Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 61–69
www.emis.de/journals

CHARACTERIZATIONS OF EFFECTIVE SETS AND NONEXPANSIVE MULTIPLIERS IN CONDITIONALLY COMPLETE AND INFINITELY DISTRIBUTIVE PARTIALLY ORDERED SETS

ISTVÁN KOVÁCS AND ÁRPÁD SZÁZ

Dedicated to Professor Árpád Varecza on his 60th birthday

Abstract. First, we establish a useful characterization of effective sets in conditionally complete partially ordered sets. Then, we prove that each maximal nonexpansive partial multiplier on a conditionally complete and infinitely distributive partially ordered set with upper bounded centre is inner. Finally, we show that some analogous results hold for T_1-families of sets partially ordered by inclusion.

Introduction

Throughout this paper, we shall assume the terminogy of [17] which may differ from that of the earlier works on some particular subsets and mappings of posets (partially ordered sets) and semilattices (posets in which any two elements have a meet).

If $\mathcal{A}$ is a poset, then in contrast to Birkhoff [2, p. 67] the family

$$\mathcal{A}_o = \{\, D \in \mathcal{A} : \forall A \in \mathcal{A} : \exists A \wedge D \,\},$$

where $A \wedge D = \inf\{\, A, D \,\}$, will be called the centre of $\mathcal{A}$. Note that $\mathcal{A}_o = \mathcal{A}$ if and only if $\mathcal{A}$ is a semilattice.

Moreover, in contrast to Cornish [4, p. 340], a subset $\mathcal{D}$ of $\mathcal{A}$ will be called effective (supereffective) if $\mathcal{D} \subset \mathcal{A}_o$ and for each $A, B \in \mathcal{A}$, with $A \neq B$ (resp. $A \not\leq B$), there exists a $D \in \mathcal{D}$ such that $A \wedge D \neq B \wedge D$ (resp. $A \wedge D \not\leq B \wedge D$). And, as some straightforward improvements of some slightly incorrect statements of Cornish, we shall prove the following assertions.

Theorem 1. *If $\mathcal{A}$ is a conditionally complete poset and $\mathcal{D}$ is a nonvoid subset of $\mathcal{A}_o$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective (supereffective);

(2) $A = \bigvee_{D \in \mathcal{D}} A \wedge D$ for all $A \in \mathcal{A}$.

Corollary 1. *If $\mathcal{A}$ is a conditionally complete semilattice and $\mathcal{D}$ is an ideal of $\mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective (supereffective);

2000 *Mathematics Subject Classification.* Primary 06A06, 06A12; Secondary 20M14, 20M15.

Key words and phrases. Partially ordered sets, conditional completeness and infinite distributivity, effective sets and nonexpansive multipliers.

The research of the second author has been supported by the grants OTKA T-030082 and FKFP 0310/1997.

(2) $A = \bigvee \{ D \in \mathcal{D} : D \leq A \}$ *for all* $A \in \mathcal{A}$.

If $\mathcal{A}$ is a poset, then in contrast to Schmid [12, p. 403] a function F form a nonvoid subset $\mathcal{D}_F$ of $\mathcal{A}_o$ into $\mathcal{A}$ will be called a partial multiplier on $\mathcal{A}$ if

$$F(D) \wedge E = F(E) \wedge D$$

for all $D, E \in \mathcal{D}_F$. And the family of all partial multipliers F on $\mathcal{A}$ will be denoted by $\mathcal{M}(\mathcal{A})$.

In particular, a multiplier $F \in \mathcal{M}(\mathcal{A})$ will be called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}_F$. Moreover, the multiplier F will be called inner if $\mathcal{D}_F = \mathcal{A}_o$ and there exists an $A \in \mathcal{A}$ such that $F(D) = A \wedge D$ for all $D \in \mathcal{A}_o$.

As some straightforward extensions of some of the results of Szász [13, p. 165], Brainerd and Lambek [3, Proposition 3] and Berthiaume [1, Theorem 17], we shall prove the following assertions.

