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# RECENT RESULTS ON THE DERIVED LENGTH OF LIE SOLVABLE GROUP ALGEBRAS

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ABSTRACT. Let $G$ be a group with cyclic commutator subgroup of order $p^n$ and $F$ a field of characteristic $p$. We obtain the description of the group algebras $FG$ of Lie derived length 3.

## 1. Introduction and results

Let us consider the group algebra $FG$ of a group $G$ over a field $F$ as a Lie algebra with the usual bracket operation and define the *Lie derived series* and the *strong Lie derived series* of $FG$ as follows: let $\delta^{[0]}(FG) = \delta^{(0)}(FG) = FG$ and

$$\delta^{[n+1]}(FG) = \big[\delta^{[n]}(FG), \delta^{[n]}(FG)\big],$$
$$\delta^{(n+1)}(FG) = \big[\delta^{(n)}(FG), \delta^{(n)}(FG)\big]FG,$$

where $[X, Y]$ is the additive subgroup generated by all Lie commutators $[x, y] = xy - yx$ with $x \in X$ and $y \in Y$. We say that $FG$ is *Lie solvable* if there exists $m$ such that $\delta^{[m]}(FG) = 0$, and similarly, if $\delta^{(n)}(FG) = 0$ for some $n$ then $FG$ is said to be *strongly Lie solvable*. The minimal integers $m, n$ for which $\delta^{[m]}(FG) = 0$ and $\delta^{(n)}(FG) = 0$ are called the *Lie derived length* and the *strong Lie derived length* of $FG$ and they are denoted by $\mathrm{dl}_L(FG)$ and $\mathrm{dl}^L(FG)$, respectively. I. B. S. Passi, D. S. Passman and S. K. Sehgal [6] proved that a group algebra $FG$ is Lie solvable if and only if one of the following conditions holds: (i) $G$ is abelian; (ii) $\mathrm{char}(F) = p$ and the commutator subgroup $G'$ of $G$ is a finite $p$-group; (iii) $\mathrm{char}(F) = 2$ and $G$ has a subgroup $H$ of index 2 whose commutator subgroup $H'$ is a finite 2-group. As it is well-known, a group algebra $FG$ is strongly Lie solvable if either $G$ is abelian or $\mathrm{char}(F) = p$ and $G'$ is a finite $p$-group.

It is obvious that $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = 1$ if and only if $G$ is abelian. The group algebras $FG$ with $\mathrm{dl}_L(FG) = 2$ are known as *Lie metabelian* group

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algebras. F. Levin and G. Rosenberger described these group algebras in [4], namely, a noncommutative group algebra $FG$ of characteristic $p$ is Lie metabelian if and only if one of the following conditions holds: (i) $p = 3$ and $G'$ is central of order 3; (ii) $p = 2$ and $G'$ is central and elementary abelian of order dividing 4. Moreover, they proved that $\mathrm{dl}_L(FG) = 2$ if and only if $\mathrm{dl}^L(FG) = 2$.

M. Sahai in [9] gave the full description of the strongly Lie solvable group algebras of strong derived length 3 for odd characteristic, and showed that the statements $\delta^{[3]}(FG) = 0$ and $\delta^{(3)}(FG) = 0$ are equivalent, provided $\mathrm{char}(F) \geq 7$. In the other cases the question is still open. Further examples can be found in R. Rossmanith's papers [7, 8] for group algebras with Lie derived length at most 3 of characteristic 2.

The introductory results on the Lie derived length of Lie solvable group algebras are in A. Shalev's papers [10, 11].

In this article we continue the study which we started in [3, 1]. In [3] the Lie solvable group algebras $FG$ whose Lie derived lengths are maximal are given in the case when $G$ is a nilpotent group with cyclic commutator subgroup of order $p^n$. Later [1], we investigated the non-nilpotent case.

