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THE SMALLEST DENOMINATOR FUNCTION AND THE RIEMANN FUNCTION

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Abstract. In this paper, we initiate a rigorous and detailed study of the smallest denominator function and the Riemann function. For this, we first establish some basic facts about real numbers and the divisibility of integers.

Introduction

In the existing literature at least three important particular functions are associated with the name of B. Riemann. The most simple and well-known one is usually defined by

$$f(x) = 0 \qquad \text{if} \qquad x \in \mathbb{R} \setminus \mathbb{Q}$$

and

$$f(x) = 1/q \qquad \text{if} \qquad x \in \mathbb{Q}, \quad x = p/q, \quad p \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad (p; q) = 1.$$

Such functions have, for instance, been studied by J. J. Benedetto [2, pp. 24–24], I. Szalay [8] and Z. Németh [6] in greater detail.

It is surprising that the above Riemann function f is frequently only touched upon in or even omitted from the standard textbooks on mathematical analysis. Moreover, it is usually not properly defined and treated in the avaiable literature. For instance, none of the above mentioned authors observed that the Riemann function f should be preceeded by the following even more important functions defined by

$$q(x) = \min\{ n \in \mathbb{N} : \quad n\,x \in \mathbb{Z} \} \qquad \text{and} \qquad p(x) = x\,q(x) \quad \text{for all} \quad x \in \mathbb{Q}.$$

In this respect, it is even more surprising that the functions q and p seem also to be negligated by the authors of the standard textbooks on number theory.

The main purpose of this paper is to initiate a rigorous and detailed study of the above functions q, p and f in order that they could gain a proper place in the teaching of mathematical analysis and number theory. For this, because of the lack of completely satisfactory references, it seems neccessary to establish first some basic facts about real numbers and the divisibility of integers in the next two preparatory sections. We think that this may also be of some use for those who are not really interested in the above functions.

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1. A few basic facts about the real number system

In the first-semester course of mathematical analysis one usually starts with an appropriate system of axioms for the real numbers instead of that for the natural or rational numbers. For this, it is most convenient to assume that $\mathbb{R}\,(\,+\,,\,\cdot\,,\,\leq\,)$ is a fixed complete, totally ordered field, which will then be called the real number system.

To briefly formulate the corresponding axioms, the students must already be well-acquainted with some basic definitions about groups and ordered sets. For instance, they must know that an ordered set is complete if each of its nonvoid subsets that is bounded below has an infimum. And the infimum of a subset is the maximum of the set of its lower bounds.

After deriving some immediate consequences of the definition of the real number system concernig the operations, order and absolute value, a precise definition of the natural numbers has to be established. This can be done in the most simple and elegant way by using the auxiliary notion of an inductive set.

Definition 1.1. A subset A of $\mathbb{R}$ is called inductive if $1 \in A$, and $x \in A$ implies $x+1 \in A$.

Moreover, the members of the set

$$\mathbb{N} = \bigcap \big\{ A \subset \mathbb{R} : \quad A \text{ is inductive} \big\}$$

are called the natural numbers.

Hence, by noticing that intersections of inductive sets are inductive, we can at once state

Theorem 1.2. *$\mathbb{N}$ is the smallest inductive subset of $\mathbb{R}$.*

Remark 1.3. Therefore, if A is an inductive subset of $\mathbb{R}$, then $\mathbb{N} \subset A$. Thus the usual proofs by mathematical induction could be applied to prove the subsequent theorems. However, we prefer to use sets of natural numbers since they are more precise means than sequences of statements.

Theorem 1.4. *If $n \in \mathbb{N}$, then $n \geq 1$.*

Proof. It is clear that the set $A = \big\{ x \in \mathbb{R} : \quad x \geq 1 \big\}$ is inductive. Therefore, we have $\mathbb{N} \subset A$, and hence $n \in A$.

Theorem 1.5. *If $m, n \in \mathbb{N}$, then $m+n, \, mn \in \mathbb{N}$.*

Proof. Note that the sets

$$A = \big\{ k \in \mathbb{N} : \quad m+k \in \mathbb{N} \big\} \qquad \text{and} \qquad B = \big\{ k \in \mathbb{N} : \quad mk \in \mathbb{N} \big\}$$

are inductive. Therefore, $\mathbb{N} \subset A$ and $\mathbb{N} \subset B$, and hence $n \in A$ and $n \in B$.

Theorem 1.6. *If* $m, n \in \mathbb{N}$, *then* $m \le n$ *or* $m - n \in \mathbb{N}$.

Proof. For each $k \in \mathbb{N}$, define

$$A_k = \{ l \in \mathbb{N} : \quad l \le k \quad \text{or} \quad l - k \in \mathbb{N} \}.$$

Then, to prove the theorem, it would be enough to show only that the set A_n is inductive. Namely, in this case, we would have $\mathbb{N} \subset A_n$. Thus, in particular, the inclusion $m \in A_n$, i. e., the assertion of the theorem would be true.

Unfortunately, we cannot prove directly that the set A_n is inductive. Therefore, we shall rather prove that the set

$$B = \{ k \in \mathbb{N} : \quad A_k \ \text{is inductive} \}$$

is inductive. Namely, in this case, we have $\mathbb{N} \subset B$. Thus, in particular, $n \in B$, i. e., the set A_n is inductive.