Theorem 2. *If* $\mathcal{A}$ *is a conditionally complete and infinitely distributive poset such that* $\mathcal{A}_o$ *is bounded above in* $\mathcal{A}$*, then for each nonexpansive multiplier* $F \in \mathcal{M}(\mathcal{A})$ *there exists an* $A \in \mathcal{A}$ *such that*

$$F(D) = A \wedge D$$

for all $D \in \mathcal{D}_F$.

Corollary 2. *If* $\mathcal{A}$ *is a conditionally complete and infinitely distributive poset such that* $\mathcal{A}_o$ *is bounded above in* $\mathcal{A}$*, then each maximal and nonexpansive multiplier* $F \in \mathcal{M}(\mathcal{A})$ *is inner.*

Remark. Whenever a multiplier $F \in \mathcal{M}(A)$ is effective in the sense that its domain $\mathcal{D}_F$ is effective, then F is nonexpansive and there exists at most one $A \in \mathcal{A}$ such that $F(D) = A \wedge D$ for all $D \in \mathcal{D}_F$.

Since an ordinary topology partially ordered by inclusion forms an unconditionally complete and infinitely distributive semilattice, we shall also show that the analogues of the above results also hold true for T_1-families. A nonvoid family $\mathcal{A}$ of sets will be called a T_1-family if $A \setminus \{ x \} \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $x \in A$.

1. A few basic facts on partially ordered sets

A nonvoid set $\mathcal{A}$ together with a reflexive, transitive and antisymmetric relation $\leq$ will be called a poset [2]. A poset $\mathcal{A}$ can always be thought of as a nonvoid family of sets partially ordered by set inclusion. Namely, each $A \in \mathcal{A}$ can be identified with the set $\{ B \in \mathcal{A} : B \leq A \}$.

The infimum (greatest lower bound) and the supremum (least upper bound) of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be understood in the usual sense. However, instead of $\inf \mathcal{D}$ and $\sup \mathcal{D}$ we shall use the lattice theoretic notations meet $\bigwedge \mathcal{D}$ and join $\bigvee \mathcal{D}$, respectively.

Concerning finite meets and arbitrary joins

$$A \wedge B = \inf\{ A, B \} \qquad \text{and} \qquad \bigvee_{i \in I} A_i = \sup\{ A_i : i \in I \},$$

we shall only need here the following theorems, partly proved in [17].

Theorem 1.1. *If* $\mathcal{A}$ *is a poset and* $A, B, C, D \in \mathcal{A}$*, then*

(1) $A \leq B$ *if and only if* $A = A \wedge B$*;*

(2) $A \leq B$ *and* $C \leq D$ *imply* $A \wedge C \leq B \wedge D$ *whenever* $A \wedge C$ *and* $B \wedge D$ *exist.*

Corollary 1.2. *If* $\mathcal{A}$ *is a poset and* $A, B, C \in \mathcal{A}$*, then*

(1) $A = A \wedge A$*;*

(2) $A = A \wedge (A \vee B)$ *whenever* $A \vee B$ *exists;*

(3) $A \leq B$ *implies* $A \wedge C \leq B \wedge C$ *whenever* $A \wedge C$ *and* $B \wedge C$ *exist.*

Theorem 1.3. *If $\mathcal{A}$ is a poset and $A, B, C \in \mathcal{A}$, then*

(1) $A \wedge B = B \wedge A$ *whenever either* $B \wedge A$ *or* $A \wedge B$ *exist;*
(2) $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ *whenever* $A \wedge B$ *and* $B \wedge C$ *and moreover either* $(A \wedge B) \wedge C$ *or* $A \wedge (B \wedge C)$ *exist.*

Theorem 1.4. *If $\mathcal{A}$ is a poset and $A_i, B_i \in \mathcal{A}$ for all $i \in I$ such that $A_i \leq B_i$ for all $i \in I$, then*

$$\bigvee_{i\in I} A_i \leq \bigvee_{i\in I} B_i$$

whenever both $\bigvee_{i\in I} A_i$ and $\bigvee_{i\in I} B_i$ exist.

Theorem 1.5. *If $\mathcal{A}$ is a poset, $A_i \in \mathcal{A}$ for all $i \in I$ and $B \in \mathcal{A}$, then*

$$\bigvee_{i\in I} A_i \wedge B \leq \left(\bigvee_{i\in I} A_i\right) \wedge B$$

whenever both $\bigvee_{i\in I} A_i \wedge B$ and $\left(\bigvee_{i\in I} A_i\right) \wedge B$ exist.