To describe the Lie solvable group algebras of derived length 3 seems a difficult problem. A partial solution can be found here; we indeed prove the following

**Theorem 1.** *Let $G$ be a group with cyclic commutator subgroup of order $p^n$ and let $F$ be a field of characteristic $p$. Then $\mathrm{dl}_L(FG) = 3$ if and only if one of the following conditions holds:*

(i) *$p = 7$, $n = 1$ and $G$ is nilpotent;*
(ii) *$p = 5$, $n = 1$ and either $x^g = x^{-1}$ for all $x \in G'$ and $g \notin C_G(G')$ or $G$ is nilpotent;*
(iii) *$p = 3$, $n = 1$ and $G$ is not nilpotent;*
(iv) *$p = 2$ and*
  a) *$n = 2$;*
  b) *$n = 3$ and $G$ is of class 4;*
  c) *$G$ has an abelian subgroup of index 2.*

A. Shalev proved (see Proposition C in [11]): if $G$ is an abelian-by-cyclic $p$-group of class two with $p > 2$ and $\mathrm{char}(F) = p$, then $\mathrm{dl}_L(FG) = \lceil \log_2(t(G')+1)\rceil$, where $t(G')$ denotes the nilpotent index of the augmentation ideal of $FG'$ and $\lceil r\rceil$ the upper integral part of a real number $r$. We generalize this result for the case when the nilpotency class of $G$ is not necessary two.

**Theorem 2.** *Let $G$ be an abelian-by-cyclic $p$-group with $p > 2$ such that $\gamma_3(G) \subseteq (G')^p$ and let $F$ be a field of characteristic $p$. Then*

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2 t(G') + 1\rceil.$$

In this article $\omega(FG)$ denotes the augmentation ideal of $FG$; for a normal subgroup $H \subseteq G$ we denote by $\mathfrak{I}(H)$ the ideal $FG \cdot \omega(FH)$. For $x, y \in G$

let $x^y = y^{-1}xy$, $(x, y) = x^{-1}x^y$. By $\zeta(G)$ we mean the center of the group $G$, by $\gamma_n(G)$ the $n$-th term of the lower central series of $G$ with $\gamma_1(G) = G$. Furthermore, we denote by $C_n$ the cyclic group of order $n$.

## 2. Preliminaries and proofs

**Proposition 1.** *Let $G$ be a group and* $\mathrm{char}(F) = 2$. *If $H$ is a subgroup of index 2 of $G$ whose commutator subgroup $H'$ is a finite 2-group, then*

$$\mathrm{dl}_L(FG) \leq \lceil \log_2 t(H') \rceil + 3.$$

*Proof.* Firstly, suppose that $H$ is an abelian subgroup of index 2 of $G$. Then $G = \langle H, b\rangle$ for some $b$ and every $x \in FG$ has a unique representation in the form $x = x_1 + x_2b$, where $x_1, x_2 \in FH$. It is easy to see that the map $u \mapsto \overline{u} = b^{-1}ub$ $(u \in FH)$ is an automorphism of order 2 of $FH$ and for all $x, y \in FG$

$$\begin{aligned}[x, y] &= [x_1 + x_2b, y_1 + y_2b] \\ &= (x_2\overline{y_2} + \overline{x_2}y_2)b^2 + \big((x_1 + \overline{x_1})y_2 + x_2(\overline{y_1} + y_1)\big)b \\ &\equiv w_1b \pmod{\zeta(FG)},\end{aligned}$$

where $w_1 \in FH$ and $\zeta(FG)$ denotes the center of $FG$. Similarly, for $u, v \in FG$ we have $[u, v] \equiv w_2b \pmod{\zeta(FG)}$ for some $w_2 \in FH$. Hence

$$\big[[x, y], [u, v]\big] = [w_1b, w_2b] = (w_1\overline{w_2} + \overline{w_1}w_2)b^2 \in FH.$$

Since the elements of the form $\big[[x, y], [u, v]\big]$ with $x, y, u, v \in FG$ generate $\delta^{[2]}(FG)$ and $FH$ is a commutative algebra, $\delta^{[3]}(FG) = 0$, as asserted.

