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ON SOME SYMMETRIC DESIGNS WITH CLASSICAL PARAMETERS

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Abstract. We describe a construction of sixty-seven symmetric $(31, 15, 7)$ designs, thirty-eight symmetric $(63, 31, 15)$ designs, and two symmetric $(127, 63, 31)$ designs. The orders and the structures of the full automorphism groups of the constructed designs are given, as well as their 2-ranks. The designs are constructed with the help of tactical decompositions.

1. Introduction

A 2-(v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = v$;
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3. every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks. If $|\mathcal{P}| = |\mathcal{B}| = v$ and $2 \le k \le v - 2$, then a 2-(v, k, λ) design is called a symmetric (v, k, λ) design.

Given two designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$, an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence relation. An isomorphism from a symmetric design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of the design $\mathcal{D}$ forms a group; it is called the full automorphism group of $\mathcal{D}$ and denoted by $\mathrm{Aut}(\mathcal{D})$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and G a subgroup of $\mathrm{Aut}(\mathcal{D})$. The action of G produces the same number of point and block orbits (see [9, Theorem 3.3, p. 79]). We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$, (i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. The numbers γ_{ir} are independent of the choice of the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [5]):

$$(1) \qquad \sum_{r=1}^{t} \gamma_{ir} = k\,,$$

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$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} \quad = \quad \lambda\Omega_j + \delta_{ij} \cdot (k-\lambda). \tag{2}$$

Definition 1. Let $(\mathcal{D})$ be a symmetric (v,k,λ) design and $G \leq \mathrm{Aut}(0\mathcal{D})$. Further, let $\mathcal{P}_1,\ldots,\mathcal{P}_t$ be the point orbits and $\mathcal{B}_1,\ldots,\mathcal{B}_t$ the block orbits with respect to G, and let $\omega_1,\ldots,\omega_t$ and $\Omega_1,\ldots,\Omega_t$ be the respective orbit lengths. We call $(\mathcal{P}_1,\ldots,\mathcal{P}_t)$ and $(\mathcal{B}_1,\ldots,\mathcal{B}_t)$ the orbit distributions, and $(\omega_1,\ldots,\omega_t)$ and $(\Omega_1,\ldots,\Omega_t)$ the orbit size distributions for the design and the group G. A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v,k,λ) and orbit distributions $(\mathcal{P}_1,\ldots,\mathcal{P}_t)$ and $(\mathcal{B}_1,\ldots,\mathcal{B}_t)$.

The first step – when constructing designs for given parameters and orbit distributions – is to find all compatible orbit structures (γ_{ir}). The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a chosen representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir}. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 2. The set of all indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i,r) of the orbit structure and the given representative.

Definition 3. Let G be an additively written group of order v not necessarily Abelian. A $k-$subset D of G is a $(v,k,\lambda;n)-$difference set of order $n = k-\lambda$ if every nonzero element of G has exactly λ representations as a difference $d-d'$ with elements from D. The difference set is Abelian, cyclic etc. if the group G has the respective property.

The development of a difference set D is the incidence structure $\mathrm{dev}(D)$ whose points are the elements of G and whose blocks are the translates $D+g = \{d+g|\ d \in D\}$. The existence of a $(v,k,\lambda;n)-$difference set is equivalent to the existence of a symmetric (v,k,λ) design $\mathcal{D}$ admitting G as a point regular automorphism group, i.e. for any two points P and Q there is a unique element of G which maps P to Q. The design $\mathcal{D}$ is isomorphic to $\mathrm{dev}(D)$. The design $\mathcal{D}$ is called cyclic when the difference set is cyclic.

Definition 4. Let $\mathcal{D}$ be an incidence structure with incidence matrix N. The p-rank of $\mathcal{D}$ is defined as the rank of N over a field F of characteristic p. Without loss of generality, we may assume $F = GF(p)$.

For further basic definitions and properties of symmetric designs and difference sets we refer the reader to [1] and [9].

In this paper we describe a construction of symmetric designs with the classical parameters $(2^d-1, 2^{d-1}-1, 2^{d-2}-1)$ for $d = 5,6$ and 7. It is known that there are a lot of designs with these parameters (see [6] and [8]). This article contributes to the classification of such designs which allow certain automorphism groups. We explicitly construct the designs, determine their 2-ranks, and compute the orders of their full automorphism groups. In addition, the structures of the automorphism groups of the designs are given.

For the definition of the basic group theoretic terminology and concepts used in this paper, such as the direct product $N \times H$ of groups N and H, a semidirect product (split extension) $N:H$ of N by H, the derived group G' of G, or elementary Abelian groups, the reader may consult any standard book on group theory, for example [7] or [11].

2. Symmetric (31,15,7) Designs

We shall construct all symmetric (31,15,7) designs having an automorphism group isomorphic to $\mathrm{Frob}_{7\cdot 3}$ or $\mathrm{Frob}_{31\cdot 5}$. The Frobenius group $\mathrm{Frob}_{p\cdot q}$, where p and q are primes, is a non-Abelian group of order $p \cdot q$ which – up to isomorphisms – is unique.

