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SOME INEQUALITIES CONCERNING THE EXISTENCE OF (k, λ, l)-CHORDAL POLYGONS

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Abstract. The paper deals with some inequalities concerning the existence of k-chordal polygons and its generalized variant, the (k, λ, l)-chordal polygons, i.e. the k-chordal polygons which sides $a_1, \dots, a_n$ have the property that $a_1^\lambda, \dots, a_n^\lambda; \lambda \in \mathbf{R}_+$ are the side lengths of an another l-chordal polygon. In fact, the hypothesis and results obtained in the paper [1] are generalized and discussed.

1. Introduction

This article is an addendum to the paper [1] and primarily deals with the hypothesis given there. For convenience we shall first repeat briefly some definitions and the results of the Corollary 1.2. from the quoted article.

Definition 1.1. Let $\underline{A} = A_1A_2 \cdots A_n$ be a chordal polygon and let $\mathcal{C}^n_{\underline{A}}$ be its circumcircle. By S_{A_i} and $\widehat{S}_{A_i}$ we denote the semicircles such that

$$S_{A_i} \cup \widehat{S}_{A_i} = \mathcal{C}^n_{\underline{A}}, \qquad A_i \in S_{A_i} \cap \widehat{S}_{A_i}.$$

The polygon $\underline{A}$ is said to be of the first kind if the following is fulfilled:

(1) not all vertices $A_1, A_2, \cdots, A_n$ lie on the same semicircle;
(2) for every three consecutive vertices A_i, A_{i+1}, A_{i+2} it is valid

$$A_i \in S_{A_{i+1}} \Longrightarrow A_{i+2} \in \widehat{S}_{A_i};$$

(3) any two consecutive vertices A_i, A_{i+1} do not lie on the same diameter.

Definition 1.2. Let $\underline{A} = A_1A_2 \cdots A_n$ be a chordal polygon and let k be a positive integer. The polygon $\underline{A}$ is said to be k-inscribed and called k-chordal polygon if it is of the first kind and if $\sum_{i=1}^n \angle A_iCA_{i+1} = 2k\pi$, where C denotes the centre of the circumscribed circle to the polygon $\underline{A}$.

For the sake of the simplicity we write β_i for $\angle CA_iA_{i+1}$ in the sequel according to the notations introduced in [1]. Now, it is easy to see that if $\underline{A}$ is k-chordal, then

$$\sum_{i=1}^n \beta_i = (n - 2k)\pi. \tag{1}$$

It is important to report on the following result concerning the sum of the side lengths in the k-chordal polygons, as well.

Denote in the sequel $a^\star = \max_{1\le j\le n} a_j$.

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Corollary 1.1. *([1], Corollary 1.2.) If $a_1, a_2, \ldots, a_n$ are the side lengths of the k-chordal polygon $\underline{A}$, then*

$$\sum_{i=1}^{n} a_i > 2ka^\star. \tag{2}$$

Remark 1.1. The following question immediately arises: *If $a_1, a_2, \cdots, a_n$ are the given lengths so that (2) holds, is there always a k-chordal polygon with the sides of prescribed lengths?* The answer is negative! For example, when we specify $a_i = 9 + i$; $i = \overline{1,5}$ and we are looking for the 2-chordal pentagon, then (2) is indeed satisfied, but no angles $\beta_1, \cdots, \beta_5$ are there so that

$$\begin{aligned} \beta_1 + \cdots + \beta_5 &= \frac{\pi}{2}, \qquad 0 < \beta_i < \frac{\pi}{2} \\ \frac{\cos\beta_1}{a_1} &= \cdots = \frac{\cos\beta_5}{a_5}. \end{aligned}$$

2. Existence of k-chordal polygons

At first we give a very probable new hypothesis reading as follows:

Hypothesis. *Let the lengths $a_1, a_2, \ldots, a_n$ be such that*

$$\sum_{i=1}^{n} a_i^{2m-1} > 2m(a^\star)^{2m-1}, \tag{3}$$

where $m = \left[\frac{n-1}{2}\right]$, i.e. $m = \frac{n-1}{2}$ if n is odd and $m = \frac{n}{2} - 1$ as n is even. Then for each $k = 1, 2, \ldots, m$ there exists a k-chordal polygon, sides of which possess the already given length $a_1, a_2, \ldots, a_n$.