Let now $H$ be nonabelian. It is clear that $H'$ is normal in $G$ and $H/H'$ is an abelian subgroup of index 2 of $G/H'$, so we can use the result proved above to get $\delta^{[3]}\big(F(G/H')\big) = 0$. In view of $F(G/H') \cong FG/\mathfrak{I}(H')$ we have $\delta^{[3]}(FG) \subseteq \mathfrak{I}(H')$. Hence an easy induction on $k$ yields $\delta^{[3+k]}(FG) \subseteq \mathfrak{I}(H')^{2^k}$ for all $k \geq 0$. Consequently, if $2^k \geq t(H')$, that is $k \geq \lceil \log_2 t(H') \rceil$, then $\delta^{[3+k]}(FG) = 0$, which implies the statement. □

Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. It is well known that the automorphism group $\mathrm{aut}(G')$ of $G'$ is a direct product of the cyclic group $\langle\alpha\rangle$ of order 2 and the cyclic group $\langle\beta\rangle$ of order $2^{n-2}$ where the action of these automorphisms on $G'$ is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. For $g \in G$, let $\tau_g$ denote the restriction to $G'$ of the inner automorphism $h \mapsto h^g$ of $G$. The map $G \to \mathrm{aut}(G)$, $g \mapsto \tau_g$ is a homomorphism whose kernel coincides with the centralizer $C = C_G(G')$. Clearly, the map $\varphi : G/C \to \mathrm{aut}(G')$ given by $\varphi(gC) = \tau_g$ is a monomorphism. In [3] we introduced the subset

$$G_\beta = \big\{g \in G \mid \varphi(gC) \in \langle\beta\rangle\big\}$$

of $G$. Evidently, $G_\beta$ is a subgroup of index not greater than 2. It is shown in [3] that $G = G_\beta$ if and only if $G$ has nilpotency class at most $n$, furthermore under this condition $\mathrm{dl}_L(FG) = n+1$. Combining this fact with Proposition 1 we obtain the following statement.

**Corollary 1.** *Let $G$ be a group with cyclic commutator subgroup of order $2^n$ and let $\mathrm{char}(F) = 2$. If $G'_\beta$ has order $2^r$, then*

$$r + 1 \le \mathrm{dl}_L(FG) \le r + 3.$$

*Proof.* If $G = G_\beta$ then Lemma 3 and Theorem 1 in [3] say $\mathrm{dl}_L(FG) = r + 1$. Otherwise, $G_\beta$ is of index 2 in $G$ and we can apply Proposition 1 to get $\mathrm{dl}_L(FG) \le r + 3$. Furthermore, Lemma 3 and Theorem 1 of [3] ensure that $\mathrm{dl}_L(FG_\beta) = r + 1$. Since $\mathrm{dl}_L(FG_\beta) \le \mathrm{dl}_L(FG)$, the corollary is true. □

Let $\mathrm{char}(F) = 2$ and $H = \langle x \mid x^{2^n} = 1\rangle$. We claim that if $r > 0$ and the $k_j$'s are odd positive integers for $1 \le j \le r$ then the element

$$\varrho = (x^{k_1} + 1)(x^{k_2} + 1)\cdots(x^{k_r} + 1) \in FH$$

is equal to zero if and only if $r \ge 2^n$.

Indeed, $\varrho \in \omega^r(FH)$ and if $r \ge 2^n$ then $\varrho = 0$, because $t(H) = 2^n$. Assume now $r < 2^n$. Applying the identity

$$(x^{k_j} + 1) = (x^{k_j-1} + 1)(x + 1) + (x^{(k_j-1)/2} + 1)^2 + (x + 1)$$

for every $1 \le j \le r$, we can write $\varrho = (x + 1)^r + \varrho_1$, where $\varrho_1$ is the sum of elements of weight greater than $r$. Clearly, $(x+1)^r \in \omega^r(FH) \setminus \omega^{r+1}(FH)$ and $\varrho_1 \in \omega^{r+1}(FH)$, hence $\varrho \in \omega^r(FH) \setminus \omega^{r+1}(FH)$ and $\varrho \ne 0$.

In the sequel we shall use freely this fact.

In the proof of the next lemmas we will use that $C' \subseteq G' \cap \zeta(G)$. This inclusion is indeed valid, because for $a, b, c \in G$ the well-known Hall-Witt identity states that

$$(a, b^{-1}, c)^b(b, c^{-1}, a)^c(c, a^{-1}, b)^a = 1.$$

Evidently, if $b, c \in C$ then this formula yields that $(b, c, a) = 1$, which guarantees our statement.