In connection with posets, we shall assume here a rather particular terminology. A nonvoid subset $\mathcal{B}$ of a poset $\mathcal{A}$ will be called a semilattice in $\mathcal{A}$ if $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{B}$ for all $D, E \in \mathcal{B}$. Moreover, a nonvoid subset $\mathcal{D}$ of $\mathcal{B}$ will be called an ideal of $\mathcal{B}$ if $D \wedge E$ is in $\mathcal{D}$ for all $D \in \mathcal{D}$ and $E \in \mathcal{B}$.

If $\mathcal{A}$ is a poset, then in contrast to Birkhoff [2, p. 67] the family

$$\mathcal{A}_o = \left\{\, D \in \mathcal{A} : \forall A \in \mathcal{A} : \exists A \wedge D \,\right\}$$

will be called the centre of $\mathcal{A}$. By using Theorem 1.3 (2), it is easy to see that $\mathcal{A}_o$ is a semilattice in $\mathcal{A}$ whenever $\mathcal{A}_o \neq \emptyset$.

A poset $\mathcal{A}$ will be called conditionally complete if $\bigvee \mathcal{D}$ exists for every nonvoid subset $\mathcal{D}$ of $\mathcal{A}$ which is bounded above. Note that in this case each nonvoid subset of $\mathcal{A}$ which is bounded below has a meet [6, p. 14], but $\mathcal{A}$ still need not even be a semilattice.

A conditionally complete poset $\mathcal{A}$ will be called infinitely $\mathcal{A}_o$-distributive if

$$A \wedge \bigvee_{i\in I} D_i = \bigvee_{i\in I} A \wedge D_i$$

for every $A \in \mathcal{A}$ and for every nonvoid family $(D_i)_{i\in I}$ in $\mathcal{A}_o$ which is bounded above in $\mathcal{A}$.

Moreover, a conditionally complete poset $\mathcal{A}$ will be called infinitely $\mathcal{A}$-distributive if

$$\left(\bigvee_{i\in I} A_i\right) \wedge D = \bigvee_{i\in I} A_i \wedge D$$

for every nonvoid family $(A_i)_{i\in I}$ in $\mathcal{A}$ which is bounded above and for every $D \in \mathcal{A}_o$.

2. Characterizations of effective and supereffective sets

In contrast to Cornish [4, p. 340], we shall assume here the following definition of [17].

Definition 2.1. A subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be called effective if $\mathcal{D} \subset \mathcal{A}_o$ and for each $A, B \in \mathcal{A}$, with $A \neq B$ (resp. $A \not\leq B$), there exists a $D \in \mathcal{D}$ such that $A \wedge D \neq B \wedge D$ (resp.$A \wedge D \not\leq B \wedge D$).

Moreover, a poset $\mathcal{A}$ will be called effective (supereffective) if its centre $\mathcal{A}_o$ is effective (supereffective).

Remark 2.2. In [17], it has been proved that a suppereffective subset of a poset is in particular effective. And an effective subset of a semilattice is necessary supereffective.

Moreover, the effective (supereffective) subsets of a poset $\mathcal{A}$ are closely related to those subsets $\mathcal{D}$ of $\mathcal{A}_o$ which are cofinal in $\mathcal{A}$ in the sense that for each $A \in \mathcal{A}$ there exists a $D \in \mathcal{D}$ such that $A \leq D$.

Now, by assuming the conditional completeness of $\mathcal{A}$, we can prove a more satisfactory characterization of the effective (suppereffective) sets which improves a slightly incorrect statement of Cornish [4, p. 340].

Theorem 2.3. *If $\mathcal{A}$ is a conditionally complete poset and $\mathcal{D}$ is a nonvoid subset of $\mathcal{A}_o$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective (supereffective);

(2) $A = \bigvee_{D \in \mathcal{D}} A \wedge D$ for all $A \in \mathcal{A}$.