It is very difficult to find out some convenient approximation for proving the Hypothesis. Anyway, the following result gives some progress in this direction.

Theorem 2.1. *Let (3) be valid and let k be a fixed positive integer so that*

$$(2m-1)\mathbf{S}_m + \mathbf{R}_m > (2k-1)\frac{\pi}{2}, \tag{4}$$

where

$$\mathbf{S}_m := 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \cdots + \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\cdot\frac{1}{2m+1}, \tag{5}$$

$$\mathbf{R}_m := \sum_{i=m+1}^{\infty} \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i}\cdot\frac{1}{2i+1}\cdot\frac{a_2^{2i-1} + \cdots + a_n^{2i-1}}{a_1^{2i-1}}. \tag{6}$$

Then there is a k-chordal polygon the sides of which have the length $a_1, a_2, \ldots, a_n$.

Proof. Suppose $a^\star = a_1$. From (3) it follows that for each $k = 1, 2, \ldots, m$ it is

$$\sum_{i=2}^{n} \left(\frac{a_i}{a_1}\right)^{2m-1} > 2m - 1 \quad\Longrightarrow\quad \sum_{i=2}^{n} \left(\frac{a_i}{a_1}\right)^{2k-1} > 2m - 1.$$

Hence, we get the inequality

$$a_2^{2k-1} + \cdots + a_n^{2k-1} > (2m-1)a_1^{2k-1}. \tag{7}$$

Now, we will show that for each positive integer k satisfying (4) there are the angles $\beta_1, \beta_2, \ldots, \beta_n$ so that

$$\begin{aligned} \sum_{i=1}^{n} \beta_i = (n-2k)\frac{\pi}{2}; &\qquad 0 < \beta_i < \frac{\pi}{2}, \\ \frac{\cos\beta_1}{a_1} = \cdots &= \frac{\cos\beta_n}{a_n}. \end{aligned}$$

For this goal it is sufficient to show that the angles $\gamma_1, \gamma_2, \ldots, \gamma_n$ exist with the properties

$$\cos\gamma_i = \frac{a_i}{a_1}\cos\gamma_1,\ \gamma_1 = 0,$$

$$\sum_{i=2}^{n}\arccos\left(\frac{a_i}{a_1}\right) < (n-2k)\frac{\pi}{2}. \tag{8}$$

To prove this inequality, we use the well-known equality

$$\sum_{i=2}^{n}\arccos\left(\frac{a_i}{a_1}\right) = (n-1)\frac{\pi}{2} - \sum_{i=2}^{n}\arcsin\left(\frac{a_i}{a_1}\right).$$

The power-series expansion of the sum of arcsines in the above equality and (7) give us:

$$\begin{aligned}\frac{a_2+\cdots+a_n}{a_1} \quad &+ \quad \frac{1}{2}\cdot\frac{a_2^3+\cdots+a_n^3}{3a_1^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{a_2^5+\cdots+a_n^5}{5a_1^5} + \cdots \\ &> \quad (2m-1)\mathbf{S}_m + \mathbf{R}_m > (2k-1)\frac{\pi}{2},\end{aligned}$$

an equivalent form of (8). This ends the proof of the Theorem 2.1. □

Remark 2.1. Here we have to point out that for each positive integer $m < 10$ and for all positive integers $k < 9$ it is valid

$$(2m-1)\mathbf{S}_m > (2k-1)\frac{\pi}{2}. \tag{9}$$

Consequently, if $n \le 20$ having in mind that (3) holds, there exists a k-chordal polygon with n sides for at least $k = 1, 2, \ldots, m-1$.

For $k = m-1$ we define $\mathbf{V}_m := \frac{(2m-1)\mathbf{S}_m}{2m-1}$, and we are looking for its minimum. It is not difficult to see that $\min_{m\in\mathbf{N}}\mathbf{V}_m = \mathbf{V}_{33}$.