To verify that $1 \in B$, i. e., the set

$$A_1 = \{ l \in \mathbb{N} : \quad l \le 1 \quad \text{or} \quad l - 1 \in \mathbb{N} \}$$

is inductive, note that $1 \in A_1$. Moreover, if $l \in A_1$, then

$$(l+1) - 1 = l \in A_1 \subset \mathbb{N},$$

and thus $l + 1 \in A_1$.

Now, we need only show that if $k \in B$, then $k + 1 \in B$. That is, if A_k is inductive, then A_{k+1} is also inductive. For this, note that $1 \in A_{k+1}$. Moreover, if $l \in A_{k+1}$, then since A_k is inductive, and hence $\mathbb{N} \subset A_k$, we also have $l \in A_k$. Therefore,

$$l \le k \qquad \text{or} \qquad l - k \in \mathbb{N}.$$

Hence, it is clear that

$$l + 1 \le k + 1 \qquad \text{or} \qquad (l+1) - (k+1) \in \mathbb{N}.$$

Therefore, $l + 1 \in A_{k+1}$, and thus A_{k+1} is also inductive.

Now, by observing that $m - n \notin \mathbb{N}$ whenever $m \le n$, we may also introduce the following

Definition 1.7. The members of the set

$$\mathbb{Z} = \{ m - n : \quad m, n \in \mathbb{N} \}$$

are called the integers.

Hence, by Theorem 1.5, it is clear that we have

Theorem 1.8. *If* $k, l \in \mathbb{Z}$, *then* $k + l$, $k - l$, $k\,l \in \mathbb{Z}$.

Moreover, by Theorem 1.6, it is clear that we also have

Theorem 1.9. $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$.

Now, as a useful consequence of the above two theorems, we can also prove

Theorem 1.10. *If* $k, l \in \mathbb{Z}$, *then* $l \leq k$ *or* $k+1 \leq l$.

Proof. In this case, by Theorem 1.8, we have $l-k \in \mathbb{Z}$. Hence, by Theorem 1.9, it follows that $l-k \in \mathbb{N}$ or $l-k=0$ or $l-k \in -\mathbb{N}$. Therefore, by Theorem 1.4, we can also state that $l-k \geq 1$ or $l-k=0$ or $-(l-k) \geq 1$. Hence, it is already clear that $l \leq k-1$ or $l=k$ or $k+1 \leq l$. And thus the statement of the theorem is also true.

Now, by using the latter theorem, we can also prove the following basic

Theorem 1.11. *If* A *is a nonvoid subset of* $\mathbb{Z}$ *such that* A *is bounded below in* $\mathbb{R}$, *then* $\min(A)$ *exists.*

Proof. Because of the completeness of $\mathbb{R}$, we may define

$$\alpha = \inf(A).$$

Hence, since $\alpha < \alpha + 1$, it is clear that there exists an $l \in A$, such that

$$\alpha \leq l < \alpha + 1.$$

Now, to prove the theorem, we need only show that $\alpha = l$.

For this, note that if $\alpha < l$, then again by the equality $\alpha = \inf(A)$ there exists a $k \in A$ such that $\alpha \leq k < l < \alpha + 1$. Hence, it follows that $0 < l - k < 1$, i.e., $k < l < k+1$. And this contradicts Theorem 1.10.

Remark 1.12. Hence, in particular, it is clear that $\mathbb{Z}$ is also a complete ordered set.

Moreover, from Theorems 1.4 and 1.11, we can also see that $\mathbb{N}$ is, in turn, a well-ordered set.

On the other hand, as another immediate consequence of Theorem 1.11, we can also prove

Theorem 1.13. *If* A *is a nonvoid subset of* $\mathbb{Z}$ *such that* A *is bounded above in* $\mathbb{R}$, *then* $\max(A)$ *exists.*

Proof. Note that $-A$ is now a nonvoid subset of $\mathbb{Z}$ which is bounded below in $\mathbb{R}$. Therefore, by Theorem 1.11, $\alpha = \min(-A)$ exists. And hence, it is clear that $-\alpha = \max(A)$.

Remark 1.14. From Theorems 1.2 and 1.4, we know that $\min(\mathbb{N}) = 1$. Moreover, by using Theorem 1.13, we can easily see that $\sup(\mathbb{N}) = +\infty$. Therefore, by Theorem 1.9, we also have

$$\inf(\mathbb{Z}) = -\infty \qquad \text{and} \qquad \sup(\mathbb{Z}) = +\infty.$$

Note that if $\mathbb{N}$ is bounded above in $\mathbb{R}$, then from Theorem 1.13 we can see $n = \max(\mathbb{N})$ exists. Moreover, from Theorem 1.2 we can see that $n+1 \in \mathbb{N}$. Therefore, $n+1 \leq n$, i.e., $1 \leq 0$. And this contradicts an earlier consequence of the axioms of $\mathbb{R}$ that $0 < 1$.

Because of the equality $\inf(\mathbb{Z}) = -\infty$ and Theorem 1.13, it is clear that the following definition is correct.

Definition 1.15. If $x \in \mathbb{R}$, then the numbers

$$[\,x\,] = \max\big\{\, k \in \mathbb{Z} : \quad k \leq x \,\big\}, \qquad \text{and} \qquad \langle\, x\,\rangle = x - [x]$$

are called the integral and the fractional parts of x, respectively.

Hence, it is clear that we have

Theorem 1.16. *If $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, then the following asssertions are equivalent:*

(1) $k = [x]$; *(2)* $k \leq x < k+1$; *(3)* $x - 1 < k \leq x$.