Lemma 1. *Let ρ be an automorphism of a symmetric $(31, 15, 7)$ design $\mathcal{D}$. If $|\langle\rho\rangle| = 7$, then ρ fixes exactly three points and blocks of $\mathcal{D}$.*

Proof. By [9, Theorem 3.1, p. 78], $\langle\rho\rangle$ fixes the same number of points and blocks. Denote that number by f. Obviously, $f \equiv 1\,(mod\,7)$. Using the inequality $f \leq v - 2(k - \lambda)$ (see [9, Corollary 3.7 p. 82]) we get $f \in \{3, 10\}$. Suppose that $f = 10$. Since a fixed block must be a union of $\langle\rho\rangle$-orbits of points, every fixed block contains 1 or 8 fixed points. Two fixed blocks must intersect in 0 or 7 fixed points, since $\lambda = 7$. Therefore, two fixed blocks having one fixed point intersect in an orbit of length 7. So there are at most three fixed blocks which contain only one fixed point. Similarly, there are at most three fixed blocks which have precisely 8 fixed points. Therefore $f \neq 10$. $\square$

Lemma 2. *Let the group $\mathrm{Frob}_{7\cdot 3}$ act as an automorphism group of a symmetric $(31, 15, 7)$ design $\mathcal{D}$. Then $\mathrm{Frob}_{7\cdot 3}$ acts on $\mathcal{D}$ semistandardly with orbit size distribution $(1, 1, 1, 7, 7, 7, 7)$ or $(3, 7, 21)$.*

Proof. Let the group G be isomorphic to the Frobenius group $\mathrm{Frob}_{7\cdot 3}$. Since there is only one isomorphism class of such groups of order 21 we may write

$$G = \langle \rho, \sigma |\ \rho^7 = 1,\, \sigma^3 = 1,\, \rho^\sigma = \rho^2 \rangle.$$

The Frobenius kernel $\langle\rho\rangle$ of order 7 acts on $\mathcal{D}$ semistandardly with three fixed blocks and points and 4 orbits of length 7. Since $\langle\rho\rangle$ is a normal subgroup of G, the element σ of order 3 maps $\langle\rho\rangle$-orbits onto $\langle\rho\rangle$-orbits. Therefore, the group $\mathrm{Frob}_{7\cdot 3}$ acts on $\mathcal{D}$ semistandardly with orbit size distribution $(1, 1, 1, 7, 7, 7, 7)$ or $(3, 7, 21)$. $\square$

The stabilizer of each block from a block orbit of length 7 is conjugate to $\langle\sigma\rangle$. Therefore, the entries of the orbit structures corresponding to point and block orbits of length 7 must satisfy the condition $\gamma_{ir} \equiv 0, 1$ $(mod\ 3)$. Solving equations (1) and (2), we get – up to isomorphism and duality – exactly two solutions for the orbit size distribution $(1, 1, 1, 7, 7, 7, 7)$, the orbit structures OS1 and OS2, and two solutions for the orbit size distribution $(3, 7, 21)$, the orbit structures OS3 and OS4:

OS1	1	1	1	7	7	7	7
1	1	0	0	7	7	0	0
1	0	1	0	7	0	7	0
1	0	0	1	7	0	0	7
7	1	1	1	3	3	3	3
7	1	0	0	3	3	4	4
7	0	1	0	3	4	3	4
7	0	0	1	3	4	4	3

OS2	1	1	1	7	7	7	7
1	1	0	0	7	7	0	0
1	0	1	0	7	0	7	0
1	0	0	1	7	0	0	7
7	0	0	0	3	4	4	4
7	0	1	1	3	4	3	3
7	1	0	1	3	3	4	3
7	1	1	0	3	3	3	4

OS3	3	7	21
3	1	7	7
7	3	3	9
21	1	3	11

OS4	3	7	21
3	1	7	7
7	0	3	12
21	2	3	10

Lemma 3. *Up to isomorphism there are exactly* 54 *symmetric* $(31, 15, 7)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{7\cdot 3}$ *acting with orbit size distribution* $(1, 1, 1, 7, 7, 7, 7)$ *for blocks and points. Among them there are* 4 *self-dual designs and* 25 *pairs of mutually dual designs.*

Proof. The designs have been constructed by the method described in [2] and [4]. We denote the points by $1_0, 2_0, 3_0, 4_i, 5_i, 6_i, 7_i,\ i = 0, 1, \ldots, 6$ and put $G = \langle \rho, \sigma \rangle$ where the generators for G are permutations defined as follows:

$$\rho = (1_0)(2_0)(3_0)(I_0, I_1, \ldots, I_6),\ I = 4, 5, 6, 7,$$
$$\sigma = (1_0)(2_0)(3_0)(K_0)(K_1, K_2, K_4)(K_3, K_6, K_5),\ K = 4, 5, 6, 7.$$

Indexing the fixed part of an orbit structure is a trivial task. Therefore, we shall consider only the right-lower part of order 4 of the orbit structures OS1 and OS2. To eliminate isomorphic structures during the indexing process we have used the permutation which – on each $\langle \rho \rangle$-point-orbit – acts as $x \mapsto 3x$ (mod 7), and certain automorphisms of the orbit structures OS1 and OS2.

As representatives for the block orbits we chose blocks fixed by $\langle \sigma \rangle$. Therefore, the index sets – numbered from 0 to 3 – which could occur in the designs are among the following:

$$0 = \{1, 2, 4\}, \quad 1 = \{3, 5, 6\}, \quad 2 = \{0, 1, 2, 4\}, \quad 3 = \{0, 3, 5, 6\}.$$

The indexing process of the orbit structure OS1 led to 18 designs, denoted by $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{18}$. Among them there are 4 self-dual designs and 7 pairs of mutually dual designs. Duality and self-duality have been determined with the help of C-programs based on the program library Nauty (see [10]) and by comparing the statistics of intersections of any three blocks. The designs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{18}$ are ordered lexicographically. We write down base blocks for the designs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{18}$ in terms of the index sets defined above:

$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
0 0 3 3	0 0 3 3	0 0 3 3	0 0 3 3	0 0 3 3	0 1 2 3
0 3 0 3	0 3 0 3	0 3 1 2	1 2 0 3	1 2 1 2	0 2 1 3
0 3 3 0	1 2 2 1	0 3 2 1	1 2 3 0	1 2 2 1	1 2 2 1