So it is quite possible that there is a m-chordal polygon as well, since the inequality (9) replaces now the inequality (4).

At this point we are focusing to the following important question. Assume that $\underline{A}$ is k-chordal polygon. It is natural to ask for the radius of the circumcircle $\mathcal{C}_{\underline{A}}^n$ of $\underline{A}$ in terms of its side lengths $a_1, a_2, \ldots, a_n$. (The convex case, i.e. the 1-chordal case is well-known for triangles and quadrilaterals. It is well-known that no explicite formula is there for the radius of $\mathcal{C}_{\underline{A}}^n$ for $n \ge 5$ in general). The results in continuation completely describe the computation method of the mentioned radius of the circumcircle $\mathcal{C}_{\underline{A}}^n$.

Theorem 2.2. *Let $\mathcal{C}_{\underline{A}}^n$ be the circumcircle of the k-chordal polygon $\underline{A}$, with given side lengths $a_1, a_2, \ldots, a_n$. Then no permutation of the sides of $\underline{A}$ changes the radius ϱ of the depending circumcircles $\mathcal{C}_{\underline{A}}^n$.*

Moreover, let $\mathcal{K}$ be a circle with the fixed radius $R > \frac{a^\star}{2}$ and let φ_i be the central angle corresponding to certain chord in $\mathcal{K}$ of length a_i. Then ϱ is the unique solution of the equation

$$\sum_{i=1}^{n}\arcsin\left(\frac{R}{\varrho}\sin\frac{\varphi_i}{2}\right) = k\pi. \tag{10}$$

Proof. Denote α_i the central angle of $\mathcal{C}_{\underline{A}}^n$ corresponding to the side with the length $a_i,\ i = 1, 2, \ldots, n$ and let ϱ be its radius. Then taking a circle $\mathcal{K}$ with radius $R > \frac{a^\star}{2}$, we inscribe the lengths a_i consecutively into $\mathcal{K}$. Obviously, knowing a_i and R, the central angles φ_i are uniquely determined in $\mathcal{K}$. Now, it is easy to see that $R\sin\frac{\varphi_i}{2} = \varrho\sin\frac{\alpha_i}{2}$, and therefore, bearing in mind that $\underline{A}$ is k-chordal, it immediately follows (10). Since the sums in (10) are invariant with respect to any permutation of its addenda, the proof of the first assertion of the theorem is complete.

Now, it remains to show that (10) possesses unique solution in ϱ. In this goal write

$$f_k(\varrho) = \sum_{i=1}^{n} \arcsin\left(\frac{R}{\varrho}\sin\frac{\varphi_i}{2}\right) - k\pi.$$

Because of the radius $R > \frac{a^\star}{2}$ can be taken so that the endpoint of the last length inscribed into $\mathcal{K}$ does not coincide with the first endpoint of the first inscribed length we have

$$\sum_{i=1} \varphi_i < 2k\pi.$$

Therefore it is clear that

$$f_k(R) = \frac{1}{2}\sum_{i=1}^{n} \varphi_i - k\pi < 0.$$

Put

$$r = \frac{R}{k\pi}\sum_{i=1}^{n} \sin\frac{\varphi_i}{2}.$$

By the help of the well-known result $\arcsin x > x$ we clearly get

$$f_k(r) > \frac{R}{r}\sum_{i=1}^{n} \sin\frac{\varphi_i}{2} - k\pi = 0.$$

On the other hand $f_k(\varrho)$ monotonously decreases on the interval $[r, R]$, since its first derivative is negative:

$$f_k'(\varrho) = -\frac{R}{\varrho^2}\sum_{i=1}^{n} \frac{\sin\frac{\varphi_i}{2}}{\sqrt{1 - \left(\frac{R}{\varrho}\right)^2 \sin^2\frac{\varphi_i}{2}}} < 0.$$

This proves together with $f_k(r)f_k(R) < 0$ that the equation (10) has the unique root in ϱ on (r, R). $\square$

3. On (k, λ, l)-chordal polygons

The following result is in fact a converse of the Theorem 2.1.