Moreover, by using the latter theorem, we can also easily prove

Theorem 1.17. *If $k \in \mathbb{R}$, then the following assertions are equivalent:*

(1) $k \in \mathbb{Z}$; *(2)* $[k] = k$; *(3)* $\langle\, k\,\rangle = 0$;

(4) $[\,x+k\,] = [x] + k$ *if* $x \in \mathbb{R}$; *(5)* $\langle\, x+k\,\rangle = \langle\, x\,\rangle$ *if* $x \in \mathbb{R}$.

Hint. If $x \in \mathbb{R}$, then by Theorem 1.16 we have $[x] \leq x < [x]+1$, i.e.,

$$[x] + k \leq x + k < ([x]+k) + 1\,.$$

Hence, if $k \in \mathbb{Z}$, then again by Theorem 1.16 it is clear that $[\,x+k\,] = [x]+k$. Therefore, the implication $(1) \Longrightarrow (4)$ is true.

To prove the implication $(5) \Longrightarrow (1)$, note that if the assertion (5) holds, then in particular we have $\langle\, k\,\rangle = \langle\, 0+k\,\rangle = \langle\, 0\,\rangle = 0$. And thus the assertion (1) also holds.

On the other hand, by using Theorem 1.11, we can also prove the next fundamental

Theorem 1.18. *If Γ is a nonzero additive subgroup of $\mathbb{Z}$, then there exists an $n \in \mathbb{N}$ such that*

$$\Gamma = \mathbb{Z}\,n = \big\{\, k\,n : \quad k \in \mathbb{Z} \,\big\}.$$

Proof. Since $\Gamma \neq \{0\}$, there exists $m \in \Gamma$ such that $m \neq 0$. Hence, since $m < 0$ implies $-m > 0$, and $m \in \Gamma$ implies $-m \in \Gamma$, it is clear that $m \in \Gamma \cap \mathbb{N}$ or $-m \in \Gamma \cap \mathbb{N}$. Therefore, $\Gamma \cap \mathbb{N} \neq \emptyset$. Thus, since $\mathbb{N}$ is well-ordered, the minimum

$$n = \min(\,\Gamma \cap \mathbb{N})$$

exists. Moreover, since Γ is closed under addition and subtraction, we can see by induction that $\mathbb{Z}\,n \subset \Gamma$.

To prove the converse inclusion, note that if $m \in \Gamma$ such that $m > 0$, then because of the inclusion $m \in \Gamma \cap \mathbb{N}$ and the definition of the number n, we have $n \leq m$, and hence $1 \leq m/n$. Therefore, under the notation

$$k = [\,m/n\,]$$

we have $1 \le k$. On the other hand, by Theorem 1.16, it is clear that $k \le m/n < k+1$, and hence $kn \le m < kn+n$. Therefore, under the notation

$$r = m - kn$$

we have $0 \le r < n$. Moreover, since Γ is closed under addition and subtraction, it is clear that $r \in \Gamma$ is also true. Hence, by the definition of the number n, we can see that only $r = 0$ can hold. Therefore, $m = kn \in \mathbb{Z}n$. Now, since $\mathbb{Z}n$ is a group with respect to addition, it is clear that the inclusion $\Gamma \subset \mathbb{Z}n$ is also true.

Now, by noticing that $m/n \notin \mathbb{N}$ whenever $m, n \in \mathbb{N}$ such that $m < n$, we may also introduce the following

Definition 1.19. The members of the set

$$\mathbb{Q} = \{ m/n : \quad m, n \in \mathbb{Z}, \quad n \neq 0 \}$$

are called the rational numbers.

Hence, by Theorem 1.8, it is clear that we have

Theorem 1.20. *If $r, s \in \mathbb{Q}$, then $r+s$, $r-s$, $rs \in \mathbb{Q}$. Moreover, if $s \neq 0$, then $r/s \in \mathbb{Q}$.*

Remark 1.21. Therefore, if $r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $r + x \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, if $r \neq 0$, then $rx \in \mathbb{R} \setminus \mathbb{Q}$.

Moreover, concerning the rational numbers, we can also prove the following important

Theorem 1.22. *If $x, y \in \mathbb{R}$ such that $x < y$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. Define

$$n = \left[1/(y-x) \right] + 1 \qquad \text{and} \qquad m = \left[nx + 1 \right].$$

Then, by Theorem 1.16, it is clear that

$$nx < m \le nx + 1 \qquad \text{and} \qquad 1/(y-x) < n,$$

i. e.,

$$0 < m - nx \le 1 \qquad \text{and} \qquad 1 < n(y-x).$$

Hence, we can see that

$$0 < m - nx < n(y-x), \qquad \text{i. e.,} \qquad x < m/n < y.$$

Therefore, the number $r = m/n$ has the required properties.

Remark 1.23. Note that the definitions of the numbers m and n have been found by using the converse argument.

By applying another natural argument, we can also easily get to the definitions

$$n = \big[2/(y-x)\big] + 1 \qquad \text{and} \qquad m = \big[n\,(x+y)/2\big]\,.$$

2. A few basic fact about the divisibility of integers

The question of divisibility is usually not discussed in an introductory course of analysis, despite that some results of this type may be needed there. For instance, it may be useful to prove that $\sqrt[n]{n}$ is irrational whenever $n = 2\,,\,3\,,\ldots\,.$

Therefore, it may also be of some interest to present a treatment of the divisibility of integers based upon the real number system. In the sequel, we shall mainly list and prove only those results which are necessary to precisely work out the subjects of the forthcoming sections.