Proof. If $A \in \mathcal{A}$, then $A \wedge D \leq A$ for all $D \in \mathcal{D}$. Therefore, by the conditional completeness of $\mathcal{A}$, the join

$$B = \bigvee_{D \in \mathcal{D}} A \wedge D$$

exists. And we evidently have $B \leq A$, and hence $B \wedge D \leq A \wedge D$ for all $D \in \mathcal{D}$.

Moreover, by the corresponding properties of $\wedge$ and the definition of B, we also have

$$A \wedge D = A \wedge (D \wedge D) = (A \wedge D) \wedge D \leq B \wedge D$$

for all $D \in \mathcal{D}$. Hence, it is clear that $A \wedge D = B \wedge D$ for all $D \in \mathcal{D}$. Therefore, if $\mathcal{D}$ is effective, then $A = B$, and thus the assertion (2) also holds.

On the other hand, if the assertion (2) holds and $A, B \in \mathcal{A}$ such that $A \wedge D \leq B \wedge D$ for all $D \in \mathcal{D}$, then by Theorem 1.4 we also have

$$A = \bigvee_{D \in \mathcal{D}} A \wedge D \leq \bigvee_{D \in \mathcal{D}} B \wedge D = B.$$

And thus $\mathcal{D}$ is supereffective. □

Remark 2.4. Note that the implication (2) $\Longrightarrow$ (1) does not require the conditional completeness of $\mathcal{A}$.

Now, as a useful consequence of Theorem 2.3, we can also prove

Corollary 2.5. *If $\mathcal{A}$ is a conditionally complete and infinitely $\mathcal{A}_o$-distributive poset with a greatest element X and $\mathcal{D}$ is nonvoid subset of $\mathcal{A}_o$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective (supereffective);

(2) $X = \bigvee \mathcal{D}$.

Proof. If the assertion (1) holds, then by Theorems 2.3 we have

$$X = \bigvee_{D \in \mathcal{D}} X \wedge D = \bigvee_{D \in \mathcal{D}} D = \bigvee \mathcal{D}.$$

That is, the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then by the infinite $\mathcal{A}_o$-distributivity of $\mathcal{A}$, we have

$$A = A \wedge X = A \wedge \bigvee \mathcal{D} = A \wedge \bigvee_{D \in \mathcal{D}} D = \bigvee_{D \in \mathcal{D}} A \wedge D$$

for all $A \in \mathcal{A}$. And thus, by Theorem 2.3, the assertion (1) also holds. □

Remark 2.6. Note that the implication (1) $\Longrightarrow$ (2) does not require the infinite $\mathcal{A}_o$-distributivity of $\mathcal{A}$.

And the converse implication (2) $\Longrightarrow$ (1) does not really require the conditional completeness of $\mathcal{A}$.

From Theorem 2.3, we can also easily get the following improvement of an other slightly incorrect statement of Cornish [4, p. 340].

Corollary 2.7. *If $\mathcal{A}$ is a conditionally complete semilattice and $\mathcal{D}$ is an ideal of $\mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective (supereffective);
(2) $A = \bigvee \{ D \in \mathcal{D} : D \leq A \}$ for all $A \in \mathcal{A}$.

Proof. If $A \in \mathcal{A}$, then by the conditional completeness of $\mathcal{A}$ the joins

$$B = \bigvee \{ A \wedge D : D \in \mathcal{D} \} \text{ and } C = \bigvee \{ D \in \mathcal{D} : D \leq A \}$$

exist. Moreover, if $D \in \mathcal{D}$ such that $D \leq A$, then we evidently have $D = A \wedge D \leq B$. Therefore, $C \leq B$.

On the other hand, if $D \in \mathcal{D}$, the since $\mathcal{D}$ is an ideal in $\mathcal{A}$ we also have $A \wedge D \in \mathcal{D}$. Hence, since $A \wedge D \leq A$, it is clear that $A \wedge D \leq C$. Therefore, $B \leq C$ is also true. Consequently, we have $B = C$, and thus by Theorem 2.3, the assertions (1) and (2) are equivalent. □

3. Nonexpansive, effective and maximal multipliers

In contrast to Schmid [12, p. 403], we assume here the following definition of [17].