Theorem 3.1. *Let $\mathcal{C}^n_{\underline{A}}$ be any given k-chordal polygon with $a_1, a_2, \ldots, a_n$ as being the lengths of its sides. If $l \in \mathbf{N}$, $\lambda \in \mathbf{Q}_+$ so that*

$$(n - 2k)\lambda = n - 2l, \tag{11}$$

then

$$a_1^\lambda + a_2^\lambda + \cdots + a_n^\lambda > 2l(a^\star)^\lambda. \tag{12}$$

Proof. Denote ϱ the radius of the circumcircle $\mathcal{C}^n_{\underline{A}}$. For the angles $\beta_i \in (0, \frac{\pi}{2})$; $i = \overline{1, n}$ then it is $\sum_{i=1}^n \beta_i = (n - 2k)\frac{\pi}{2}$ and

$$\frac{\cos\beta_1}{a_1} = \cdots = \frac{\cos\beta_n}{a_n} = \frac{1}{2\varrho} \tag{13}$$

because $\mathcal{C}^n_{\underline{A}}$ is k-chordal by the assumption. By reason of simplicity put $t_i := \frac{2}{\pi}\beta_i$. That means $\sum_{i=1}^n t_i = n - 2k$. It is well-known that $\cos\left(\frac{\pi}{2}t_i\right) > 1 - t_i$ on $(0, 1)$.

It is also not hard to show (e.g. by the properties of the weighted means and the Bernoulli inequality) that using (11) we get

$$\sum_{i=1}^{n} \cos^{\lambda}\left(\frac{\pi}{2}t_i\right) > \sum_{i=1}^{n}(1-t_i)^{\lambda} \geq n\left(\frac{2k}{n}\right)^{\lambda} \geq 2l.$$

It is straightforward that $2l \geq 2l\cos^{\lambda}\beta_j$ for all $j = \overline{1,n}$ therefore

$$\sum_{i=1}^{n} \cos^{\lambda}\beta_i > 2l\cos^{\lambda}\beta_j.$$

Now, multiplying the last display with $(2\varrho)^{\lambda}$, by means of (13), the inequality (12) follows. □

Remark 3.1. Taking e.g. $n = 11, k = 5, \lambda = 3$ it is $l = 4$, so

$$a_1^3 + \cdots + a_{11}^3 > 8(a^{\star})^3.$$

In this section of the paper we introduce the so-called (k,λ,l)-chordal polygons and we extend the already given results to this very general geometrical concept according to the Definitions 1.1 and 1.2.

Definition 3.1. When $\underline{A}$ is a k-chordal with the sides of the length $a_1,\dots,a_n$ and in the same time $a_1^{\lambda}, a_2^{\lambda},\dots,a_n^{\lambda};\ \lambda \in \mathbf{R}_+$ are the side lengths of a l-chordal polygon, then $\underline{A}$ is called (k,λ,l)-chordal polygon.

The following theorem is related in some way to the Theorems 2.1. and 3.1. and in many aspects may be considered as the main result of the paper.

Theorem 3.2. *Let $a_1, a_2, \dots, a_n$ be any given lengths satisfying (2) for some fixed positive integer $k \leq \left[\frac{n-1}{2}\right]$. If $l \in \mathbf{N},\ \lambda \in (1,\infty)$ satisfy the condition*

$$\arcsin\left(\frac{2k}{n}\right)^{\lambda} > \frac{l\pi}{n}, \tag{14}$$

then there is a (j,λ,l)-chordal polygon, $j = \overline{1,l}$, the side lengths of which are $a_1,\dots,a_n$.

