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REPRESENTATION OF PRODUCT SYSTEMS ON THE INTERVAL $[0,1]$

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ABSTRACT. The aim of this paper is establish a natural relation between the Haar integration on the complete direct product of finite discrete topological groups and the Lebesgue integration on the interval $[0,1]$. We use this relation to plot known representative product systems which are given in abstract way.

Throughout this work denote by $\mathbf{N}$, $\mathbf{P}$, $\mathbf{C}$ the set of nonnegative, positive integers and complex numbers respectively. In order to simplicity we always use the multiplication to denote the group operation and use the symbol e to denote the identity of the groups. The notation which we used in this paper is similar to the one appeared in [4].

1. Representative product systems

Let $m := (m_k,\ k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k, $(k \in \mathbf{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, \ldots)$, where $x_k \in G_k$, $(k \in \mathbf{N})$. We call this sequence the *expansion* of x. The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

Define G^0 the set of sequences of G terminating in e's (i.e. the set of "finite" sequences), $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\} \quad (x \in G, n \in \mathbf{N})$$

$I_n := I_n(e)$. We say that every set $I_n(x)$ is an *interval*. The set of intervals I_n is a countable neighborhood base at the identity of the product topology on G.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \ldots)$ is the expansion of n with respect to m. In this case let $n^* = (n_0, n_1, \ldots) \in G$. We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and $n_{(k)} := \sum_{j=0}^{k-1} n_k M_k$, $n^{(k)} = \sum_{j=k}^{\infty} n_k M_k$.

Now we denote by Σ_k the dual object of G_k. Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be the set of all *normalized coordinate functions* of the group G_k and suppose that $\varphi_k^0 \equiv 1$. Thus for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \ldots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

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Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \ldots)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's theorem (see [4]) secure that the system ψ is orthonormal and complete on $\mathcal{L}^2(G_m)$

The functions ψ_n $(n \in \mathbf{N})$ are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N}).$$

Ψ_k is the multiplication of the greatest product of the square root of the dimension and $\mathcal{L}^1$-norm of the functions φ appeared in all group G_{m_j} for $0 \le j < k$. It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

For an integrable complex function f defined in G we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f\overline{\psi}_k \, d\mu \qquad (k \in \mathbf{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{P}).$$

The Dirichlet kernels:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P}).$$

It is clear that

$$S_n f(x) = \int_{G_m} f(y) D_n(x,y) d\mu(y).$$

The following result appears in [3] as the Paley lemma.

Lemma. *If* $n \in \mathbf{N}$, $x, y \in G_m$, *then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

2. Examples

2.1. **The Walsh system.** Let $m_k = 2$ for all $k \in \mathbf{N}$ and $\mathcal{Z}_2$ be the cyclic group of order 2. Thus $G_k = \mathcal{Z}_2$. The characters of $\mathcal{Z}_2$ are the *Rademacher functions*:

$$\varphi_k^s(x) = (-1)^{sx} \qquad (s \in \{0,1\},\ x \in \mathcal{Z}_2)$$

The product system of φ is called the *Walsh system.* It is easy to see that in this case $\Psi_k \equiv 1$.

2.2. **Vilenkin systems.** Let the sequence m be an arbitrary sequence of integers greater than 1 and $\mathcal{Z}_n$ be the cyclic group of order n, where n is an integer greater than 1. Let $G_k = \mathcal{Z}_{m_k}$ for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_{m_k}$ are the *generalized Rademacher functions*:

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k) \qquad (s \in \{0, \ldots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The product system of φ is called a *Vilenkin system.* We also obtain that $\Psi_k \equiv 1$.

2.3. **The complete product of $\mathcal{S}_3$.** Let $m_k = 6$ for all $k \in \mathbf{N}$ and $\mathcal{S}_3$ be the *symmetric group* on 3 elements. Let $G_k = \mathcal{S}_3$ for all $k \in \mathbf{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation ($6 = 1^2 + 1^2 + 2^2$). Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the φ_k^s functions that do not depend on k:

	e	(12)	(13)	(23)	(123)	(132)	$\lVert\varphi^s\rVert_1$	$\lVert\varphi^s\rVert_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$\varphi^2,\dots,\varphi^5$ correspond to the 2-dimensional representation. Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems. In other hand $\max\{\lVert\varphi^s\rVert_1\lVert\varphi^s\rVert_\infty : 0 \le s < 6\} = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$ if $k \to \infty$