Definition 3.1. If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then a function F from a nonvoid subset $\mathcal{D}_F$ of $\mathcal{A}_o$ into $\mathcal{A}$ will be called a partial multiplier on $\mathcal{A}$ if

$$F(D) \wedge E = F(E) \wedge D$$

for all $D, E \in \mathcal{D}_F$. And the family of all partial multipliers F on $\mathcal{A}$ will be denoted by $\mathcal{M}(\mathcal{A})$.

Remark 3.2. A multiplier $F \in \mathcal{M}(\mathcal{A})$ may be called total if $\mathcal{D}_F = \mathcal{A}_o$. Clearly, the identity function $\Delta_{\mathcal{A}_o}$ of $\mathcal{A}_o$ is a total member of $\mathcal{M}(\mathcal{A})$.

Moreover, by using Theorem 1.3 (2), we can easily establish the following

Proposition 3.3. *If $\mathcal{A}$ is a poset, with $\mathcal{A}_o \neq \emptyset$, and $A \in \mathcal{A}$, then the function F_A, defined by*

$$F_A(D) = A \wedge D$$

for all $D \in \mathcal{A}_o$, is a total member of $\mathcal{M}(\mathcal{A})$.

Remark 3.4. A total multiplier $F \in \mathcal{M}(\mathcal{A})$ may be called inner if there exists an $A \in \mathcal{A}$ such that $F = F_A$.

Moreover, concerning multipliers, we can also naturally introduce the following terminology.

Definition 3.5. A multiplier $F \in \mathcal{M}(\mathcal{A})$ will be called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}_F$.

Moreover, a multiplier $F \in \mathcal{M}(\mathcal{A})$ will be called effective (supereffective) if its domain $\mathcal{D}_F$ is effective (supereffective).

Remark 3.6. By Theorem 1.1 (1), it is clear that $F \in \mathcal{M}(\mathcal{A})$ is nonexpansive if and only if $F(D) = F(D) \wedge D$ for all $D \in \mathcal{D}_F$.

Moreover, in [17] it has been proved the following

Theorem 3.7. *If $F \in \mathcal{M}(\mathcal{A})$ is effective, then F is nonexpansive.*

Remark 3.8. The importance of nonexpansive multipliers lies mainly in the fact that if $F \in \mathcal{M}(\mathcal{A})$ is nonexpansive, then $F(D \wedge E) = F(D) \wedge E$ for all $D \in \mathcal{D}_F$ and $E \in \mathcal{A}_o$ with $E \wedge D \in \mathcal{D}_F$.

Definition 3.9. A multiplier $F \in \mathcal{M}(\mathcal{A})$ is called maximal if $G \in \mathcal{M}(\mathcal{A})$ and $F \subset G$ imply $F = G$.

Moreover, if $F, G \in \mathcal{M}$ such that $F \subset G$ and G is maximal, then G is called a maximal extension of F.

Remark 3.10. By using the Hausdorff maximality principle, it can be shown that each multiplier $F \in \mathcal{M}(\mathcal{A})$ has a maximal extension.

Moreover, in [17] it has been proved the following

Theorem 3.11. *If $\mathcal{A}$ is an effective poset and $\mathcal{D}$ is a nonvoid subset of $\mathcal{A}_o$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is effective;

(12) each $F \in \mathcal{M}(\mathcal{A})$ with domain $\mathcal{D}$ has a unique maximal extension.

Remark 3.12. The importance of maximal effective multipliers lies mainly in the fact if $F, G \in \mathcal{M}(\mathcal{A})$ are maximal and effective, then $F = G$ if and only if there exists an effective subset $\mathcal{D}$ of $\mathcal{A}$ such that $F(D) = G(D)$ for all $D \in \mathcal{D}$.

4. Characterizations of nonexpansive and effective multipliers

As a straightforward extension of some of the results of Szász [13, p. 165], Brainerd and Lambek [3, Proposition 3] and Berthiaume [1, Theorem 17], we can now prove the following

Theorem 4.1. *If $\mathcal{A}$ is a conditionally complete and infinitely $\mathcal{A}$-distributive poset such that $\mathcal{A}_o$ is bounded above in $\mathcal{A}$, then for each nonexpansive multiplier $F \in \mathcal{M}(\mathcal{A})$ there exists an $A \in \mathcal{A}$ such that*

$$F(D) = A \wedge D$$

for all $D \in \mathcal{D}_F$.