**Lemma 1.** *Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n > 3$, let $\mathrm{char}(F) = 2$ and assume that $\exp(G/C) \le 2$. Then $\mathrm{dl}_L(FG) = 3$ if and only if $C$ is abelian and $G/C = \langle aC\rangle$, where $x^a = x^{-1}$.*

*Proof.* Since $\exp(G/C) \le 2$, only the following cases are possible:

*Case 1*: either $G/C$ is trivial or $G/C = \langle bC\rangle$ where $x^b = x^{2^{n-1}+1}$. Clearly, $G$ has nilpotency class at most 3, therefore by Theorem 1 in [3] we have $\mathrm{dl}_L(FG) = n + 1$.

*Case 2*: $G/C = \langle aC\rangle$, where $x^a = x^{-1}$. Then $C' \subseteq G' \cap \zeta(G) = \langle x^{2^{n-1}}\rangle$. If $C' = \langle 1\rangle$ then $C$ is an abelian subgroup of index 2 of $G$ and Proposition 1 implies that $\mathrm{dl}_L(FG) = 3$. Now, let $C' = \langle x^{2^{n-1}}\rangle$. Then we can choose $b, c \in C$ such that

$$(c, a) = x, \qquad (c, b) = x^{2^{n-1}}, \qquad (a, b) \in \langle x^2\rangle.$$

Indeed, let us consider the map $\varphi : C \to G'$, where $\varphi(c) = (c, a)$, which is an epimorphism because $G' = (a, C)$. Of course, $H = \varphi^{-1}\big(\langle x^2\rangle\big)$ is a proper subgroup of $C$. Let $u \in C \setminus \zeta(C)$ and $c \in C \setminus \big(H \cup C_C(u)\big)$ be such that $(c, a) = x$. Obviously, $(c, u) = x^{2^{n-1}}$. If $(a, u) \in \langle x^2\rangle$ then set $b = u$, otherwise $b = cu$. It is easy to see that the elements $b$ and $c$ satisfy the conditions stated. Then

$$\begin{aligned}&\Big[\big[[c,a],[c^{-1}a,c]\big],\ \big[[c,a],[c^{-1}ba,c]\big]\Big]\\ &\quad=\big[[ac(x+1),a(x^{-1}+1)],\ [ac(x+1),ba(x^{2^{n-1}-1}+1)]\big]\\ &\quad=\big[a^2cx^{-1}(x+1)^3,\ ba^2c\big((b,a)x^{-1}+1\big)(x^{2^{n-1}+1}+1)(x+1)\big]\\ &\quad= a^4bc^2x^{-1}\big((b,a)x^{-1}+1\big)(x^{2^{n-1}+1}+1)(x+1)^{2^{n-1}+4}\end{aligned}$$

belongs to $\delta^{[3]}(FG)$ and is not equal to zero, thus $\mathrm{dl}_L(FG) > 3$.

*Case 3*: $G/C = \langle dC\rangle$, where $x^d = x^{2^{n-1}-1}$. Since $G' = (d, C)$, similarly as before, we can choose $c \in C$ such that $(c, d) = x$. Then

$$\begin{aligned}&\Big[\big[[c,d],[d^{-1}c,d]\big],\ \big[[c,d],[c,dc]\big]\Big]\\ &\qquad=\Big[\big[dc(x+1),c(x+1)\big],\ \big[dc(x+1),dc^2(x+1)\big]\Big]\\ &\qquad=\big[dc^2(x+1)^{2^{n-1}+1},\ d^2c^3x(x^{2^{n-1}-1}+1)(x+1)^2\big]\\ &\qquad= d^3c^5x(x^{2^{n-1}-1}+1)(x^{2^{n-2}-1}+1)^2(x+1)^{2^{n-1}+2}\end{aligned}$$

is a nonzero element in $\delta^{[3]}(FG)$ so $\mathrm{dl}_L(FG) > 3$.

*Case 4*: $G/C = \langle aC, bC\rangle$, where $x^a = x^{-1}$ and $x^b = x^{2^{n-1}+1}$. Then

$$G' = \langle (ab, b)\rangle(ab, C)(b, C)C' = \langle (a, b)\rangle(ab, C)(b, C),$$

because $C' \subseteq \langle x^{2^{n-1}}\rangle$. Since $G'$ is cyclic, $G'$ coincides with either $\langle (a, b)\rangle$ or $(ab, C)$ or $(b, C)$.

