

with R regular. Then X can't have a preinjective component , since if I is such a component we would have an exact sequence

$$0 \to \operatorname{Hom}(I, P_m) \to \operatorname{Hom}(I, X) \to \operatorname{Hom}(I, R),$$

with $\operatorname{Hom}(I, P_m) = \operatorname{Hom}(I, R) = 0$ and $\operatorname{Hom}(I, X) \neq 0$ (see Lemma 1.1). Moreover $\partial X = \partial P_m + \partial R = -1$ and so we obtain that X must be isomorphic with $P_k \oplus R'$ with P_k preprojective indecomposable and R' regular. Denote by p_1 (respectively p_2) the projection of $P_k \oplus R'$ on P_k (respectively on R'). Then one can see that we must have $p_1 u \neq 0$ which means that $k \geq m$ (since $\operatorname{Hom}(P_i, P_j) = 0$ for $i > j$).

On the other hand for an exact sequence of the form

$$0 \longrightarrow P_m \xrightarrow{f} P_{m+n-i} \oplus R_i \xrightarrow{g} Y \longrightarrow 0$$

we have Y not regular iff $p_1 f = 0$. Indeed, the if part is trivial. For the only if part suppose that Y is not regular. Since $\partial Y = 0$ this means that Y is of the form $P_l \oplus Z$, where P_l is indecomposable preprojective and Z has at least a preinjective component.

Denote by p_1' the projection of $P_l \oplus Z$ on P_l and by q_1 the injection of P_{m+n-i} into $P_{m+n-i} \oplus R_i$. Then $p_1' g$ is an epimorphism and since $\operatorname{Hom}(R, P) = 0$ we have that $p_1' g q_1 : P_{m+n-i} \to P_l$ is also an epimorphism, so it is an isomorphism (by Lemma 1.1). But then $p_1' g$ is a split epimorphism, $\operatorname{Ker} p_1' g = R_i$ and so $P_m \cong \operatorname{Im} f = \operatorname{Ker} g \subseteq \operatorname{Ker} p_1' g = R_i$.

Our conclusion is that

$$\begin{aligned} A(m,n,i) :=& \sum_{[R_n]} F_{R_n P_m}^{P_{m+n-i} \oplus R_i} = \sum_{[Y]} F_{Y P_m}^{P_{m+n-i} \oplus R_i} - \sum_{[Y] \text{ not regular}} F_{Y P_m}^{P_{m+n-i} \oplus R_i} \\ =& |\{U \subseteq P_{m+n-i} \oplus R_i : U \cong P_m\}| - |\{V \subseteq R_i : V \cong P_m\}|. \end{aligned}$$

Using Lemma 1.1 one can easily see that

$$\begin{aligned} A(0,n,i) =& \frac{|\{\text{monomorphisms } S_1 \to P_{n-i} \oplus R_i\}|}{|\operatorname{Aut}(S_1)|} \\ &- \frac{|\{\text{monomorphisms } S_1 \to R_i\}|}{|\operatorname{Aut}(S_1)|} \\ =& \frac{|\operatorname{Hom}(S_1, P_{n-i} \oplus R_i)|}{|\operatorname{Aut}(S_1)|} - \frac{|\operatorname{Hom}(S_1, R_i)|}{|\operatorname{Aut}(S_1)|} \\ =& \frac{q^{n+1} - 1}{q - 1} - \frac{q^i - 1}{q - 1} = \frac{q^{n+1} - q^i}{q - 1}. \end{aligned}$$

Also we have that

$$\begin{aligned} A(1,n,i) =& \frac{|\{\text{monomorphisms } P_1 \to P_{1+n-i} \oplus R_i\}|}{|\operatorname{Aut}(P_1)|} \\ -& \frac{|\{\text{monomorphisms } P_1 \to R_i\}|}{|\operatorname{Aut}(P_1)|} \\ =& \frac{|\operatorname{Hom}(P_1, P_{1+n-i} \oplus R_i)| - |\mathcal{A}|}{|\operatorname{Aut}(P_1)|} - \frac{|\operatorname{Hom}(P_1, R_i)| - |\mathcal{B}|}{|\operatorname{Aut}(P_1)|}, \end{aligned}$$

where $\mathcal{A}$ ={morphisms $P_1 \to P_{1+n-i} \oplus R_i$ with kernel S_1} and $\mathcal{B}$ ={morphisms $P_1 \to R_i$ with kernel S_1}. But observe that $\mathcal{A} = \mathcal{B}$ so by Lemma 1.1 we obtain that

$$A(1,n,i) = \frac{q^{n+1} - q^i}{q-1} = A(0,n,i).$$

Lemma 1.2 gives us

$$A(m,n,i) = \frac{q^{n+1} - q^i}{q-1}.$$

Finally using all the previous observations we have

$$\begin{aligned} r_n[P_m] =& \sum_{[R_n]} \sum_{0 \le i \le n} \sum_{[R_i]} F^{P_{m+n-i} \oplus R_i}_{R_n P_m} [P_{m+n-i} \oplus R_i] \\ =& \sum_{0 \le i \le n} \sum_{[R_i]} A(m,n,i)[P_{m+n-i}][R_i] = \sum_{0 \le i \le n} \frac{q^{n+1} - q^i}{q-1} [P_{m+n-i}] r_i. \end{aligned}$$

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Received 27 August, 2006.

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