Definition 2.1. If $m\,,\,n \in \mathbb{Z}$ such that there exists an $k \in \mathbb{Z}$ with $m = k\,n\,,$ then we say that n divides $m\,,$ and we write $n\,|\,m\,.$

Remark 2.2. Note that $1|m\,,$ $m|m$ and $n|0\,.$ Moreover,

$$n\,|\,m \iff m \in \mathbb{Z}\,n \iff \mathbb{Z}\,m \subset \mathbb{Z}\,n\,.$$

Concerning the division in $\mathbb{Z}\,,$ it is also important to note the following simple but important theorems.

Theorem 2.3. *If $m\,,\,n \in \mathbb{Z}$ such that $n\,|\,m$ and $m \neq 0\,,$ then $|\,n\,| \leq |\,m|\,.$*

Remark 2.4. Therefore, if $m \in \mathbb{Z}$ such that $m \neq 0\,,$ then set of all divisors of m is bounded.

Theorem 2.5. *If $m\,,\,n\,,\,k \in \mathbb{Z}$ such that $k\,|\,m$ and $k\,|\,n\,,$ and moreover $\alpha\,,\,\beta \in \mathbb{Z}\,,$ then $k\,|\,(\alpha\,m + \beta\,n)\,.$*

By Remark 2.4 and Theorem 1.13, it is clear that the following definition is correct.

Definition 2.6. If $m\,,\,n \in \mathbb{Z}$ such that $m \neq 0$ or $n \neq 0\,,$ then the number

$$(\,m\,;\,n\,) \,=\, \max\big\{\,k \in \mathbb{Z}\,: \quad k\,|\,m\,, \quad k\,|\,n\,\big\}$$

is called the greatest common divisor of m and $n\,.$

Remark 2.7. If in particular $(\,m\,;\,n) = 1\,,$ then we say that the numbers m and n are relatively prime.

The existence of the greatest common divisor is usually proved by the Euclidean algorithm. However, we prefer to use the subsequent more precise proof.

Theorem 2.8. *If $m\,,\,n \in \mathbb{Z}$ such that $m \neq 0$ or $n \neq 0\,,$ then there exist $\alpha\,,\,\beta \in \mathbb{Z}$ such that*

$$(\,m\,;\,n\,) \,=\, \alpha\,m + \beta\,n\,.$$

Proof. Let $k = (m\,;\,n)$ and

$$\Gamma = \big\{ \alpha m + \beta n : \quad \alpha , \beta \in \mathbb{Z} \big\} .$$

Then, it is easy to see that Γ is a nonzero additive subgroup $\mathbb{Z}$. Therefore, by Theorem 1.18, there exists an $l \in \mathbb{N}$ such that

$$\Gamma = \mathbb{Z}\, l .$$

Hence, since $l \in \mathbb{Z}\, l$, it is clear that there exist $\alpha , \beta \in \mathbb{Z}$ such that

$$l = \alpha m + \beta n .$$

Now, since $k \mid m$ and $k \mid n$, it is clear that $k \mid l$, and thus $k \leq l$. On the other hand, since $m , n \in \Gamma = \mathbb{Z}\, l$, it is clear that $l \mid m$ and $l \mid n$, and thus $l \leq k$ is also true. Therefore, we have $l = k$, and thus the required equality is also true.

Now, by using Theorem 2.8, we can also easily prove the following

Theorem 2.9. *If $m , n \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $k \mid m$ and $k \mid n$, then the following assertions are equivalent:*

(1) $k = (m \,;\, n)$; *(2)* $\exists\, \alpha , \beta \in \mathbb{Z} : \quad k = \alpha m + \beta n$;

(3) $l \mid m ,\ l \mid n \implies l \mid k$; *(4)* $\big(m/k \,;\, n/k \big) = 1$.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (1) are quite obvious by Theorems 2.8, 2.5 and 2.3.

To prove the equivalence of the assertions (1) and (4), it is now enough to note only that for each $\alpha , \beta \in \mathbb{Z}$ we have $k = \alpha m + \beta n$ if and only if $1 = \alpha \big(m/k \big) + \beta \big(n/k \big)$.

Remark 2.10. From the above theorem we can easily see that if $m , n \in \mathbb{Z}$, then the following assertions are equivalent:

(1) $(m \,;\, n) = 1$; (2) $\exists\, \alpha , \beta \in \mathbb{Z} : \quad \alpha m + \beta n = 1$.

Moreover, by using Theorem 2.8, we can also prove the following

Theorem 2.11. *If $m_1 , m_2 , n \in \mathbb{Z}$ such that $n \mid m_1 m_2$, and $k = (m_1 ;\, n)$, then $n \mid k\, m_2$.*

Proof. In this case, by Theorem 2.8, there exist $\alpha , \beta \in \mathbb{Z}$ such that

$$k = \alpha m_1 + \beta n , \qquad \text{and thus} \qquad k\, m_2 = \alpha m_1 m_2 + \beta n\, m_2 .$$

And hence, by Theorem 2.5, it is clear that the reqiured assertion is also true.

Moreover, by using Remark 2.10 and Theorem 2.11, we can also prove

Theorem 2.12. *If $m , n \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $k \mid m$ and $k \mid n$, and $\alpha , \beta \in \mathbb{Z}$ such that $k = \alpha m + \beta n$, then $(\alpha ; \beta) = 1$.*

Proof. In this case, for the integers $m_1 = m/k$ and $n_1 = n/k$, we have $1 = \alpha m_1 + \beta n_1$. Therefore, by Remark 2.10, we also have $(\alpha , \beta) = 1$.