2.4. **The complete product of $\mathcal{Q}_2$.** Let $m_k = 8$ for all $k \in \mathbf{N}$ and $\mathcal{Q}_2$ be the the *quaternion group* of order 8, i.e.

$$\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$$

Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbf{N}$. $\mathcal{Q}_2$ has four characters and a 2-dimensional representation ($8 = 1^2+1^2+1^2+1^2+2^2$). Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the φ_k^s functions that do not depend on k:

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\lVert\varphi^s\rVert_1$	$\lVert\varphi^s\rVert_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$\varphi^4,\dots,\varphi^7$ correspond to the 2-dimensional representation. Notice that values of $|\varphi^s|$ are only 0 or the square of the corresponding dimension. Hence the absolute value of the coordinate functions are only 0 or 1 respectively. A representation of this form is called a *monomial representation.* If all of the representations are monomial, then $\Psi_k = 1$ for $k \in \mathbf{N}$, but the group G is not necessarily Abelian.

3. Relation with the interval [0,1]

The theorems appeared in this section are proved by Morgenthaler [5] for the Walsh group and also appear in the book [7].

The topology of G is metrizable. Moreover, the metric concerned is induced by a norm as follows. Order the elements of all G_k, ($k \in \mathbf{N}$) groups in some way such that the first is always their identity. In fact, the ordering is a bijection between G_k and $\{0, 1, \dots, m_k - 1\}$ which give to every $x \in G_k$ the integer $0 \le \overline{x} < m_k$ ($\overline{e} = 0$). Define

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x \in G).$$

It is easy to see that $|.|$ is a norm and the proceeded metric $d(x,y) := |xy^{-1}|$ induces the topology of G. In addition, $0 \le |x| \le 1$ for all $x \in G$. Using this fact we represent the group G in the interval $[0,1]$.

Any $x \in [0,1]$ can be written

$$x := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (0 \le \overline{x_k} \le m_k - 1),$$

but there are numbers with two expressions of this form. They are all numbers in the set

$$\mathbf{Q} := \left\{ \frac{p}{M_n} : 0 \le p < M_n,\ n, p \in \mathbf{N} \right\}$$

called *m-adic rational numbers* (Note that 1 is not an m-adic rational number). The other numbers have only one expression. The m-adic rational numbers have an expression terminates in 0's and other terminates in $m_k - 1$'s. We choose the first one to make an unique relation for all numbers in the interval $[0,1]$ with their expression, named de *m-adic expansion* of the number. In this manner we assign to a number in the interval $[0,1]$ having an m-adic expansion $(\overline{x_0}, \overline{x_1}, \dots)$ an element of G with expansion $(x_0, x_1, \dots)$ denoting this relation by ρ. ρ is called the *Fine's map*. Fine's map is an injective map satisfying:

$$\begin{array}{lll} \rho(x+) = \rho(x-) = \rho(x) & & (x \in (0,1) \setminus \mathbf{Q}) \\ \rho(x+) = \rho(x), & \rho(x-) = \rho^*(x) & (x \in \mathbf{Q}) \\ \rho(0+) = \rho(0) = (e, e, \dots), & \rho(1-) = \rho(1) & \end{array}$$

where $\rho^*(x)$ denotes the element of G terminates in $m_k - 1$'s with norm x. $\rho(x+)$ and $\rho(x-)$ denote respectively the right and left limit of ρ at x under the usual metric.

Using Fine's map we introduce a new operation in the interval $[0,1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x, y \in [0,1[.)$$

We remark that the interval $[0,1[$ is not a group under the new operation since it is not associative, but commutative having the identity 0.

An *m-adic interval* always mean an interval of the form

$$I(n,p) := \left[\frac{p}{M_n}, \frac{p+1}{M_n} \right[\qquad (0 \le p < M_n,\ n, p \in \mathbf{N}).$$