Also, whenever it holds

$$\arcsin\left(\frac{2k}{n}\right) > \frac{l\pi}{n}, \tag{15}$$

then there is a (l,μ,j)-chordal polygon with the side lengths $a_1,a_2,\dots,a_n$, where $j = \overline{1,l}$ and $\mu \in (1,\mu_0]$. Here μ_0 is the solution in μ of the equation

$$\arcsin\left(\frac{2k}{n}\right)^{\mu} = \frac{l\pi}{n}.$$

Proof. Assume that $a^{\star} = \max_{1\leq i\leq n} a_i$ (cf. [1], Theorem 1.). By (2) it follows that, for at least $l = \lambda = 1$ it is valid

$$\sum_{i=1}^{n} \arcsin\left(\frac{a_i}{a^{\star}}\right)^{\lambda} > l\pi. \tag{16}$$

To show this consider the left-hand expression L in (16). Because of the convexity and the monotony of arcsine and the function $x^{\lambda},\ \lambda \geq 1$ it follows that:

$$\begin{aligned} L &= n\sum_{i=1}^{n}\frac{1}{n}\arcsin\left(\frac{a_i}{a^{\star}}\right)^{\lambda} \geq n\arcsin\left\{\sum_{i=1}^{n}\frac{1}{n}\left(\frac{a_i}{a^{\star}}\right)^{\lambda}\right\} \\ &\geq n\arcsin\left(\frac{1}{n}\sum_{i=1}^{n}\frac{a_i}{a^{\star}}\right)^{\lambda} > n\arcsin\left(\frac{2k}{n}\right)^{\lambda}. \end{aligned}$$

Finally we get by (14) that

$$L > n \arcsin\left(\frac{2k}{n}\right)^{\lambda} > l\pi.$$

Thus, (16) holds for all $l \in \mathbf{N},\ \lambda \in (1,\infty)$, that satisfy (14). But, this is equivalent to

$$\sum_{i=1}^{n} \arccos\left(\frac{a_i}{a^\star}\right)^{\lambda} < (n-2l)\frac{\pi}{2}.$$

Now it is not hard to see that there is an unique $\theta \in (0,1)$ so that

$$\sum_{i=1}^{n} \arccos\left[\left(\frac{a_i}{a^\star}\right)^{\lambda}\theta\right] = (n-2l)\frac{\pi}{2}.$$

Also, there is unique $\beta^\star \in (0,\frac{\pi}{2})$ so that $\theta = \cos\beta^\star$. Consecutively putting

$$\beta_i = \arccos\left[\left(\frac{a_i}{a^\star}\right)^{\lambda}\cos\beta^\star\right],\quad i=\overline{1,n},$$

we get the angles β_i of a l-chordal polygon which sides are of the length $a_1^\lambda,\ldots,a_n^\lambda$. Finally, as

$$\arcsin\left(\frac{2k}{n}\right) > \arcsin\left(\frac{2k}{n}\right)^{\lambda} > \frac{j\pi}{n};\qquad \lambda>1;\quad j=\overline{1,l},$$

it follows that $a_1,a_2,\ldots,a_n$ are the side lengths of an l-chordal polygon for the same reasons as before.

Now, it remains to prove the second part of the theorem, when (15) is assumed. Precisely speaking the condition

$$\arcsin\left(\frac{2k}{n}\right) > \frac{l\pi}{n}$$

enables the interpolation

$$\arcsin\left(\frac{2k}{n}\right) > \arcsin\left(\frac{2k}{n}\right)^{\mu} > \frac{l\pi}{n},$$

using some real parameter $\mu > 1$.

Thus, the proof of the Theorem is complete. □

Example 3.1. It may be interesting that, for example

n	k	λ	l
5	2	1	1
7	3	1	1,2
15	7	1	1,2,3,4,5
15	7	2	1,2,3,4,5
15	7	4.5	1,2,3

Theorem 3.3. *Let $a_1,a_2,\ldots,a_n$ be any given lengths satisfying*

$$|a_j - M| < \frac{M}{2k},\qquad j=1,2,\ldots,n, \tag{17}$$

where $nM := \sum_{i=1}^{n} a_i$, k positive integer so that $k \le \left[\frac{n-1}{2}\right]$. Then it holds that

$$a_1+a_2+\cdots+a_n > 2ka^\star.$$

Proof. By the direct computation we get

$$\sum_{i=1}^{n} a_i = nM \geq (2k+1)M = 2k \cdot \frac{2k+1}{2k} M > 2ka^\star .$$

The assertion is proved. □

The following special case of the Theorem 3.2. is of particular interest, but the proving procedure uses just the efforts of the Theorem 3.3. therefore we list it in the continuation of the exposition of the matter.