Proof. If Y is an upper bound of $\mathcal{A}_o$ in $\mathcal{A}$, then by the nonexpansivity of F we have $F(D) \leq D \leq Y$. Therefore, by the conditional completeness of $\mathcal{A}$, the join

$$A = \bigvee_{D \in \mathcal{D}_F} F(D)$$

exists. Hence, since F is nonexpansive, it is clear that

$$F(D) = F(D) \wedge D \leq A \wedge D$$

for all $D \in \mathcal{D}_F$.

On the other hand, by the infinite $\mathcal{A}$-distributivity and the multiplier property of F it is clear that

$$A \wedge D = \Big(\bigvee_{E \in \mathcal{D}_F} F(E) \Big) \wedge D = \bigvee_{E \in \mathcal{D}_F} F(E) \wedge D = \bigvee_{E \in \mathcal{D}_F} F(D) \wedge E \leq F(D)$$

for all $D \in \mathcal{D}_F$. Therefore, the equality $F(D) = A \wedge D$ is also true for all $D \in \mathcal{D}_F$. □

Now, as an immediate consequence of Theorem 4.1, we can also state

Corollary 4.2. *If $\mathcal{A}$ is a conditionally complete and infinitely $\mathcal{A}$-distributive poset such that $\mathcal{A}_o$ is bounded above in $\mathcal{A}$, then each maximal and nonexpansive multiplier $F \in \mathcal{M}(\mathcal{A})$ is inner.*

Proof. If $F \in \mathcal{M}(\mathcal{A})$ is nonexpansive, then by Theorem 4.1, there exists an $A \in \mathcal{A}$ such that $F(D) = A \wedge D = F_A(D)$ for all $D \in \mathcal{D}_F$. Hence, it is clear that F_A is a maximal extension of F. Therefore, if F is, in addition, maximal, then we necessarily have $F = F_A$. □

Remark 4.3. In this respect, it is also worth mentioning that if $\mathcal{A}$ is a poset such that each total and nonexpansive member of $\mathcal{M}(\mathcal{A})$ is inner, then $\mathcal{A}_o$ is necessarily bounded above in $\mathcal{A}$.

Namely, the identity function $\Delta_{\mathcal{A}_o}$ of $\mathcal{A}_o$ is a total and nonexpansive member of $\mathcal{M}(\mathcal{A})$. Therefore, by the assumption, there exists an $A \in \mathcal{A}$ such that $\Delta_{\mathcal{A}_o} = F_A$. Hence, it follows that $D = A \wedge D$, and thus $D \leq A$ for all $D \in \mathcal{A}_o$.

In addition to Theorem 4.1, we can also easily establish the following

Theorem 4.4. *If $\mathcal{A}$ is a conditionally complete and infinitely $\mathcal{A}$-distributive poset such that $\mathcal{A}_o$ is bounded above in $\mathcal{A}$, then for each effective multiplier $F \in \mathcal{M}(\mathcal{A})$ there exists a unique $A \in \mathcal{A}$ such that*

$$F(D) = A \wedge D$$

for all $D \in \mathcal{D}_F$.

Proof. In this case, by Theorem 3.7, F is, in particular, nonexpansive. Thus, by Theorem 4.1, there exists an $A \in \mathcal{A}$ such that $F(D) = A \wedge D$ for all $D \in \mathcal{D}_F$.

On the other hand, if $B \in \mathcal{A}$ such that $F(D) = B \wedge D$ for all $D \in \mathcal{D}_F$, then we have $A \wedge D = B \wedge D$ for all $D \in \mathcal{D}_F$. Therefore, by the effectiveness of $\mathcal{D}_F$, we also have $A = B$. □

Hence, it is clear that in particular we also have

Corollary 4.5. *If $\mathcal{A}$ is a conditionally complete and infinitely $\mathcal{A}$-distributive poset such that $\mathcal{A}_o$ is bounded above in $\mathcal{A}$, then for each maximal and effective multiplier $F \in M(\mathcal{A})$ then there exists a unique $A \in \mathcal{A}$ such that $F = F_A$.*