Remark 2.13. From the latter theorem we can also at once see that if $m, n \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{Z}$ such that $\alpha m + \beta n = 1$, then $(\alpha ; \beta) = 1$.

Theorem 2.14. *If $m, n \in \mathbb{Z}$ such that $m \neq 0$ or $n \neq 0$, and $\alpha, \beta \in \mathbb{Z}$, then the following assertions are equivalent:*

(1) $\alpha m + \beta n = 0$;

(2) $\exists\, l \in \mathbb{Z} : \quad \alpha = l n/(m ; n), \quad \beta = -l m/(m ; n)$.

Proof. If the assertion (1) holds, then

$$m \alpha = -\beta n .$$

Hence, by Theorem 2.11, it is clear that under the notation $k = (m ; n)$ there exists an $l \in \mathbb{Z}$ such that

$$k \alpha = l n , \qquad \text{and thus} \qquad \alpha = l n/k .$$

Moreover,

$$k \beta = k\,(-m \alpha/n) = -l m , \qquad \text{and thus} \qquad \beta = -l m/k ,$$

provided that $n \neq 0$. Therefore, in this case, the assertion (2) is also true.

The converse implication $(2) \Longrightarrow (1)$ is quite obvious.

By using Theorem 2.9, we can also prove the following

Theorem 2.15. *If $m, n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $\alpha_i , \beta_i \in \mathbb{Z}$, for each $i = 1, 2$, such that $\alpha_1 \beta_2 - \alpha_2 \beta_1 = 1$, then the following assertions are equivalent:*

(1) $k = (m ; n)$; *(2)* $k = (\alpha_1 m + \beta_1 n ;\ \alpha_2 m + \beta_2 n)$.

Hint. If the assertion (1) holds, then by Theorem 2.9, there exist $\alpha, \beta \in \mathbb{Z}$ such that

$$\alpha m + \beta n = k .$$

Hence, since for the integers $\alpha' = \alpha \beta_2 - \beta \alpha_2$ and $\beta' = \alpha_1 \beta - \beta_1 \alpha$ we have

$$\alpha_1 \alpha' + \alpha_2 \beta' = \alpha \qquad \text{and} \qquad \beta_1 \alpha' + \beta_2 \beta' = \beta ,$$

it is easy to see that

$$\alpha' (\alpha_1 m + \beta_1 n) + \beta' (\alpha_2 m + \beta_2 n) = k .$$

Therefore, again by Theorem 2.9, it is clear that the assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can now be derived from the implication $(1) \Longrightarrow (2)$.

Remark 2.16. From Theorem 2.15 we can at once see that if $m, n, k \in \mathbb{Z}$, then the following assertions are equivalent:

(1) $(m ; n) = 1$; (2) $(m + k n ;\ n) = 1$.

3. The smallest denominator function

To precisely define and easily investigate the Riemann function, it is necessary to consider first the following two interesting functions defined only for rational numbers.

Definition 3.1. For each $x \in \mathbb{Q}$, we define

$$q(x) = \min\{\, n \in \mathbb{N} : \quad n\,x \in \mathbb{Z}\,\} \qquad \text{and} \qquad p(x) = x\,q(x)\,.$$

Remark 3.2. Note that the existence of $q(x)$ strongly depends upon the definition of $\mathbb{Q}$ and the well-orderedness of $\mathbb{N}$.

From Definition 3.1 we can easily get the following

Theorem 3.3. *$p(\mathbb{Q}) = \mathbb{Z}$ and $q(\mathbb{Q}) = \mathbb{N}$.*

Proof. By Definition 3.1, it is clear that if $m \in \mathbb{Z}$, then $q(m) = 1$, and thus $p(m) = m$. Moreover, if $n \in \mathbb{N}$, then we can easily see that $q(1/n) = n$. Namely, if $l \in \mathbb{N}$ such that $l < n$, then it is clear that $0 < l\,(1/n) < 1$. And thus, by Theorem 1.10, we have $l\,(1/n) \notin \mathbb{Z}$.

Moreover, by computing some futher values of the functions p and q, we can also easily establish the following

Theorem 3.4. *If $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then*

$$p\left(\frac{m}{n}\right) = \frac{m}{(m\,;\,n)} \qquad \text{and} \qquad q\left(\frac{m}{n}\right) = \frac{n}{(m\,;\,n)}\,.$$

Proof. Define

$$x = m/n\,, \qquad k = (m\,;\,n)\,, \qquad m_1 = m/k\,, \qquad n_1 = n/k\,.$$

Then, from the equality $q(x)\,x - p(x) = 0$, it is clear that

$$q(x)\,m + (-p(x))\,n \;=\; 0\,.$$

Therefore, by Theorem 2.14, thee exists an $l \in \mathbb{Z}$ such that

$$q(x) = l\,n_1 \qquad \text{and} \qquad p(x) = l\,m_1\,.$$

Hence, since $q(x)\,,\, n_1 \in \mathbb{N}$, it is clear that $l \in \mathbb{N}$. On the other hand, since

$$n_1 x = m_1 \in \mathbb{Z}\,,$$

by the definition of $q(x)$, we can easily see that

$$l\,n_1 = q(x) \le n_1\,.$$

Therefore, only $l \le 1$, and hence $l = 1$ can hold.

Remark 3.5. From Theorem 3.4, because of the equivalence of the assertions (1) and (4) in Theorem 2.9, it is clear that

$$\big(p(x)\,;\,q(x)\big) \,=\, 1$$

for all $x \in \mathbb{Q}$.