We name the *m-adic topology* the one induced by the m-adic intervals on $[0,1[$. This topology is totally disconnect because the m-adic intervals are both open and closed forming a countable basis. The m-adic topology is issued by the metric:

$$d(x,y) := |\rho(x)\rho(y)^{-1}| \qquad (x, y \in [0,1[.)$$

Fine's map give a natural relation between the new structure of $[0,1[$ and the structure of G. ρ is a continuous map under the m-adic topology since for any $x \in G$ and $n \in \mathbf{N}$ we have $\rho^{-1}(I_n(x)) = I(n,p)$, where $p := M_n \sum_{k=0}^{n-1} \frac{\overline{x_k}}{M_{k+1}}$, but this property is not true for $|.|$. In addition

$$|\rho(x)| = x \qquad (x \in [0,1]), \tag{1}$$

$$\rho(|x|) = x \ \text{ a.e} \qquad (x \in G). \tag{2}$$

(2) is only not true for the elements of G with expansion terminates in $m_k - 1$'s. From similar reason Fine's map is not a homomorphism but the

$$\rho(x \odot y) = \rho(|\rho(x)\rho(y)|) = \rho(x)\rho(y) \tag{3}$$

equality is true for all of elements $x, y \in [0,1[$ such that $x \odot y$ is not a m-adic rational.

Let $L^0(G)$ denote the set of all measurable functions on G which are a.e. finite. In some way denote by L^0 the set of all Lebesgue measurable functions on $[0,1]$ which are a.e. finite. According to the Paley lemma the set of all representative functions on G coincide with the set of all finite linear combinations of characteristics function of intervals, so a function in $L^0(G)$ is a.e the limit of representative functions.

The following theorem show the relation between the Haar integration on G and the Lebesgue integration on the interval $[0,1]$.

Theorem 1. *Let ρ denote the Fine's map.*

(a) If $f \in L^0(G)$ then $f \circ \rho \in L^0$. Conversely, if $g \in L^0$ and

$$f(x) := g(|x|) \qquad (x \in G) \tag{4}$$

then $f \in L^0(G)$.

(b) If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f\, d\mu = \int_0^1 (f \circ \rho)(x)\, dx.$$

Conversely, if g is Lebesgue integrable and f is defined by (4) then f is integrable on G and

$$\int_0^1 g(x)\, dx = \int_G f\, d\mu.$$

Proof. We can prove our statements for intervals using (1) and (2). Indeed, if $x \in G$, $n \in \mathbf{N}$ and f is the characteristic function of the interval $I_n(x)$ then g is the characteristic function of the interval $I(n,p)$, where $p := M_n \sum_{k=0}^{n-1} \frac{\overline{x_k}}{M_{k+1}}$. The conversion of the above statement is valid a.e. Then we obtain our statements for finite linear combinations of characteristic function of intervals and finally for the corresponding set of functions using the Lebesgue convergence theorem. This completes the proof of the theorem. □

The m-adic topology differs considerably from the usual topology on the interval $[0,1[$, but the Lebesgue measure (λ) is also translation invariant under the new operation. To show this statement we introduce the following notation. Let f be a complex function defined in the interval $[0,1[$ and denote by τ the *left translation operator* under the new operation, so

$$(\tau_y f)(x) := f(y \odot x) \qquad (x, y \in [0,1[)$$

and denote the *left translation of the set E* by

$$\tau_y(E) := \{y \odot x : x \in E\} \qquad (E \subseteq [0,1[,\ y \in [0,1[).$$

Theorem 2. *Let f be a complex function defined in the interval $[0,1[$, then*

(a) If the function f is Lebesgue integrable then $\tau_y f$ is also Lebesgue integrable and

$$\int_0^1 (\tau_y f)(x)\, dx = \int_0^1 f(x)\, dx \qquad (y \in [0,1[).$$

(b) In particular for all $E \subseteq [0,1[$ Lebesgue measurable set we have

$$\lambda(\tau_y(E)) = \lambda(E) \qquad (y \in [0,1[).$$

Proof. From (1), (2) and (3), using the translation invariant property of the measure μ we have

$$\int_0^1 (\tau_y f)(x)\,dx = \int_0^1 f(y \odot x)\,dx = \int_G f(y \odot |x|)\,d\mu =$$
$$= \int_G f(|\rho(y)x|)\,d\mu = \int_G f(|x|)\,d\mu = \int_0^1 f(x)\,dx$$