Assume that $G' = (ab, C)$ and set $H = \langle ab, C\rangle$. Then $H$ satisfies the hypothesis of Case 3 of this lemma, so $\mathrm{dl}_L(FG) \geq \mathrm{dl}_L(FH) > 3$. We get the same result in the case $G' = (b, C)$.

There remains the possibility that $(a, b) = y$ is of order $2^n$. Then

$$\begin{aligned}&\Big[\big[[a,b],[b^{-1}a,b]\big],\ \big[[a,b],[b,ab]\big]\Big]\\ &\quad=\Big[\big[ba(y+1),a(y+1)\big],\ \big[ba(y+1),ab^2(y^{2^{n-1}-1}+1)\big]\Big]\\ &\quad=\big[ba^2(y^{2^{n-1}-2}+1)(y+1),\ b^3a^2y^{-1}(y^{2^{n-1}-2}+1)(y+1)\big]\\ &\quad= b^4a^4y^{-1}(y^{-1}+1)^4(y^{2^{n-1}+1}+1)(y+1)^{2^{n-1}+1}\neq 0,\end{aligned}$$

and the statement is valid. □

**Lemma 2.** *Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{16} = 1\rangle$ and let* $\mathrm{char}(F) = 2$*. Then* $\mathrm{dl}_L(FG) = 3$ *if and only if $G$ has an abelian subgroup of index* 2*.*

*Proof.* By the previous lemma, the statement is true if $\exp(G/C) \leq 2$. The other possible cases are:

*Case 1*: $G/C = \langle bC\rangle$, where $x^b = x^5$. Since then $G = G_\beta$, Lemma 3 and Theorem 1 in [3] state that $\mathrm{dl}_L(FG) = 5$.

*Case 2*: $G/C = \langle dC\rangle$, where $x^d = x^{-5}$. Then $G' = (d, C)$ and, as before, we can choose $c \in C$ such that $(c, d) = x$ and

$$\begin{aligned}\Big[\big[[c,d],[d^{-1}c,d]\big],\ \big[[c,d],[c,dc]\big]\Big]&\\ &=\Big[\big[dc(x+1),c(x+1)\big],\ \big[dc(x+1),dc^2(x^{-5}+1)\big]\Big]\\ &=\big[dc^2(x^{-4}+1)(x+1),\ d^2c^3(x^{-5}+1)(x+1)^2\big]\\ &=b^3c^5x^6(x^{-5}+1)(x^9+1)(x+1)^9\end{aligned}$$

belongs to $\delta^{[3]}(FG)$ and is not zero.

*Case 3*: $G/C = \langle aC, bC\rangle$, where $x^a = x^{-1}$ and $x^b = x^5$. Then by similar arguments as in the last case of the previous lemma we can restrict ourselves to the case when $(a, b) = x$. Then

$$\begin{aligned}\Big[\big[[a,b],[b^{-1}a,b]\big],\ \big[[a,b],[b,ab]\big]\Big]&\\ &=\Big[\big[ba(x+1),a(x+1)\big],\ \big[ba(x+1),ab^2(x^{-5}+1)\big]\Big]\\ &=\big[ba^2(x^{10}+1)(x+1),\ b^3a^2(x^{10}+1)(x^7+1)\big]\\ &=b^4a^4x^3(x^5+1)^4(x+1)^6 \neq 0,\end{aligned}$$

which was to be proved. □

Now we are ready to prove our main theorem.

*Proof of Theorem 1.* Suppose first that $p > 7$. Then Theorem A in [10] states that $\mathrm{dl}_L(FG) \geq \lceil \log_2(p+1)\rceil \geq 4$. For odd $p \leq 7$ the statement follows directly from Theorem 1 in [3], Theorem 1 in [1].