Therefore, as an immediate consequence of Remarks 2.10 and 2.13, we can also state the following

Theorem 3.6. *There exist functions* $\alpha, \beta : \mathbb{Q} \to \mathbb{Z}$ *such that*

$$\alpha(x)\,p(x) + \beta(x)\,q(x) \,=\, 1$$

for all $x \in \mathbb{Q}$. *Moreover, in this case we necessarily have* $\big(\alpha(x)\,;\,\beta(x)\big) = 1$ *for all* $x \in \mathbb{Q}$.

Remark 3.7. Furthermore, from Theorem 2.14 we can see that the functions α and β are very far from being unique.

From the $(m;\,n) = 1$ particular case of Theorem 3.4, by using Remark 2.16, we can also easily get the following

Theorem 3.8. *If* $k \in \mathbb{Q}$, *then the following assertions are equivalent:*

(1) $k \in \mathbb{Z}$; *(2)* $q(k) = 1$;

(3) $q(x+k) = q(x)$ *for all* $x \in \mathbb{Q}$.

Proof. If the assertion (1) holds, then because of $1k = k \in \mathbb{Z}$ it is clear that the assertion (2) also holds. While, if the assertion (2) holds, then because of $k = 1k = q(k)\,k = p(k) \in \mathbb{Z}$ it is clear that the assertion (1) also holds.

On the other hand, if $x \in \mathbb{Q}$ and $k \in \mathbb{Z}$, then by the equality $p(x) = x\,q(x)$ and Remark 2.16 it is clear that

$$x + k \,=\, \frac{p(x) + k\,q(x)}{q(x)} \qquad \text{and} \qquad \big(p(x) + k\,q(x)\,;\,q(x)\big) \,=\, 1\,.$$

And hence, by the corresponding particular case of Theorem 3.4, it is clear that the assertion (3) also holds.

Finally, if the assertion (3) holds, then it is clear in particular we also have

$$q(k) = q(0+k) = q(0) = 1\,.$$

Therefore, because of the equivalence of the assertions (1) and (2), the assertion (1) also holds.

From Theorem 3.8, because of the equality $p(x) = x\,q(x)$, it is clear that we also have

Theorem 3.9. *If* $k \in \mathbb{Q}$, *then the following assertions are equivalent:*

(1) $k \in \mathbb{Z}$; *(2)* $p(k) = k$;

(3) $p(x+k) = p(x) + k\,q(x)$ *for all* $x \in \mathbb{Q}$.

Moreover, by using the definition of the function q, we can also easily prove the following

Theorem 3.10. *If* $x, y \in \mathbb{Q}$, *then*

(1) $q(-x) = q(x)$; (2) $q(|x|) = q(x)$;

(3) $q(x+y) \leq q(x)\,q(y)$; (4) $q(xy) \leq q(x)\,q(y)$.

Hint. For instance, note that $q(x)\,q(y) \in \mathbb{N}$ such that

$$q(x)\,q(y)\,(x+y) \;=\; p(x)\,q(y) + q(x)\,p(y) \;\in\; \mathbb{Z}\,.$$

And thus, by the definition of q, the assertion (3) is true.

From Theorem 3.10, by the equality $p(x) = x\,q(x)$, it is clear that we also have

Theorem 3.11. *If* $x, y \in \mathbb{Q}$, *then*

(1) $p(-x) = -p(x)$; (2) $p(|x|) = |p(x)|$;

(3) $|p(x+y)| \leq |p(x)|\,q(y) + |p(y)|\,q(x)$; (4) $|p(xy)| \leq |p(x)|\,|p(y)|$.

4. An important limit property of the function q

To easily establish some more delicate properties of the function q, it seems convenient to consider first the following important distance function.

Definition 4.1. For each $x \in \mathbb{R}$, we define

$$\varphi(x) \;=\; d\big(x\,,\, \mathbb{Z} \setminus \{x\}\big)\,.$$

Remark 4.2. By the corresponding definitions, it is clear that

$$\varphi(x) \;=\; \inf\big\{\, |k-x| : \quad k \in \mathbb{Z} \setminus \{x\} \,\big\}\,.$$

Moreover, it can also be easily seen that instead of the infimum we may write minimum. However, for an easier application of the function φ, it is now more important to prove the following

Theorem 4.3. $\varphi(x) = 1$ *if* $x \in \mathbb{Z}$ *and*

$$\varphi(x) \;=\; \frac{1}{2} - \left|\, \langle x \rangle - \frac{1}{2} \,\right| \qquad \textit{if} \qquad x \in \mathbb{R} \setminus \mathbb{Z}\,.$$

Hint. If $x \in \mathbb{R} \setminus \mathbb{Z}$, then it is clear that $[x] \neq x$, and hence by Theorem 1.16 we have $[x] < x < [x] + 1$. On the other hand, if $k \in \mathbb{Z} \setminus \{x\}$, then by Theorem 1.10 we have $k \leq [x]$ or $[x] + 1 \leq k$. Therefore, by Remark 4.2, we also have

$$\varphi(x) = \min\big\{\, x - [x]\,,\, [x] + 1 - x \,\big\} \;=\; \min\big\{\, \langle x \rangle\,,\, 1 - \langle x \rangle \,\big\}\,.$$

Hence, since

$$\min\{\, a\,,\, b \,\} \;=\; \frac{1}{2}\,\Big(a + b - |a-b|\Big)$$

for all $a, b \in \mathbb{R}$, it is clear that the second assertion of the theorem is true.