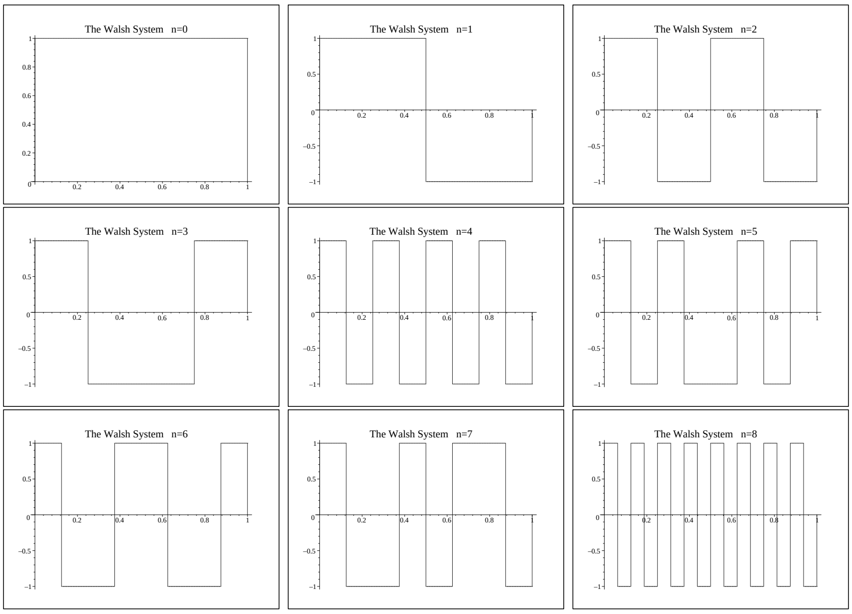
This completes the proof of the theorem. □

Finally, we represent the system ψ on the interval $[0,1]$ substituting it by the

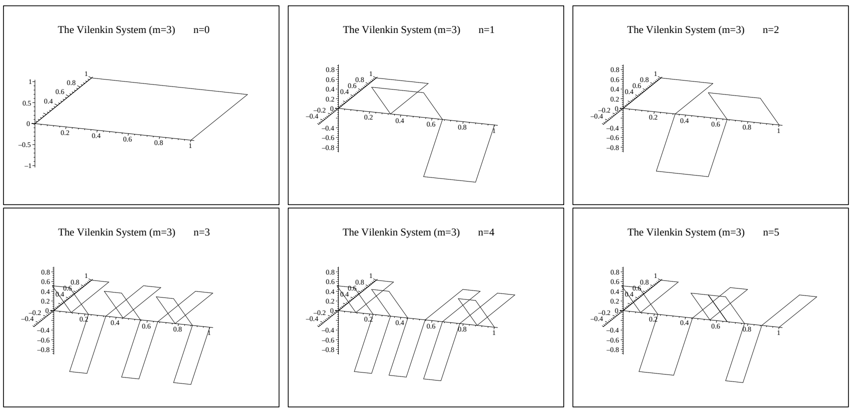
$$\upsilon_n := \psi_n \circ \rho \qquad (n \in \mathbf{N})$$

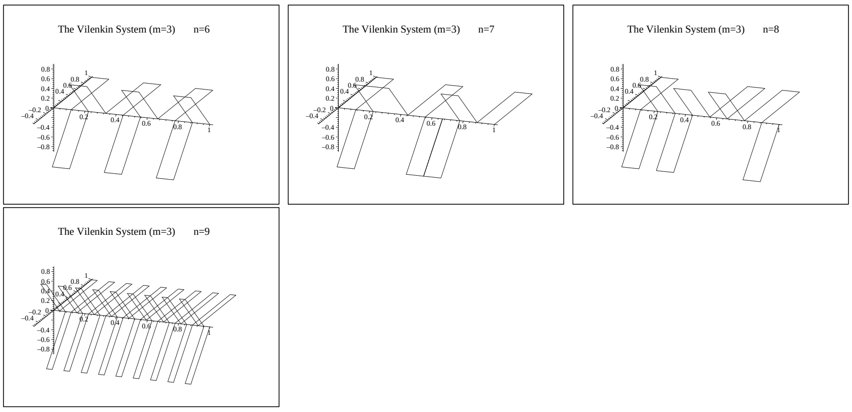
system, according to Theorem 1.

The Walsh system can takes the values 1 and -1 only.