Remark 4.6. Note that if $\mathcal{A}$ is a conditionally complete and effective poset such that $\mathcal{A}_o$ is bounded above in $\mathcal{A}$, then by Theorem 2.3 we have

$$A = \bigvee_{D \in \mathcal{A}_o} A \wedge D \leq \bigvee_{D \in \mathcal{A}_o} D = \bigvee \mathcal{A}_o$$

for all $A \in \mathcal{A}$. Therefore, $\bigvee \mathcal{A}_o$ is the greatest element of $\mathcal{A}$, and thus we also have $\bigvee \mathcal{A}_o \in \mathcal{A}_o$.

5. Characterizations of effective sets and multipliers in T_1-families of sets

Since the poset $\mathcal{A}$ in Theorems 2.3 and 4.4 may, in particular, be an ordinary topology, partially ordered by inclusion, it seems to be of some interest to point out that some analogous results hold for T_1-families too.

Definition 5.1. A nonvoid family $\mathcal{A}$ of subsets of a set X, with $X = \bigcup \mathcal{A}$, will be called a T_1-family on X if $A \in \mathcal{A}$ and $x \in A$ imply $A \setminus \{ x \} \in \mathcal{A}$.

Remark 5.2. Note that if $\mathcal{A}$ is a T_1-family on X, then for each $x, y \in X$, with $x \neq y$, there exists an $A \in \mathcal{A}$, with $x \in A$, such that $y \notin A$, but in contrast to T_1-topologies the converse statement need not be true.

Definition 5.3. If $\mathcal{A}$ is a nonvoid family of sets, then we write

$$\mathcal{A}_1 = \big\{ D \in \mathcal{A} : \forall A \in \mathcal{A} : A \cap D \in \mathcal{A} \big\}.$$

Remark 5.4. Note that, by considering $\mathcal{A}$ to be partially ordered by inclusion, we have $A \wedge D = A \cap D$ for all $A \in \mathcal{A}$ and $D \in \mathcal{A}_1$. Therefore, $\mathcal{A}_1 \subset \mathcal{A}_o$, but the equality is not, in general, true.

Now, analogously to Theorem 2.3 and Corollary 2.5, we can also prove the following

Theorem 5.5. *If $\mathcal{A}$ is a T_1-family on X and $\mathcal{D} \subset \mathcal{A}_1$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ *is an effective (supereffective) subfamily of* $\mathcal{A}$*;*
(2) $A = \bigcup_{D \in \mathcal{D}} A \cap D$ *for all* $A \in \mathcal{A}$*;*
(3) $X = \bigcup \mathcal{D}$*;*
(4) $\mathcal{D}$ *is an effective (supereffective) subfamily of* $\mathcal{P}(X)$*.*

Proof. If the assertion (2) does not hold, then there exist $A \in \mathcal{A}$ and $x \in A$ such that $x \notin D$ for all $D \in \mathcal{D}$. Therefore, we have

$$A \wedge D = A \cap D = \big(A \setminus \{\, x \,\}\big) \cap D = \big(A \setminus \{\, x \,\}\big) \wedge D$$

for all $D \in \mathcal{D}$. And thus the assertion (1) does not also hold. Consequently, the implication (1) $\Longrightarrow$ (2) is true.

On the other hand, if $x \in X$, then because of $X = \bigcup \mathcal{A}$ there exists an $A \in \mathcal{A}$ such that $x \in A$. Moreover, if the assertion (2) holds, then there exists a $D \in \mathcal{D}$ such that $x \in D$. Therefore, $X \subset \bigcup \mathcal{D}$, and thus the assertion (3) also holds.

Moreover, if $A, B \subset X$, such that $A \not\subset B$, then there exists an $x \in A$ such that $x \notin B$. Furthermore, if the assertion (3) holds, then there exists a $D \in \mathcal{D}$ such that $x \in D$. Hence, it is clear that

$$A \wedge D = A \cap D \not\subset B \cap D = B \wedge D.$$

And thus the assertion (4) also holds.

Now, since the implication (4) $\Longrightarrow$ (1) is quite obvious, the proof of the theorem is complete. □

Remark 5.6. Note that only the implication (1) $\Longrightarrow$ (2) requires the T_1-property of the family $\mathcal{A}$.