Let $G' = \langle x \mid x^{2^n} = 1\rangle$. The result follows from Theorem 1 in [3] for $n = 2$ and $n = 3$. For $n > 3$, using induction on $n$, we shall show that if $\mathrm{dl}_L(FG) = 3$ then $C$ is abelian and $G/C = \langle aC\rangle$, where $x^a = x^{-1}$ (i.e. $G$ has an abelian subgroup of index 2). Indeed, by Lemma 2, this is true for $n = 4$. Let now $n > 4$ and $\mathrm{dl}_L(FG) = 3$ and assume that the statement is true for every group with commutator subgroup of order less than $2^n$. Set $H = \langle x^{2^{n-1}}\rangle \subset G'$. Then $\mathrm{dl}_L\big(F(G/H)\big) = 3$ and $(G/H)' = G'/H = \langle xH\rangle$, and by inductive hypothesis we get

$$(xH)^{gH} = x^g H = x^{(-1)^k} H$$

for all $g \in G$. It follows that $x^g = x^i$ with $i \in \{-1, 1, 2^{n-1} - 1, 2^{n-1} + 1\}$, i.e. $\exp(G/C) \leq 2$ and the statement follows from Lemma 1. $\square$

**Example.** Let $G_i$ be a finite nonabelian 2-group of order $2^m$ and exponent $2^{m-2}$ from the list in [5]. The group algebras of $G_i$ have been examined by several authors, for example V. Bovdi [2]. Our results enable us to determine the derived length of $FG_i$ over a field $F$ of characteristic 2. Using Proposition 1 and Theorem 1 we get

$$\mathrm{dl}_L(FG_i) = \begin{cases} 2, & \text{if either } i \in \{2,3\} \text{ and } \ m = 4 \text{ or } i \in \{1,4,5,9,10\}; \\ 4, & \text{if } \quad i \in \{15,16,18,20,24,25\} \quad \text{and} \quad m > 5; \\ 3, & \text{otherwise.} \end{cases}$$

Note that $G'_{17} \cong G'_{26} \cong C_2 \times C_2$. Then we applied Theorem 3 in [4] to compute the derived length.

Now let us turn to Theorem 2.

**Lemma 3.** *Let $G$ be a group with commutator subgroup of order $p^n$ and* $\mathrm{char}(F) = p$. *If $\gamma_3(G) \subseteq (G')^p$ then for all $m \geq 1$*

$$\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}.$$

*Moreover, if $G'$ is abelian, then for all $m, k \geq 1$*

$$\big[\mathfrak{I}(G')^m, \mathfrak{I}(G')^k\big] \subseteq \mathfrak{I}(G')^{m+k+1}.$$

*Proof.* We use induction on $m$. For every $y \in G'$ and $g \in G$ we have

$$[y-1, g-1] = [y,g] = gy\big((y,g)-1\big) \in \mathfrak{I}\big(\gamma_3(G)\big) \subseteq \mathfrak{I}(G')^p.$$

This shows that the statement holds for $m = 1$, because all elements of the form $g - 1$ with $g \in G$ constitute an $F$-basis of $\omega(FG)$.

Now, assume that $\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}$ for some $m$. Then

$$\begin{aligned} &\big[\omega^{m+1}(FG'), \omega(FG)\big] \\ &\qquad \subseteq \omega^m(FG')\big[\omega(FG'), \omega(FG)\big] + \big[\omega^m(FG'), \omega(FG)\big]\omega(FG') \\ &\qquad \subseteq \omega^m(FG')\mathfrak{I}(G')^p + \mathfrak{I}(G')^{m+p-1}\omega(FG') \subseteq \mathfrak{I}(G')^{m+p}, \end{aligned}$$

and the proof of the first assertion is complete. The second one is a consequence of the first one, because

$$\mathfrak{I}(G') = \omega(FG)\omega(FG') + \omega(FG').$$

$\square$

*Proof of Theorem 2.* Write $G = \langle A, x\rangle$, where $A$ is abelian and normal in $G$. Clearly, $G' = (A, x)$ is abelian. We shall show that for all $c \in A$ and $z_1, z_2, \ldots, z_{2^n-1} \in G'$ and $j$ not divisible by $p$ there exists $\varrho \in \mathfrak{I}(G')^{2^n}$ such that

$$x^j c(1-z_1)(1-z_2)\cdots(1-z_{2^n-1}) + \varrho \in \delta^{[n]}(FG).$$

We use induction on $n$. Let first $n = 1$ and $2k \equiv j$ modulo the order of $x$. Then $G' = (A, x^k)$ and $z_1 = (a_1, x^k)\cdots(a_s, x^k)$ for some $a_1, \dots, a_s \in A$, thus

$$x^j c(1 - z_1) \equiv \sum_{i=1}^{s} x^j c\big(1 - (a_i, x^k)\big) \pmod{\mathfrak{I}(G')^2}. \tag{1}$$