Corollary 3.1. *Let the situation be the same as in the previous Theorem. Then if*

$$\arcsin\left(\frac{2k}{2k+1}\right) > \frac{k\pi}{n} \tag{18}$$

then there is the k-chordal polygon the sides of which have these lengths.

Proof. As

$$\sum_{i=1}^{n} \arcsin\left(\frac{a_i}{a^\star}\right) \geq n \arcsin\left(\frac{\sum_{i=1}^{n} a_i}{na^\star}\right),$$

using (16) and (17) it holds that

$$\sum_{i=1}^{n} \arcsin\left(\frac{a_i}{a^\star}\right) \geq n \arcsin\left(\frac{M}{na^\star}\right) > n \arcsin\left(\frac{2k}{2k+1}\right) > k\pi .$$

Since

$$\sum_{i=1}^{n} \arcsin\left(\frac{a_i}{a^\star}\right) = n\frac{\pi}{2} - \sum_{i=1}^{n} \arccos\left(\frac{a_i}{a^\star}\right),$$

repeating the proving procedure of the Theorem 3.2., we easily show the required statement. □

Example 3.2. The inequality (18) is valid e.g. under the following specifications:

$k =$	2	3	4	11	30	127	2001	3001
$n \geq$	7	10	14	30	68	270	4060	6073

It is obvious that with growing k the least upper bound for side number n of the considered polygon $\underline{A}$ converges to $2k$. Actually, the result of the Corollary 3.1. means that for the not "too large a_i with respect to the mean M", (18) is sufficient for the existence of a k-chordal polygon with the prescribed side lengths a_i.

Now we are ready to formulate a question concerning the Remark 1.1. As we know, the condition $a_1 + \cdots + a_n > 2ka^\star$ is not sufficient for the existence of the k-chordal polygon with the sides of the lengths $a_i,\ i = \overline{1,n}$. Accordingly, one asks: *Is there some constant $\gamma(k,n)$ depending just on the parameters k,n, that the condition*

$$a_1 + \cdots + a_n > \gamma(k,n)a^\star \tag{19}$$

suffices for the existence of the k-chordal polygon with the side lengths $a_i,\ i = \overline{1,n}$? Also, how close can we go with $\gamma(k,n)$ to $2k$, i.e. order $\liminf_{n\to\infty} \gamma(k,n)$, etc.

We can immediately remark that $\gamma(k,n)$ has to be from the interval $(2k,n)$ [1].

[1]The constant $\gamma(k,n)$ defined by (19) has the same meaning in the sequel, therefore we will not note this separately.

Theorem 3.4. *Let $a_1, a_2, \dots, a_n$ be any given lengths so that*

$$a_1 + a_2 + \cdots + a_n > \gamma(k,n)a^\star, \tag{20}$$

where $k \in \mathbf{N}$ and $k \le \left[\frac{n-1}{2}\right]$. If

$$\gamma(k,n) \ge n \sin\frac{k\pi}{n} \tag{21}$$

then there is a k-chordal polygon whose sides have the length $a_1, a_2, \dots, a_n$.

Also, when

$$\lim_{n\to\infty} \gamma(k,n) < k\pi,$$

then no k-chordal polygon exists the side lengths of which are $a_1, a_2, \dots, a_n$.

Proof. By the convexity of the arcsine function we conclude:

$$\sum_{i=1}^n \arcsin\left(\frac{a_i}{a^\star}\right) \ge n \arcsin\left(\frac{1}{na^\star}\sum_{i=1}^n a_i\right) \ge n \arcsin\frac{\gamma(k,n)}{n}. \tag{22}$$

When the last expression satisfies $n \arcsin\frac{\gamma(k,n)}{n} < k\pi$, then the sum of arcsines in (22) cannot achieve $k\pi$ by any transformation of the arguments, therefore it has to be

$$\gamma(k,n) \ge n \sin\frac{k\pi}{n}.$$

The second assertion in the Theorem is the straightforward consequence of (21). $\square$

The proving procedure of the following corollary is in fact the same as the already given results. Therefore we leave to the reader to prove it. Anyway it is a modest generalization of the preceeding Theorem.

