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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF SPECIAL KAEHLER SPACES

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To memory of my friend and colleague Ferenc Ilosvay. (S. B.)

Abstract. Some authors [1], [2], [3], [4], [5] have studied the holomorphically flat curves in Kaehler spaces and the mappings between Kaehler spaces preserving such curves which are called holomorphically projective mappings. In the present paper we treat a holomorphically projective mapping of a Kaehler-Codazzi space [6] onto another Kaehler space. A similar problem was investigated by Sobchuk in Riemann spaces [7].

1. Introduction

Let $K^n(g,F)$ be an $n(=2m)$ dimensional Kaehler space, where g is a Riemann metric and F is a complex structure in K^n satisfying the following conditions [4], [6]:

$$\begin{aligned} F^h_\alpha F^\alpha_i &= -\delta^h_i \\ F_{ij} &= -F_{ji}, \qquad (F_{ij} = F^\alpha_i g_{\alpha j}) \\ F^i_{j,k} &= 0 \\ F^\alpha_i F^\beta_j g_{\alpha\beta} &= g_{ij}. \end{aligned}$$

(We denote the covariant derivation in K^n by ",".)[1]

The holomorphically flat curves are defined by a differential equation of the form

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = a(t)\frac{dx^i}{dt} + b(t)F^i_\alpha\frac{dx^\alpha}{dt},$$

where Γ^i_{jk} are the Cristoffel symbols of the fundamental tensor g_{ij}.

Definition 1. ([3], [4]) A diffeomorphism of a Kaehler space $K^n(g,F)$ to another Kaehler space $\tilde{K}^n(\tilde{g},\tilde{F})$ is called a holomorphically projective mapping if it maps an arbitrary holomorphically flat curve of K^n onto a holomorphically flat curve of $\tilde{K}^n$ and $F=\tilde{F}$, i.e. if it preserves the complex structure.

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[1] The Roman and the Greek indices run over the range $1\ldots n$; the Roman indices are free but the Greek indices denote summation.

Definition 2. ([6]) A Kaehler space is called a Kaehler-Codazzi space $K - C^n$ if its Ricci tensor satisfies the condition

$$R_{ij,k} = R_{ik,j}. \tag{1}$$

The main purpose of the present paper if to prove the following

Theorem. *If an n-dimensional Kaehler-Codazzi space $K-C^n$ $(n > 2)$ has a nontrivial holomorphically projective mapping onto another n-dimensional Kaehler space (i.e. the $K - C^n$ admits a holomorphically projective mapping) then $K - C^n$ is a Kaehler-Einstein space.*

We will use the following results (V.V. Domashev and J. Mikesh [5]):

(A) A Kaehler space K^n has a nontrivial holomorphically projective mapping to another Kaehler space iff in K^n there exist a symmetric tensorfield a_{ij}, and a gradient vectorfield λ_i such that

$$a_{ij,k} = \lambda_{(i} g_{j)k} + \bar{\lambda}_{(i} F_{j)k} \qquad \text{and} \tag{2}$$

$$\bar{\lambda}_i = \lambda_\beta F_i^\beta, \tag{3}$$

where (ij) means symmetrization.

(B) A Kaehler space $K^n(g, F)$ admits a holomorphically projective mapping iff the differential equations

$$a_{\alpha(i} R^\alpha_{.j)kl} = g_{k(j}\lambda_{i)l} - F_{k(j}\bar{\lambda}_{i),l} - g_{l(j}\lambda_{i)k} + F_{l(j}\bar{\lambda}_{i),k} \tag{4}$$

$$n\lambda_{i,l} = \mu g_{il} - a_{\alpha\beta} R^{\alpha}_{.\,il.}{}^{\beta} + a_{\alpha i} R^\alpha_l \tag{5}$$

$$\mu_{,k} = 2\lambda_\alpha R^\alpha_k \tag{6}$$

have nontrivial solutions in a_{ij}, λ_i and the scalar field μ.

2. The proof of the Theorem

Let us assume that the Kaehler-Codazzi space $K-C^n$ $(n > 2)$ admits a holomorphically projective mapping.

Differentiating (5) covariantly, and applying (6) and (2), one gets

$$\begin{aligned} n\lambda_{,ilk} = {} & 2\lambda^\alpha R_{\alpha k} g_{il} + \lambda^\alpha R_{\alpha l} g_{ik} + \lambda_i R_{lk} \\ & - \lambda_\alpha R^\alpha_{.ilk} - \lambda_\beta R^\beta_{.lik} - \bar{\lambda}_\alpha F_{\beta k} R^{\alpha}_{.il.}{}^{\beta} - \bar{\lambda}_\beta F_{\alpha k} R^{\alpha}_{.il.}{}^{\beta} \\ & + \bar{\lambda}_\alpha R^\alpha_l F_{ik} + \bar{\lambda}_i F_{\alpha k} R^\alpha_l + \alpha_{\alpha i} R^\alpha_{l,k} - \alpha_{\alpha\beta} R^{\alpha}_{.il.,k}{}^{\beta}. \end{aligned}$$