$\mathcal{D}_7$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_{12}$
0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 1	0 0 0 1	0 0 1 1
1 0 2 3	1 1 2 2	0 0 3 2	0 0 3 2	0 1 2 2	0 0 2 2
1 2 1 2	1 2 1 2	0 3 0 2	1 2 0 2	0 2 1 2	0 3 0 3
1 3 2 0	1 2 2 1	0 3 3 1	1 2 3 1	0 3 3 1	0 3 3 0

$\mathcal{D}_{13}$	$\mathcal{D}_{14}$	$\mathcal{D}_{15}$	$\mathcal{D}_{16}$	$\mathcal{D}_{17}$	$\mathcal{D}_{18}$
0 0 1 1	0 0 1 1	0 0 1 1	0 0 1 1	0 1 1 1	0 1 1 1
0 0 2 2	0 0 2 2	0 0 2 2	0 0 2 2	0 0 2 3	0 1 2 2
0 3 0 3	0 3 1 2	1 2 0 3	1 2 1 2	0 2 1 2	0 2 1 2
1 2 2 1	0 3 2 1	1 2 3 0	1 2 2 1	0 3 2 0	0 2 2 1

From these "small" incidence matrices it is easy to obtain incidence matrices in the ordinary form. Pairs of mutually dual designs are $(\mathcal{D}_2, \mathcal{D}_9)$, $(\mathcal{D}_4, \mathcal{D}_{12})$, $(\mathcal{D}_5, \mathcal{D}_{14})$, $(\mathcal{D}_6, \mathcal{D}_{11})$, $(\mathcal{D}_7, \mathcal{D}_{17})$, $(\mathcal{D}_8, \mathcal{D}_{18})$ and $(\mathcal{D}_{10}, \mathcal{D}_{13})$. The designs $\mathcal{D}_1$, $\mathcal{D}_3$, $\mathcal{D}_{15}$, and $\mathcal{D}_{16}$ are self-dual.

The orbit structure OS2 produces up to isomorphism exactly 18 symmetric designs. These designs, denoted by $\mathcal{D}_{19}, \mathcal{D}_{20}, \ldots, \mathcal{D}_{36}$, are presented in terms of the index sets:

$$
\begin{array}{cccccc}
\mathcal{D}_{19} & \mathcal{D}_{20} & \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} & \mathcal{D}_{24} \\
0\,2\,2\,2 & 0\,2\,2\,2 & 0\,2\,2\,3 & 0\,2\,2\,3 & 0\,2\,2\,3 & 0\,2\,2\,3 \\
0\,2\,1\,1 & 0\,2\,1\,1 & 0\,2\,1\,0 & 0\,3\,0\,0 & 0\,3\,0\,0 & 1\,2\,1\,1 \\
0\,1\,2\,1 & 0\,1\,3\,0 & 0\,1\,2\,0 & 0\,0\,3\,0 & 1\,1\,2\,1 & 1\,1\,2\,1 \\
0\,1\,1\,2 & 0\,1\,0\,3 & 0\,1\,1\,3 & 0\,1\,1\,3 & 0\,1\,1\,3 & 0\,1\,1\,3
\end{array}
$$

$$
\begin{array}{cccccc}
\mathcal{D}_{25} & \mathcal{D}_{26} & \mathcal{D}_{27} & \mathcal{D}_{28} & \mathcal{D}_{29} & \mathcal{D}_{30} \\
0\,2\,2\,3 & 0\,2\,3\,3 & 0\,2\,3\,3 & 0\,2\,3\,3 & 0\,3\,3\,3 & 0\,3\,3\,3 \\
1\,3\,0\,1 & 0\,2\,0\,0 & 0\,2\,0\,0 & 1\,2\,0\,1 & 0\,2\,0\,1 & 0\,2\,0\,1 \\
1\,0\,3\,1 & 0\,1\,2\,1 & 0\,1\,3\,0 & 1\,1\,3\,1 & 0\,0\,3\,0 & 1\,1\,2\,1 \\
0\,1\,1\,3 & 0\,1\,1\,2 & 0\,1\,0\,3 & 0\,1\,0\,3 & 0\,1\,0\,2 & 0\,1\,0\,2
\end{array}
$$

$$
\begin{array}{cccccc}
\mathcal{D}_{31} & \mathcal{D}_{32} & \mathcal{D}_{33} & \mathcal{D}_{34} & \mathcal{D}_{35} & \mathcal{D}_{36} \\
0\,3\,3\,3 & 0\,3\,3\,3 & 0\,3\,3\,3 & 0\,3\,3\,3 & 0\,3\,3\,3 & 0\,3\,3\,3 \\
0\,3\,0\,0 & 0\,3\,0\,0 & 0\,3\,0\,0 & 0\,3\,0\,0 & 1\,2\,1\,1 & 1\,2\,1\,1 \\
0\,0\,3\,0 & 0\,0\,3\,0 & 1\,1\,2\,1 & 1\,1\,3\,0 & 1\,1\,2\,1 & 1\,1\,3\,0 \\
0\,0\,0\,3 & 1\,1\,1\,2 & 1\,1\,1\,2 & 1\,1\,0\,3 & 1\,1\,1\,2 & 1\,1\,0\,3
\end{array}
$$

Since the orbit structure OS2 is not self-dual, the dual structure of OS2 also produces 18 designs, dual to the designs constructed from OS2. Let us denote these designs by $\mathcal{D}_{37}, \mathcal{D}_{38}, \ldots, \mathcal{D}_{54}$. $\square$