Corollary 3.2. *Let $a_1, a_2, \dots, a_n$ be any given lengths so that*

$$a_1^\lambda + a_2^\lambda + \cdots + a_n^\lambda > \gamma(k,n)(a^\star)^\lambda,$$

where $k \in \mathbf{N}$, $\lambda \in [1,\infty)$ and $k \le \left[\frac{n-1}{2}\right]$. Then there is a (k,λ,k)-chordal polygon the side lengths of which are $a_1^\lambda, a_2^\lambda, \dots, a_n^\lambda$.

Corollary 3.3. *Let the situation be the same as in the previous Theorem. If $a_i = a + (i-1)d$, $i = \overline{1,n}$; $a, d > 0$ and $2\gamma(k,n) > n$, then $d \to 0$ as n goes to the infinity.*

Proof. From

$$\sum_{i=1}^n [a + (i-1)d] > \gamma(k,n)[a + (n-1)d]$$

it follows that

$$d < \frac{2a\left(1 - \frac{\gamma(k,n)}{n}\right)}{(n-1)\left(2\frac{\gamma(k,n)}{n} - 1\right)}.$$

Now it is easy to show the validity of the assertion. $\square$

Remark 3.2. Taking e.g. $\gamma(k,n) := n \sin\frac{k\pi}{n}$, the condition $2\gamma(k,n) > n$ becomes $3k > n$ in the Corollary 3.3.

Theorem 3.5. *Let $\sum_{i-1}^n \beta_i = \frac{\pi}{2}$; $0 < \beta_i < \frac{(2k+1)\pi}{4nk}$, where k is a positive integer so that $k \le \left[\frac{n-1}{2}\right]$. Then*

$$\sum_{i=1}^n \cos^k \beta_i > 2k \cos^k \beta_j, \qquad j = \overline{1,n}. \tag{23}$$

Proof. One approximates $\cos^k x$ by the secant of the cosine function on the interval $[0, \frac{(2k+1)\pi}{4nk}]$. Therefore, with by means of the Bernoulli inequality, we conclude that

$$\cos^k \beta_i \geq \left(1 - \frac{8nk\beta_i}{(2k+1)\pi} \sin^2 \frac{(2k+1)\pi}{8nk}\right)^k > 1 - \frac{8nk^2\beta_i}{(2k+1)\pi} \sin^2 \frac{(2k+1)\pi}{8nk}.$$

After that, since $\sin x <$ for all $x > 0$, we clearly get

$$\cos^k \beta_i > 1 - \frac{(2k+1)\pi}{8n}\beta_i > 1 - \frac{(2k+1)^2\pi^2}{32n^2k}.$$

Thus

$$\sum_{i=1}^{n} \cos^k \beta_i > n - \frac{(2k+1)^2\pi^2}{32nk} > n - \frac{(2k+1)\pi}{4n} > n - 1.$$

Now obvious transformations lead to the required result. □

Corollary 3.4. *Let $\underline{A}$ be m-chordal polygon, $m = \left[\frac{n-1}{2}\right]$. When $\beta_1, \ldots, \beta_n$ corresponding to the sides $a_1, \ldots, a_n$ of $\underline{A}$ satisfy $0 < \beta_i < \frac{(2m+1)\pi}{4mn}$, then*

$$a_1^m + a_2^m + \cdots + a_n^m > 2m(a^\star)^m.$$

Proof. The angles $\beta_1, \beta_2, \ldots, \beta_n$ satisfy the following identities:

$$\begin{aligned} \beta_1 + \beta_2 + \quad \cdots \quad + \beta_n &= \frac{\pi}{2} \\ \frac{\cos\beta_1}{a_1} = \quad \cdots \quad = \frac{\cos\beta_n}{a_n} &= \frac{1}{2\varrho}, \end{aligned}$$

where ϱ is the radius of the circumcircle of the polygon $\underline{A}$. □

References

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