Since $p$ is an odd prime, we can choose the elements $u_i, v_i$ such that $u_i^2 = ca_i^{-1}$ and $v_i^2 = ca_i$. Then $u_i, v_i \in A$, $(u_i v_i)^2 = c^2$ and $(u_i^{-1} v_i)^2 = a_i^2$ which implies $u_i v_i = c$ and $u_i^{-1} v_i = a_i$. Setting $w_i = x^k u_i (x^k)^{-1}$ we have

$$\begin{aligned}[x^k w_i, x^k v_i] &= x^j (w_i^{x^k} v_i - w_i v_i^{x^k}) \\ &= x^j w_i^{x^k} v_i\big(1 - (w_i^{-1} v_i, x^k)\big) = x^j c\big(1 - (a_i, x^k)\big),\end{aligned}$$

because $(w_i^{-1} v_i, x^k) = (u_i^{-1} v_i, x^k) = (a_i, x^k)$. Now by (1) it follows that

$$x^j c(1 - z_1) \equiv \sum_{i=1}^{s} [x^k w_i, x^k v_i] \pmod{\mathfrak{I}(G')^2}, \tag{2}$$

which proves our statement for $n = 1$.

Now, assume that $j, c, z_1, z_2, \dots, z_{2^n - 1}$ have already been given, and let $2k \equiv j$ modulo the order of $x$. We can apply the method above to find elements $w_i, v_i \in A$ such that the congruence (2) holds. Set

$$f_i = x^k w_i (1 - z_2)\cdots(1 - z_{2^{n-1}})$$

and

$$g_i = x^k v_i (1 - z_{2^{n-1}+1})\cdots(1 - z_{2^n - 1}).$$

for $1 \le i \le s$. By the induction hypothesis there exist $\varrho_1^{(i)}, \varrho_2^{(i)} \in \mathfrak{I}(G')^{2^{n-1}}$ such that $f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)} \in \delta^{[n-1]}(FG)$. Evidently,

$$\big[f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)}\big] = [f_i, g_i] + [f_i, \varrho_2^{(i)}] + [\varrho_1^{(i)}, g_i] + [\varrho_1^{(i)}, \varrho_2^{(i)}] \in \delta^{[n]}(FG).$$

According to Lemma 3 the last three summands are in $\mathfrak{I}(G')^{2^n}$. Furthermore,

$$\begin{aligned}\big[f_i, g_i\big] &= x^k w_i \big[(1 - z_2)\cdots(1 - z_{2^{n-1}}), x^k v_i\big](1 - z_{2^{n-1}+1})\cdots(1 - z_{2^n - 1}) \\ &\quad + x^k v_i \big[x^k w_i, (1 - z_{2^{n-1}+1})\cdots(1 - z_{2^n - 1})\big](1 - z_2)\cdots(1 - z_{2^{n-1}}) \\ &\quad + \big[x^k w_i, x^k v_i\big](1 - z_2)\cdots(1 - z_{2^n - 1})\end{aligned}$$

and the first two summands on the right-hand side belong to $\mathfrak{I}(G')^{2^n}$ by Lemma 3. So,

$$\big[f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)}\big] \equiv \big[x^k w_i, x^k v_i\big](1 - z_2)\cdots(1 - z_{2^n - 1}) \pmod{\mathfrak{I}(G')^{2^n}},$$

for all $1 \le i \le s$. Summing this over all possible $i$, we get

$$x^j c(1 - z_1)(1 - z_2)\cdots(1 - z_{2^n - 1}) + \varrho \in \delta^{[n]}(FG),$$

for some $\varrho \in \mathfrak{I}(G')^{2^n}$, as we claimed.

It follows that $\delta^{[n]}(FG)$ has nonzero elements while $2^n - 1 < t(G')$. Hence

$$\mathrm{dl}_L(FG) \ge \lceil \log_2 t(G') + 1 \rceil$$

and the result follows immediately from Proposition 1 in [3]. □

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