A computer program by Vladimir D. Tonchev [13] computes the order as well as generators of the full automorphism group for each of the designs found. Another computer program by V.D. Tonchev [13] computes 2-rank of the designs. The orders and the structures of the full automorphism groups, and the 2-ranks of the designs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{36}$ are given in the following table:

$\mathcal{D}$	$\vert\mathrm{Aut}(\mathcal{D})\vert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank	$\mathcal{D}$	$\vert\mathrm{Aut}(\mathcal{D})\vert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank
$\mathcal{D}_1$	9999360	$GL(5,2)$	6	$\mathcal{D}_{19}$	8064	$\mathrm{Aut}(\mathcal{D}_8)$	10
$\mathcal{D}_2$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	9	$\mathcal{D}_{20}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	16
$\mathcal{D}_3$	336	$Z_2.GL(3,2)$	12	$\mathcal{D}_{21}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	10
$\mathcal{D}_4$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	12	$\mathcal{D}_{22}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	13
$\mathcal{D}_5$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	9	$\mathcal{D}_{23}$	21	$\mathrm{Frob}_{7\cdot 3}$	16
$\mathcal{D}_6$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	15	$\mathcal{D}_{24}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	13
$\mathcal{D}_7$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	15	$\mathcal{D}_{25}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	16
$\mathcal{D}_8$	8064	$(\mathrm{Aut}(\mathcal{D}_8))':Z_6$	9	$\mathcal{D}_{26}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	16
$\mathcal{D}_9$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	9	$\mathcal{D}_{27}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	10
$\mathcal{D}_{10}$	21	$\mathrm{Frob}_{7\cdot 3}$	15	$\mathcal{D}_{28}$	21	$\mathrm{Frob}_{7\cdot 3}$	16
$\mathcal{D}_{11}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	15	$\mathcal{D}_{29}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	13
$\mathcal{D}_{12}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	12	$\mathcal{D}_{30}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	16
$\mathcal{D}_{13}$	21	$\mathrm{Frob}_{7\cdot 3}$	15	$\mathcal{D}_{31}$	64512	$(\mathrm{Aut}(\mathcal{D}_{31}))':Z_2$	7
$\mathcal{D}_{14}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	9	$\mathcal{D}_{32}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	10
$\mathcal{D}_{15}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	15	$\mathcal{D}_{33}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	10
$\mathcal{D}_{16}$	2688	$(E_{16}.E_8):\mathrm{Frob}_{7\cdot 3}$	12	$\mathcal{D}_{34}$	336	$E_{16}:\mathrm{Frob}_{7\cdot 3}$	13
$\mathcal{D}_{17}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	15	$\mathcal{D}_{35}$	126	$\mathrm{Frob}_{7\cdot 3}\times S_3$	10
$\mathcal{D}_{18}$	8064	$\mathrm{Aut}(\mathcal{D}_8)$	9	$\mathcal{D}_{36}$	42	$\mathrm{Frob}_{7\cdot 3}\times Z_2$	16

The group $(\mathrm{Aut}(\mathcal{D}_{31}))'$, the derived group of $\mathrm{Aut}(\mathcal{D}_{31})$, is isomorphic to $(E_{64} : GL(3,2)) : Z_3$. The group $(\mathrm{Aut}(\mathcal{D}_{31}))''$ of order 10752 is isomorphic to $E_{64} : GL(3,2)$, a semidirect product of the elementary Abelian group of order 64 by the simple group $GL(3,2)$ of order 168. That means that $(\mathrm{Aut}(\mathcal{D}_{31})''$ has a subgroup $H \cong GL(3,2)$ and a normal subgroup $N \cong E_{64}$, such that $(\mathrm{Aut}(\mathcal{D}_{31})'' \cong NH$ and $N \cap H = 1$. The group $(\mathrm{Aut}(\mathcal{D}_{31}))''$ is perfect, i.e., $(\mathrm{Aut}(\mathcal{D}_{31}))''' = (\mathrm{Aut}(\mathcal{D}_{31}))''$.

The group $(\mathrm{Aut}(\mathcal{D}_8))'$ is isomorphic to $E_{64} : \mathrm{Frob}_{7 \cdot 3}$, so $\mathrm{Aut}(\mathcal{D}_8)$, $\mathrm{Aut}(\mathcal{D}_{18})$ and $\mathrm{Aut}(\mathcal{D}_{19})$ are isomorphic to $(E_{64} : \mathrm{Frob}_{7 \cdot 3}) : Z_6$.

The group $\mathrm{Aut}(\mathcal{D}_{16})$ is a semidirect product of $(\mathrm{Aut}(\mathcal{D}_{16}))'$ by the Frobenius group $\mathrm{Frob}_{7 \cdot 3}$, where $(\mathrm{Aut}(\mathcal{D}_{16}))'$ is isomorphic to $E_{16}.E_8$, an extension of E_{16} by E_8.

The automorphism group $\mathrm{Aut}(\mathcal{D}_3)$ of order 336 is isomorphic to $Z_2.GL(3,2)$, an extension of Z_2 by $GL(3,2)$. Since $\mathrm{Aut}(\mathcal{D}_3)$ does not contain a subgroup isomorphic to $GL(3,2)$, this is not a split extension, i.e., this is not a semidirect product of Z_2 by $GL(3,2)$. The group $\mathrm{Aut}(\mathcal{D}_3)$ is perfect, i.e., $(\mathrm{Aut}(\mathcal{D}_3))' = \mathrm{Aut}(\mathcal{D}_3)$.

The full automorphism groups of the designs $\mathcal{D}_{37}, \mathcal{D}_{38}, \ldots, \mathcal{D}_{54}$ are isomorphic to the full automorphism groups of $\mathcal{D}_{17}, \mathcal{D}_{18}, \ldots, \mathcal{D}_{36}$, respectively, since the respective designs are pairwise dual.