Transvecting it by g^{lk}, and using that the form (1) implies $R_{ijk}{}^{\alpha}_{.,\alpha} = 0$ and $R_i{}^\alpha_{.,\alpha} = 0$, we have

$$n\lambda_{,i}{}^{\alpha}_{.\alpha} = R\lambda_i$$

where $R = R_{\alpha\beta} g^{\alpha\beta}$, and $R = \text{const}$ [6]. Further on ϱ denotes the quotient R/n. Then we obtain

$$\lambda_{,i}{}^{\alpha}_{.\alpha} = \varrho\lambda_i \tag{7}$$

and (3) implies the following equation

$$\bar{\lambda}_{j,}{}^{\alpha}_{.\alpha} = \varrho\bar{\lambda}_j. \tag{7'}$$

The covariant derivative of (4) gives

$$
\begin{aligned}
&a_{\alpha(i,|m|}R^{\alpha}_{.j)kl} + a_{\alpha(i}R^{\alpha}_{.j)kl,m} = \\
&g_{k(j}\lambda_{,i)lm} - F_{k(j}\bar{\lambda}_{i),lm} - g_{l(j}\lambda_{,i)km} + F_{l(j}\bar{\lambda}_{i),km}.
\end{aligned} \tag{8}
$$

Substitute (2) into (8) and transvect it by g^{lm}. After applying (7) and (7′) we get

$$
\begin{aligned}
&\lambda_{\alpha}R^{\alpha}_{.(i|k|j)} + \lambda_{(i}R_{j)k} + \bar{\lambda}_{\alpha}F^{\beta}_{(i}R^{\alpha}_{.j)k\beta} + \bar{\lambda}_{(i}F^{\alpha}_{j)}R_{\alpha k} \\
&\quad = \varrho\lambda_{(i}g_{j)k} + \varrho\bar{\lambda}_{(i}F_{j)k} - \lambda_{k(ij)} - F^{\alpha}_{(i}\bar{\lambda}_{j),k\alpha}.
\end{aligned}
$$

By using the Ricci identity we can rewrite the above equation into the following form

$$
\lambda_{(i}R_{j)k} + 2\bar{\lambda}_{\alpha}F^{\beta}_{(i}R^{\alpha}_{.j)k\beta} + \bar{\lambda}_{(i}F^{\alpha}_{j)}R_{\alpha k} = \varrho\lambda_{(i}g_{j)k} + \varrho\bar{\lambda}_{(i}F_{j)k} - 2\lambda_{k(ij)}. \tag{9}
$$

If we denote $F^{\alpha}_{i}R_{\alpha j}$ by $\tilde{R}_{ij}$ and antisymmetrize (9) in j and k we obtain

$$
\begin{aligned}
&\lambda_{[j}R_{k]i} + 2\bar{\lambda}_{\alpha}F^{\beta}_{i}(R^{\alpha}_{.jk\beta} + R^{\alpha}_{.k\beta j}) + \bar{\lambda}_{[j|}\tilde{R}_{i|k]} + 2\lambda_{i}\tilde{R}_{jk} \\
&\quad = \varrho\lambda_{[j}g_{k]i} + \varrho\bar{\lambda}_{[j}F_{|i|k]} + 2\varrho\bar{\lambda}_{i}F_{jk} + 2\lambda_{\alpha}R^{\alpha}_{.ijk}
\end{aligned} \tag{10}
$$

(where "[ ]" means the antisymmetrization).

The identity $R^{i}_{.(jkl)} = 0$ gives that $\bar{\lambda}_{\alpha}F^{\beta}_{i}(R^{\alpha}_{.jk\beta} + R^{\alpha}_{.k\beta j}) = \lambda_{\alpha}R^{\alpha}_{.ijk}$ so we get from (10) that

$$
\lambda_{[j}A_{k]i} + \bar{\lambda}_{[j|}\tilde{A}_{i|k]} + 2\bar{\lambda}_{i}\tilde{A}_{jk} = 0 \tag{11}
$$

where

$$
\begin{aligned}
A_{ij} &= \varrho g_{ij} - R_{ij}, \\
\tilde{A}_{ij} &= F^{\alpha}_{i}A_{\alpha j} = \varrho F_{ij} - \tilde{R}_{ij}.
\end{aligned} \tag{12}
$$

If we transvect (9) by F^{k}_{l} and apply (3), we have

$$
\lambda_{(i|}\tilde{R}_{l|j)} + 2\bar{\lambda}_{\alpha}R^{\alpha}_{.(i|l|j)} + \bar{\lambda}_{(i}R_{j)l} = \varrho\lambda_{(i|}F_{l|j)} + \varrho\bar{\lambda}_{(i}g_{j)l} - 2\bar{\lambda}_{l,(ij)}. \tag{9'}
$$

Symmetrizing (9′) in i and l, exchanging indices j and l, and adding the original equation to the result we get

$$
\bar{\lambda}_{(i}A_{j)l} + \bar{\lambda}_{l}A_{ij} = 0. \tag{13}
$$

Further contraction with $F^{l}_{k}F^{j}_{m}F^{i}_{p}$ gives us

$$
\lambda_{p}A_{mk} + \lambda_{m}A_{pk} + \lambda_{k}A_{pm} = 0. \tag{14}
$$

After transvecting (13) by F^{l}_{k}, antisymmetrizing in j and k and using (11) we immediately obtain

$$
\lambda_{j}A_{ki} = \lambda_{k}A_{ij}.
$$

Thus there exist a scalar field A such that

$$
A_{ij} = A\lambda_{i}\lambda_{j}. \tag{15}
$$

From (14) and (15) we get

$$
3A\lambda_{i}\lambda_{j}\lambda_{k} = 0
$$

thus $A = 0$. So (15) gives us

$$A_{ij} = 0.$$

Conversely from (12) we have

$$R_{ij} = \varrho g_{ij}$$

i.e. $K - C^n$ is an Einstein space. □

Remark. G. Tian and S.T. Yau [8] proved that there exist Kaehler-Einstein metrics in the compact case.

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