From Theorem 4.3, by using Theorem 1.17, we can easily get the following

Theorem 4.4. *If* $k \in \mathbb{R}$, *then the following assertions are equivalent:*

(1) $k \in \mathbb{Z}$; *(2)* $\varphi(x+k) = \varphi(x)$ *for all* $x \in \mathbb{R}$.

Hint. To prove the implication $(2) \Longrightarrow (1)$, note that if the assertion does not hold, then from the assertion (2) by taking $x = 0$ we can get

$$1/2 - \left|\, \langle\, k\, \rangle - 1/2 \,\right| \;=\; \varphi(k) = \varphi(0) = 1\,,$$

whence the contradiction $\left|\, \langle\, k\, \rangle - 1/2 \,\right| \;=\; -1/2 < 0$ follows.

Moreover, by using Theorems 4.3 és 4.4, we can also easily prove the following

Theorem 4.5. $\varphi(\mathbb{R}) \;=\;]\,0\,,\,1/2\,] \cup \{1\}$.

Hint. To prove the less obvious inclusion, note that if $0 < x \le 1/2$, then $[x] = 0$, and thus $\langle\, x\, \rangle = x - [x] = x$. Moreover, $x - 1/2 \le 0$, and thus $\left|\, x - 1/2 \,\right| = 1/2 - x$. Therefore, $\varphi(x) = 1/2 - \left|\, \langle\, x\, \rangle - 1/2 \,\right| \;=\; x$.

Remark 4.6. Hence, we can see that in particular the equality $\varphi \circ \varphi = \varphi$ is also true.

Now, by using the function φ, we can also prove the next fundamental

Theorem 4.7. *If* $x_0 \in \mathbb{R}$ *and* $x \in \mathbb{Q} \setminus \{\, x_0 \}$, *then*

$$\left|\, x - x_0 \,\right| \;\ge\; \frac{1}{q(x)}\; \varphi\big(\, q(x)\, x_0 \big)\,.$$

Proof. By defining $m = p(x)$ and $n = q(x)$, we can at once see that

$$\begin{aligned} \left|\, x - x_0 \right| \;=\; \left|\, \frac{m}{n} - x_0 \right| \;=\; \frac{1}{n}\, \left|\, m - n\, x_0 \,\right| \ge \\ \ge\; \frac{1}{n}\; \inf \Big\{ \left|\, k - n\, x_0 \,\right| : \quad k \in \mathbb{Z} \setminus \{\, n\, x_0 \} \Big\} \;=\; \frac{1}{n}\; \varphi(\, n\, x_0)\,. \end{aligned}$$

From Theorem 4.7, we can now easily get the following

Theorem 4.8. *If* $x_0 \in \mathbb{R}$, $n \in \mathbb{N}$ *and* $x \in \mathbb{Q} \setminus \{\, x_0 \}$ *such that* $q(x) \le n$, *then*

$$\left|\, x - x_0 \,\right| \;\ge\; \min \left\{ \frac{1}{i}\; \varphi(\, i\, x_0) \right\}_{i=1}^{n}\,.$$

Proof. If $k = q(x)$, then by Theorem 4.7 and the condition $k \le n$, it is clear that

$$\left|\, x - x_0 \,\right| \;\ge\; \frac{1}{k}\; \varphi(\, k\, x_0) \;\ge\; \min \left\{ \frac{1}{i}\; \varphi(\, i\, x_0) \right\}_{i=1}^{n}\,.$$

Now, we can also easily prove the following remarkable

Theorem 4.9. *If* $x_0 \in \mathbb{R}$, *then*

$$\lim_{x \to x_0} q(x) = +\infty .$$

Proof. By Theorem 1.22, it is clear that x_0 is an accumulation point of the domain $\mathbb{Q}$ of the function q. On the other hand, if $\alpha \in \mathbb{R}$,

$$n = \big[\, |\alpha| \,\big] + 1 \qquad \text{and} \qquad \delta = \min \Big\{ (1/i)\ \varphi(\, i\, x_0) \Big\}_{i=1}^{n},$$

then by using Theorems 1.16, 4.5 and 4.8 we can easily see that $n \in \mathbb{N}$ such that $\alpha < n$, and $\delta > 0$ such that

$$x \in \mathbb{Q} \setminus \{ x_0 \}, \qquad |x - x_0| < \delta \qquad \text{imply that} \qquad n < q(x),$$

i. e., $\alpha < q(x)$.

Remark 4.10. Note that if $x_0 \in \mathbb{R}$ and (x_n) is a sequence in $\mathbb{Q} \setminus \{x_0\}$ such that $\lim_{n \to \infty} x_n = x_0$, then by Theorem 4.9 we have $\lim_{n \to \infty} q(x_n) = +\infty$.

While if (k_n) is an arbitrary sequence in $\mathbb{Z}$ and $x_n = (1 + k_n n)/n$ for all $n \in \mathbb{N}$, then by Remark 2.16 and Theorem 3.4 it is clear that

$$\lim_{n \to \infty} q(x_n) = \lim_{n \to \infty} n = +\infty,$$

despite that the sequence (x_n) need not even be bounded.