The group structures have been determined with the help of GAP [12].

Remark 1. The design $\mathcal{D}_1$ is a point-hyperplane design in the projective geometry $PG(4,2)$.

Remark 2. In 1975 Hamada and Ohmori had proved (see [3]) that a symmetric $(2^d-1, 2^{d-1}-1, 2^{d-2}-1)$ design $\mathcal{D}$ satisfies

$$\mathrm{rank}_2 \mathcal{D} \geq d+1,$$

with equality if and only if $\mathcal{D}$ is a point-hyperplane design in $PG(d-1,2)$.

It is known (see [1, Lemma 11.5, p. 153]) that if $\mathcal{D}$ is a symmetric (v,k,λ) design and p a prime number dividing $k-\lambda$, then one has the following results:

(1) if p divides k, then $\mathrm{rank}_2 \mathcal{D} \leq \frac{v}{2}$,

(2) if p does not divide k, then $\mathrm{rank}_2 \mathcal{D} \leq \frac{v+1}{2}$.

So $\mathcal{D}_1$ is the unique symmetric (31,15,7) design with $2-$rank equal to 6, and $\mathcal{D}_{20}$, $\mathcal{D}_{23}$, $\mathcal{D}_{25}$, $\mathcal{D}_{26}$, $\mathcal{D}_{28}$, $\mathcal{D}_{30}$, $\mathcal{D}_{36}$ and their duals have maximal 2-rank among all symmetric (31,15,7) designs.

Lemma 4. *Up to isomorphism there are exactly* 21 *symmetric* $(31,15,7)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{7 \cdot 3}$ *acting with orbit size distribution* $(3,7,21)$ *for blocks and points. Among them there are* 3 *self-dual designs and* 9 *pairs of mutually dual designs.*

Proof. Put $G = \langle \rho, \sigma \rangle$ where the generators for G are permutations defined as follows:

$$\rho = (1_0)(2_0)(3_0)(I_0, I_1, \ldots, I_6), \; I = 4,5,6,7,$$
$$\sigma = (1_0, 2_0, 3_0)(4_0)(4_1, 4_2, 4_4)(4_3, 4_6, 4_5)(5_i, 6_{2i}, 7_{4i}), \; i = 0, \ldots, 6.$$

In order to index the row and column of orbit structures OS3 and OS4 that correspond to the orbits of length 21, we shall decompose these orbits in 3 $\langle \rho \rangle$-orbits of length 7, knowing that σ acts on the set of $\langle \rho \rangle$-orbits of points and blocks as the permutation

$$(1,2,3)(4)(5,6,7).$$

That decomposition leeds us to the orbit structures OS1 and OS2 which are computed with respect to the normal subgroup $\langle \rho \rangle$. We shall proceed with indexing for the structures OS1 and OS2, having in mind the action of σ on the sets of points and blocks. We shall omit the trivial task of indexing the fixed part of the orbit structures and take into consideration only the right-lower (4×4)-submatrices. The index sets – numbered from 0 to 69 – which could occur in the designs are among the following:

$$0=\{0,1,2\},\dots,34=\{4,5,6\},35=\{0,1,2,3\},\dots,69=\{3,4,5,6\}.$$

The indexing process for the orbit structure OS1 led to 7 designs. Among them there are 3 self-dual designs and 2 pairs of mutually dual designs. Three of the constructed designs are isomorphic to $\mathcal{D}_1$, $\mathcal{D}_8$ or $\mathcal{D}_{18}$. The other 4 designs are non-isomorphic to the designs $\mathcal{D}_1,\dots,\mathcal{D}_{54}$. Denote them by $\mathcal{D}_{55},\dots,\mathcal{D}_{58}$. These designs are presented in terms of the index sets as follows:

$\mathcal{D}_{55}$	$\mathcal{D}_{56}$	$\mathcal{D}_{57}$	$\mathcal{D}_{58}$
16 0 6 2	16 1 8 12	16 1 8 12	16 3 5 13
0 26 60 62	1 2 48 38	3 10 35 67	1 10 69 42
6 60 32 65	8 48 0 39	5 69 11 49	8 35 11 63
2 62 65 24	12 38 39 6	13 42 63 14	12 67 49 14

The designs $\mathcal{D}_{55}$ and $\mathcal{D}_{56}$ are self-dual, and the designs $\mathcal{D}_{57}$ and $\mathcal{D}_{58}$ are dual mutually.

The indexing process for OS2 led to 7 designs. Three of them are isomorphic to $\mathcal{D}_{19}$, $\mathcal{D}_{31}$ or $\mathcal{D}_{35}$. We denote the other four designs by $\mathcal{D}_{59},\dots,\mathcal{D}_{62}$.

$\mathcal{D}_{59}$	$\mathcal{D}_{60}$	$\mathcal{D}_{61}$	$\mathcal{D}_{62}$
16 35 49 42	16 36 36 36	16 37 45 43	16 37 45 43
0 61 22 6	1 49 31 17	1 40 34 22	3 48 31 30
6 2 57 15	8 25 42 21	8 15 47 20	5 31 39 21
2 29 0 68	12 30 23 35	12 27 29 54	13 30 21 38

The dual structure of OS4 is decomposed to the dual structure of OS2. The indexing process for that orbit structure leads to 7 designs, dual to the designs constructed from OS2. It is clear that three of these designs are isomorphic to the designs described in Lemma 3. We denote the other four designs by $\mathcal{D}_{63},\dots,\mathcal{D}_{66}$. $\square$

The orders and the structures of the full automorphism groups, as well as the 2-ranks of the designs $\mathcal{D}_{55},\dots,\mathcal{D}_{62}$ are given in the following table:

$\mathcal{D}$	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank	$\mathcal{D}$	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank
$\mathcal{D}_{55}$	21	$\mathrm{Frob}_{7\cdot3}$	12	$\mathcal{D}_{59}$	21	$\mathrm{Frob}_{7\cdot3}$	13
$\mathcal{D}_{56}$	21	$\mathrm{Frob}_{7\cdot3}$	12	$\mathcal{D}_{60}$	21	$\mathrm{Frob}_{7\cdot3}$	16
$\mathcal{D}_{57}$	21	$\mathrm{Frob}_{7\cdot3}$	15	$\mathcal{D}_{61}$	21	$\mathrm{Frob}_{7\cdot3}$	13
$\mathcal{D}_{58}$	21	$\mathrm{Frob}_{7\cdot3}$	15	$\mathcal{D}_{62}$	21	$\mathrm{Frob}_{7\cdot3}$	16

Lemma 1, Lemma 2, Lemma 3 and Lemma 4 lead us to the following conclusion:

Theorem 1. *Up to isomorphism there are exactly* 66 *symmetric* $(31,15,7)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{7\cdot3}$. *Among them there are* 6 *self-dual designs and* 30 *pairs of mutually dual designs.*

Theorem 2. *Up to isomorphism there are exactly two symmetric* $(31,15,7)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{31\cdot5}$, *a point-hyperplane design and a self-dual symmetric design* $\mathcal{D}_{67}$. *The full automorphism group of* $\mathcal{D}_{67}$ *is isomorphic to* $\mathrm{Frob}_{31\cdot15}$ *and its 2-rank is 16. The design* $\mathcal{D}_{67}$ *is cyclic.*

Proof. There is only one orbit structure for the parameters (31,15,7) and the group $\mathrm{Frob}_{31\cdot 5}$, namely the orbit structure OS5:

OS5	31
31	15

Indexing for OS5 produces only one design, denoted by $\mathcal{D}_{67}$. The base block of $\mathcal{D}_{67}$ is:

$$1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28.$$

The base block of $\mathcal{D}_{67}$ is a (31,15,7;8)-difference set, so $\mathcal{D}_{67}$ is a cyclic design. □

Theorem 3. *Up to isomorphism there is exactly one symmetric* $(31, 15, 7)$ *design admitting an automorphism group isomorphic to* $\mathrm{Frob}_{31\cdot 3}$. *That design is isomorphic to* $\mathcal{D}_{67}$.

Proof. The orbit structure OS5 is the only orbit structure for the parameters (31,15,7) and the group $\mathrm{Frob}_{31\cdot 3}$. Indexing for OS5 produces only one design, which is isomorphic to $\mathcal{D}_{67}$. □

3. Symmetric (63,31,15) Designs

Theorem 4. *Up to isomorphism there are exactly* 38 *symmetric* $(63, 31, 15)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{31\cdot 5}$. *Among them there are* 2 *self-dual designs and* 18 *pairs of mutually dual designs.*

Proof. Let the group G_1 be isomorphic to the Frobenius group $\mathrm{Frob}_{31\cdot 5}$. We may put

$$G_1 = \langle \rho, \sigma |\ \rho^{31} = 1,\ \sigma^5 = 1,\ \rho^\sigma = \rho^2 \rangle.$$

The orbit structure

OS'	1	31	31
1	0	31	0
31	1	15	15
31	0	15	16

is up to isomorphism the only orbit structure for the parameters (63,31,15) and the group $\mathrm{Frob}_{31\cdot 5}$. We denote the points of a design by $1_0, 2_i, 3_i,\ i = 0, 1, \ldots, 30$ and put $G_1 = \langle \rho, \sigma \rangle$ where the generators for G_1 are permutations defined as follows:

$$\begin{aligned}
\rho &= (1_0)(2_0, 2_1, \ldots, 2_{30})(3_0, 3_1, \ldots, 3_{30}),\\
\sigma &= (1_0)(K_0)(K_1, K_2, K_4, K_8, K_{16})(K_3, K_6, K_{12}, K_{24}, K_{17})\\
&\quad (K_5, K_{10}, K_{20}, K_9, K_{18})(K_7, K_{14}, K_{28}, K_{25}, K_{19})(K_{11}, K_{22}, K_{13}, K_{26}, K_{21})\\
&\quad (K_{15}, K_{30}, K_{29}, K_{27}, K_{23}),\ K = 2, 3.
\end{aligned}$$

The index sets which could occur in the designs are:

$$
\begin{aligned}
0 &= \{1,2,3,4,5,6,8,9,10,12,16,17,18,20,24\},\\
1 &= \{1,2,3,4,6,7,8,12,14,16,17,19,24,25,28\},\\
2 &= \{1,2,3,4,6,8,11,12,13,16,17,21,22,24,26\},\\
3 &= \{1,2,3,4,6,8,12,15,16,17,23,24,27,29,30\},\\
4 &= \{1,2,4,5,7,8,9,10,14,16,18,19,20,25,28\},\\
5 &= \{1,2,4,5,8,9,10,11,13,16,18,20,21,22,26\},\\
6 &= \{1,2,4,5,8,9,10,15,16,18,20,23,27,29,30\},\\
7 &= \{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28\},\\
8 &= \{1,2,4,7,8,14,15,16,19,23,25,27,28,29,30\},\\
9 &= \{1,2,4,8,11,13,15,16,21,22,23,26,27,29,30\},\\
10 &= \{3,5,6,7,9,10,12,14,17,18,19,20,24,25,28\},\\
11 &= \{3,5,6,9,10,11,12,13,17,18,20,21,22,24,26\},\\
12 &= \{3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\},\\
13 &= \{3,6,7,11,12,13,14,17,19,21,22,24,25,26,28\},\\
14 &= \{3,6,7,12,14,15,17,19,23,24,25,27,28,29,30\},\\
15 &= \{3,6,11,12,13,15,17,21,22,23,24,26,27,29,30\},\\
16 &= \{5,7,9,10,11,13,14,18,19,20,21,22,25,26,28\},\\
17 &= \{5,7,9,10,14,15,18,19,20,23,25,27,28,29,30\},\\
18 &= \{5,9,10,11,13,15,18,20,21,22,23,26,27,29,30\},\\
19 &= \{7,11,13,14,15,19,21,22,23,25,26,27,28,29,30\},\\
20 &= \{0,1,2,3,4,5,6,8,9,10,12,16,17,18,20,24\},\\
21 &= \{0,1,2,3,4,6,7,8,12,14,16,17,19,24,25,28\},\\
22 &= \{0,1,2,3,4,6,8,11,12,13,16,17,21,22,24,26\},\\
23 &= \{0,1,2,3,4,6,8,12,15,16,17,23,24,27,29,30\},\\
24 &= \{0,1,2,4,5,7,8,9,10,14,16,18,19,20,25,28\},\\
25 &= \{0,1,2,4,5,8,9,10,11,13,16,18,20,21,22,26\},\\
26 &= \{0,1,2,4,5,8,9,10,15,16,18,20,23,27,29,30\},\\
27 &= \{0,1,2,4,7,8,11,13,14,16,19,21,22,25,26,28\},\\
28 &= \{0,1,2,4,7,8,14,15,16,19,23,25,27,28,29,30\},\\
29 &= \{0,1,2,4,8,11,13,15,16,21,22,23,26,27,29,30\},\\
30 &= \{0,3,5,6,7,9,10,12,14,17,18,19,20,24,25,28\},\\
31 &= \{0,3,5,6,9,10,11,12,13,17,18,20,21,22,24,26\},\\
32 &= \{0,3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\},\\
33 &= \{0,3,6,7,11,12,13,14,17,19,21,22,24,25,26,28\},\\
34 &= \{0,3,6,7,12,14,15,17,19,23,24,25,27,28,29,30\},\\
35 &= \{0,3,6,11,12,13,15,17,21,22,23,24,26,27,29,30\},\\
36 &= \{0,5,7,9,10,11,13,14,18,19,20,21,22,25,26,28\},\\
37 &= \{0,5,7,9,10,14,15,18,19,20,23,25,27,28,29,30\},\\
38 &= \{0,5,9,10,11,13,15,18,20,21,22,23,26,27,29,30\},\\
39 &= \{0,7,11,13,14,15,19,21,22,23,25,26,27,28,29,30\}.
\end{aligned}
$$

Indexing for the orbit structure OS' leads us to 38 mutually non-isomorphic symmetric designs, denoted by $\mathcal{D}_1^1, \dots, \mathcal{D}_{38}^1$ and listed below.

$\mathcal{D}_1^1$ 3 3 3 36	$\mathcal{D}_2^1$ 3 3 4 35	$\mathcal{D}_3^1$ 3 3 5 34	$\mathcal{D}_4^1$ 3 3 8 31	$\mathcal{D}_5^1$ 3 3 10 29	$\mathcal{D}_6^1$ 3 3 13 26	$\mathcal{D}_7^1$ 3 3 15 24
$\mathcal{D}_8^1$ 3 3 18 21	$\mathcal{D}_9^1$ 3 4 3 35	$\mathcal{D}_{10}^1$ 3 4 5 26	$\mathcal{D}_{11}^1$ 3 4 10 21	$\mathcal{D}_{12}^1$ 3 4 15 31	$\mathcal{D}_{13}^1$ 3 5 3 34	$\mathcal{D}_{14}^1$ 3 5 4 26
$\mathcal{D}_{15}^1$ 3 5 10 24	$\mathcal{D}_{16}^1$ 3 5 18 31	$\mathcal{D}_{17}^1$ 3 8 3 31	$\mathcal{D}_{18}^1$ 3 10 3 29	$\mathcal{D}_{19}^1$ 3 10 4 21	$\mathcal{D}_{20}^1$ 3 10 5 24	$\mathcal{D}_{21}^1$ 3 10 13 31
$\mathcal{D}_{22}^1$ 3 13 3 26	$\mathcal{D}_{23}^1$ 3 13 10 31	$\mathcal{D}_{24}^1$ 3 15 3 24	$\mathcal{D}_{25}^1$ 3 15 4 31	$\mathcal{D}_{26}^1$ 3 18 3 21	$\mathcal{D}_{27}^1$ 3 18 5 31	$\mathcal{D}_{28}^1$ 4 3 4 36
$\mathcal{D}_{29}^1$ 4 3 8 24	$\mathcal{D}_{30}^1$ 4 3 13 34	$\mathcal{D}_{31}^1$ 4 3 18 29	$\mathcal{D}_{32}^1$ 4 4 3 36	$\mathcal{D}_{33}^1$ 4 4 4 35	$\mathcal{D}_{34}^1$ 4 4 5 34	$\mathcal{D}_{35}^1$ 4 4 15 24
		$\mathcal{D}_{36}^1$ 4 5 4 34	$\mathcal{D}_{37}^1$ 4 5 18 24	$\mathcal{D}_{38}^1$ 4 15 4 24		