Now, from Theorem 4.4, by using Theorem 5.5, we can easily get the following

Theorem 5.7. *If* $\mathcal{A}$ *is a* T_1*-family on* X*, then for each effective multiplier* $F \in \mathcal{M}(\mathcal{A})$*, with* $D_F \subset \mathcal{A}_1$*, then there exists a unique subset* A *of* X *such that*

$$F(D) = A \cap D$$

for all $D \in \mathcal{D}_F$*.*

Proof. In this case, by Remark 5.4 and Theorem 5.5, it is clear that F is also an effective member of $\mathcal{M}\big(\mathcal{P}(X)\big)$. Therefore, Theorem 4.4 can be applied to get the required conclusion. □

Moreover, as a similar consequence of Theorem 4.1, we can also state

Theorem 5.8. *If* $\mathcal{A}$ *is a nonvoid family of subsets of a set* X*, then for each nonexpansive multiplier* $F \in \mathcal{M}(\mathcal{A})$*, with* $D_F \subset \mathcal{A}_1$*, there exists a subset* A *of* X *such that*

$$F(D) = A \cap D$$

for all $D \in \mathcal{D}_F$*.*

Proof. In this case, by Remark 5.4, it is clear that F is also a nonexpansive member of $\mathcal{M}\big(\mathcal{P}(X)\big)$. Therefore, Theorem 4.1 can be applied to get the required conclusion. □

References

[1] P. Berthiaume. The injective envelope of S-sets. *Canad. Math. Bull.*, 10:261–273, 1967.
[2] G. Birkhoff. *Lattice Theory*. Amer. Math. Soc., Providence, 1973.
[3] B. Brainerd and J. Lambek. On the ring of quotients of a Boolean ring. *Canad. Math. Bull.*, 2:25–29, 1959.
[4] W. H. Cornish. The multiplier extension of a distributive lattice. *Journal of Algebra*, 32:339–355, 1974.
[5] A. Figá-Talamanca and S. P. Franklin. Multipliers of distributive lattices. *Indian J. Math.*, 12:153–161, 1970.

[6] J. L. Kelley. *General Topology.* Van Nostrand Reinhold, New York, 1955.
[7] M. Kolibiar. Bemerkungen über Translationen der Verbände. *Acta Fac. Rerum. Natur. Univ. Comenian. Math.*, 5:455–458, 1961.
[8] J. Lambek. *Lectures on Rings and Modules.* Blaisdell Publishing Company, London, 1966.
[9] R. Larsen. *An Introduction to the Theory of Multipliers.* Springer-Verlag, Berlin, 1971.
[10] G. Pataki and Á. Száz. Characterizations of nonexpansive multipliers on partially ordered sets. *Math. Slovaca*, 52, (2002), to appear.
[11] J. Schmid. Distributive lattices and rings of quotients. In *Colloq. Math. Soc. János Bolyai*, volume 33, pages 675–696, 1980.
[12] J. Schmid. Multipliers on distributive lattices and rings of quotients I. *Houston J. Math.*, 6:401–425, 1980.
[13] G. Szász. Die Translationen der Halbverbände. *Acta Sci. Math. (Szeged)*, 17:165–169, 1956.
[14] G. Szász. Translationen der Verbände. *Acta Fac. Rerum. Natur. Univ. Comenian. Math.*, 5:449–453, 1961.
[15] G. Szász and J. Szendrei. über die Translation der Halbverbände. *Acta Sci. Math. (Szeged)*, 18:44–47, 1957.
[16] Á. Száz. The multiplier extensions of admissible vector modules and the Mikusiński-type convergences. *Serdica*, 3:82–87, 1977.
[17] Á. Száz. Partial multipliers on partially ordered sets. Technical Report 98/8, Inst. Math. Inf., Univ. Debrecen, 1998. pages 1–28.

Received November 21, 2000

Department of Mathematics,
University of Kentucky,
Lexington, KY 40506–0027, USA
E-mail address: kovacs@ms.uky.edu

Institute of Mathematics and Informatics,
University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary
E-mail address: szaz@math.klte.hu