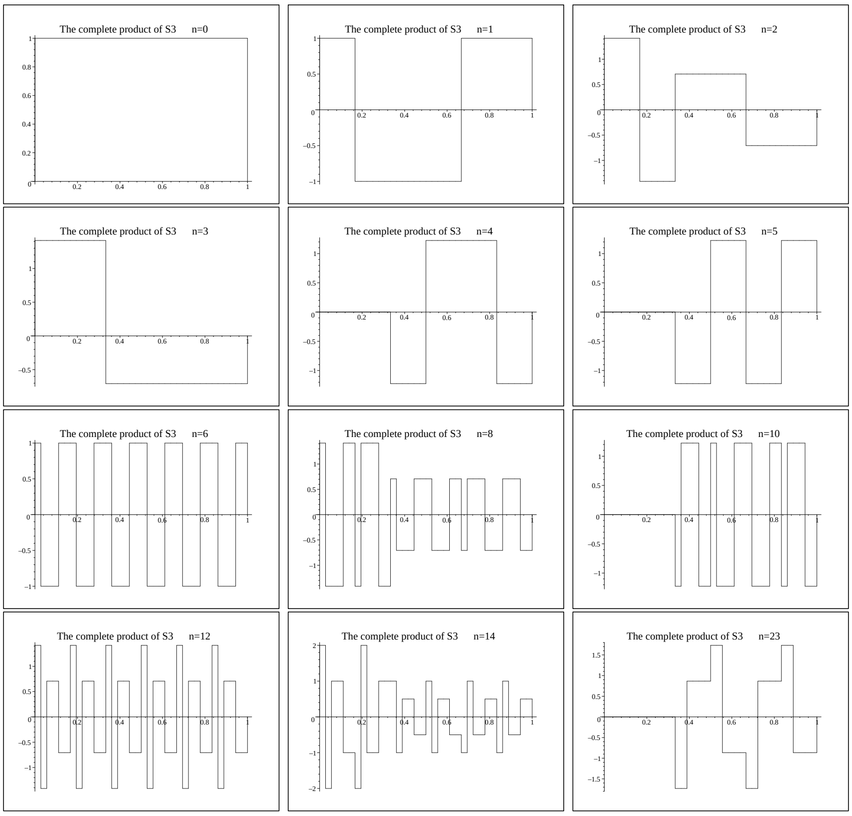


However, the Vilenkin system can takes the values of the complex unit roots.

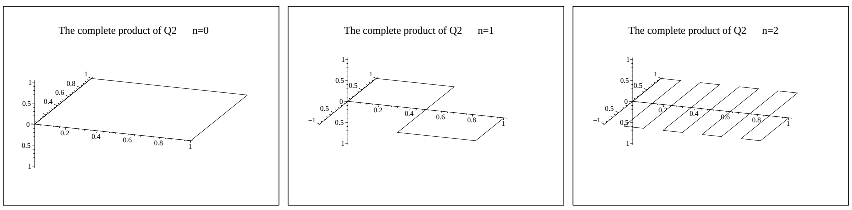


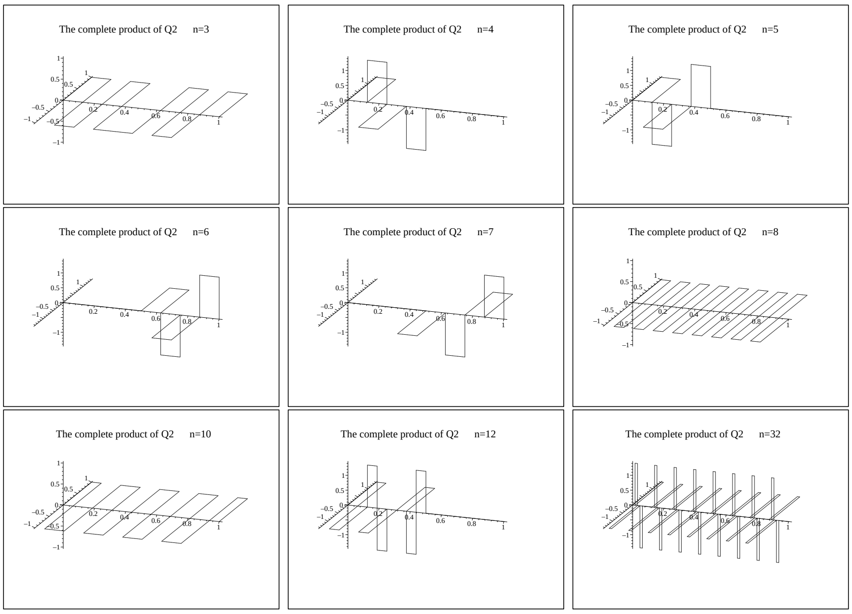


The system on the complete product of $\mathcal{S}_3$ takes only real values.



Finally, we can observe the system on the complete product of $\mathcal{Q}_2$ can takes the values of the complex 4-th unit roots and zero.





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