Pairs of dual designs are: $(\mathcal{D}_2^1, \mathcal{D}_9^1)$, $(\mathcal{D}_3^1, \mathcal{D}_{13}^1)$, $(\mathcal{D}_4^1, \mathcal{D}_{17}^1)$, $(\mathcal{D}_5^1, \mathcal{D}_{18}^1)$, $(\mathcal{D}_6^1, \mathcal{D}_{22}^1)$, $(\mathcal{D}_7^1, \mathcal{D}_{24}^1)$, $(\mathcal{D}_8^1, \mathcal{D}_{26}^1)$, $(\mathcal{D}_{10}^1, \mathcal{D}_{14}^1)$, $(\mathcal{D}_{11}^1, \mathcal{D}_{19}^1)$, $(\mathcal{D}_{12}^1, \mathcal{D}_{25}^1)$,$(\mathcal{D}_{15}^1, \mathcal{D}_{20}^1)$, $(\mathcal{D}_{16}^1, \mathcal{D}_{27}^1)$, $(\mathcal{D}_{21}^1, \mathcal{D}_{23}^1)$, $(\mathcal{D}_{28}^1, \mathcal{D}_{32}^1)$, $(\mathcal{D}_{29}^1, \mathcal{D}_{37}^1)$, $(\mathcal{D}_{30}^1, \mathcal{D}_{31}^1)$, $(\mathcal{D}_{34}^1, \mathcal{D}_{36}^1)$ and $(\mathcal{D}_{35}^1, \mathcal{D}_{38}^1)$. The designs $\mathcal{D}_1^1$ and $\mathcal{D}_{33}^1$ are self-dual. □

The orders and the structures of the full automorphism groups, as well as the 2-ranks of the designs $\mathcal{D}_1^1, \dots, \mathcal{D}_{38}^1$ are given in the following table:

$\mathcal{D}$	$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank
$\mathcal{D}^1_1$	20158709760	$GL(6,2)$	7
$\mathcal{D}^1_2$	155	$\mathrm{Frob}_{31\cdot 5}$	17
$\mathcal{D}^1_3$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_4$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_5$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_6$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_7$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_8$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_9$	155	$\mathrm{Frob}_{31\cdot 5}$	17
$\mathcal{D}^1_{10}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{11}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{12}$	155	$\mathrm{Frob}_{31\cdot 5}$	32
$\mathcal{D}^1_{13}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_{14}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{15}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{16}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{17}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_{18}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_{19}$	155	$\mathrm{Frob}_{31\cdot 5}$	22

$\mathcal{D}$	$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	Structure of $\mathrm{Aut}(\mathcal{D})$	2-rank
$\mathcal{D}^1_{20}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{21}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{22}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_{23}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{24}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{25}$	155	$\mathrm{Frob}_{31\cdot 5}$	32
$\mathcal{D}^1_{26}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	12
$\mathcal{D}^1_{27}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{28}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	17
$\mathcal{D}^1_{29}$	155	$\mathrm{Frob}_{31\cdot 5}$	32
$\mathcal{D}^1_{30}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{31}$	155	$\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{32}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	17
$\mathcal{D}^1_{33}$	465	$\mathrm{Frob}_{31\cdot 15}$	17
$\mathcal{D}^1_{34}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{35}$	465	$\mathrm{Frob}_{31\cdot 15}$	32
$\mathcal{D}^1_{36}$	4960	$E_{32}:\mathrm{Frob}_{31\cdot 5}$	22
$\mathcal{D}^1_{37}$	155	$\mathrm{Frob}_{31\cdot 5}$	32
$\mathcal{D}^1_{38}$	465	$\mathrm{Frob}_{31\cdot 15}$	32

Remark 3. The design $\mathcal{D}_1^1$ is a point-hyperplane design in the projective geometry $PG(5,2)$.

4. Symmetric (127,63,31) Designs

Theorem 5. *Up to isomorphism there are exactly two symmetric* $(127,63,31)$ *designs admitting an automorphism group isomorphic to* $\mathrm{Frob}_{127\cdot 21}$*. Let us denote these designs by* $\mathcal{D}_1^2$ *and* $\mathcal{D}_2^2$*. Both designs are self-dual and cyclic. The* 2*-ranks of* $\mathcal{D}_1^2$ *and* $\mathcal{D}_2^2$ *are* 22 *and* 64*, respectively.*

Proof. The orbit structure

OS"	127
127	63

is the only orbit structure for the parameters (127,63,31) and the group $\mathrm{Frob}_{127\cdot 21}$. Indexing for OS" produces two self-dual designs, denoted by $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$.

The base block of $\mathcal{D}_1^2$ is:

$$1,2,3,4,5,6,7,8,10,12,14,16,19,20,23,24,25,27,28,32,33,38,40,46,47,$$
$$48,50,51,54,56,57,61,63,64,65,66,67,73,75,76,77,80,87,89,92,94,95,$$
$$96,97,100,101,102,107,108,111,112,114,117,119,122,123,125,126.$$

and the base block of $\mathcal{D}_2^2$ is:

$$1,2,4,8,9,11,13,15,16,17,18,19,21,22,25,26,30,31,32,34,35,36,37,38,$$
$$41,42,44,47,49,50,52,60,61,62,64,68,69,70,71,72,73,74,76,79,81,82,$$
$$84,87,88,94,98,99,100,103,104,107,113,115,117,120,121,122,124.$$

The base blocks of $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ are (127,63,31;32)-difference sets, so $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ are cyclic designs. □

The full automorphism groups of the designs $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ are isomorphic to $\mathrm{Frob}_{127\cdot 21}$ and $\mathrm{Frob}_{127\cdot 63}$, respectively.

Theorem 6. *Up to isomorphism there is exactly one symmetric* $(127,63,31)$ *design admitting an automorphism group isomorphic to* $\mathrm{Frob}_{127\cdot 9}$*. That design is isomorphic to* $\mathcal{D}_2^2$*.*

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