Finaly, in addition to Theorem 4.7, we also prove the following

Theorem 4.11. *If* $x_0 \in \mathbb{R}$, *then there exists a sequence* (x_n) *in* $\mathbb{Q}$ *such that*

$$\lim_{n \to \infty} x_n = x_0$$

and

$$|\, x_n - x_0 \,| \leq 1/q(x_n) \qquad \textit{and} \qquad x_{2n-1} \leq x_0 \leq x_{2n}$$

for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, define

$$x_{2n-1} = \big[\, (2\,n - 1)\, x_0 \,\big] \,/\, (2n - 1) \qquad \text{and} \qquad x_{2n} = \big[\, 2\,n\,x_0 + 1 \,\big] \,/\, (2n).$$

Then, from the assertion (3) of Theorem 1.16, we can easily see that

$$x_0 - 1/(2n-1) < x_{2n-1} \leq x_0 < x_{2n} \leq x_0 + 1/(2n).$$

Moreover, by the corresponding definitions, it is clear that $x_n \in \mathbb{Q}$ such that $q(x_n) \leq n$. Therefore

$$|\, x_n - x_0 \,| \leq 1/n \leq 1/q(x_n)$$

Remark 4.12. From the above proof, we can also see that for each $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists an $m \in \mathbb{Z}$ such that

$$|\, m/n - x_0 \,| \leq 1 \,/\, 2n \,.$$

In this respect, it is also worth mentioning that if for instance $x_0 = 3/(2n)$, then $|\, m/n - x_0 \,| \geq 1/(2n)$ for all $m \in \mathbb{Z}$. But if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then the approximation can be substantially improved. (See, for instance, [5, p. 277] and [4, pp. 317 and 343].)

5. The Riemann function

Now, by using the function q, we can easily introduce and investigate the Riemann function.

Definition 5.1. We define

$$f(x) = \frac{1}{q(x)} \quad \text{if} \quad x \in \mathbb{Q} \qquad \text{and} \qquad f(x) = 0 \quad \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}.$$

By using this definition of the Riemann function f, from Theorems 3.4, 3.6, 3.8 and 3.10, we can easily get the following theorems.

Theorem 5.2. *If $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then*

$$f\left(\frac{m}{n}\right) = \frac{(m\,;\,n)}{n}\,.$$

Theorem 5.3. *There exist functions $\alpha\,, \beta \colon \mathbb{R} \to \mathbb{Z}$ such that*

$$f(x) = \alpha(x)\, x + \beta(x)$$

for all $x \in X$. Moreover, in this case, we necessarily have

$$\big(\alpha(x)\,;\,\beta(x)\big) = 1 \quad \textit{if} \quad x \in \mathbb{Q} \qquad \textit{and} \qquad \alpha(x) = \beta(x) = 0 \quad \textit{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}.$$

Remark 5.4. Furthermore, from Theorem 2.14 we can see that the above functions α and β are very far from being unique.

Theorem 5.5. *If $k \in \mathbb{R}$, then the following assertions are equivalent:*

(1) $k \in \mathbb{Z}$; *(2)* $f(k) = 1$;

(3) $f(x+k) = f(x)$ *for all* $x \in \mathbb{R}$.

Theorem 5.6. *If $x\,, y \in \mathbb{R}$, then*

(1) $f(-x) = f(x)$; *(2)* $f(|\,x\,|) = f(x)$;

(3) $f(x+y) \geq f(x)\, f(y)$; *(4)* $f(x\,y) \geq f(x)\, f(y)$.

Moreover, from Theorem 4.9 we can also easily get the following remarkable

Theorem 5.7. *If* $x_0 \in \mathbb{R}$, *then*

$$\lim_{x \to x_0} f(x) = 0 .$$

Proof. If $\varepsilon > 0$, then by Theorem 4.9 there exists $\delta > 0$ such that

$$x \in \mathbb{Q} \setminus \{x_0\}, \quad |x - x_0| < \delta \implies 1/\varepsilon < q(x) .$$

Hence, by Definition 5.1, it is clear that

$$x \in \mathbb{R} \setminus \{x_0\}, \quad |x - x_0| < \delta \implies f(x) < \varepsilon .$$

From Theorem 5.7 we can now also easily get the next fundamental

Theorem 5.8. *The function* f *is continuous only at the points of* $\mathbb{R} \setminus \mathbb{Q}$. *Moreover,* f *is neither right or left continuous at the points of* $\mathbb{Q}$.

Proof. By Theorem 5.7 and Definition 5.1, it is clear that for each $x_0 \in \mathbb{R}$ we have

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0) \iff x_0 \in \mathbb{R} \setminus \mathbb{Q}.$$

and

$$\lim_{\substack{x \to x_0 \\ x_0 < x}} f(x) = f(x_0) \iff x_0 \in \mathbb{R} \setminus \mathbb{Q}.$$

Therefore, by a useful limit criterion for continuity, the required statement is also true.

Theorem 5.9. *The function* f *is neither right or left differentiable at the points of* $\mathbb{R}$.

Proof. If $x_0 \in \mathbb{Q}$, then from Theorem 5.8 we can immediately infer that the function f cannot be either right or left differentiable at the point x_0.

While, if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then by using the sequence (x_n) given in Theorem 4.11, we can easily see that

$$\left| \frac{f(x_n) - f(x_0)}{x_n - x_0} \right| = \frac{1}{|x_n - x_0|\, q(x_n)} \geq 1$$

for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n \to \infty} \frac{f(x_{2n-1}) - f(x_0)}{x_{2n-1} - x_0} \neq 0 \qquad \text{and} \qquad \lim_{n \to \infty} \frac{f(x_{2n}) - f(x_0)}{x_{2n} - x_0} \neq 0 .$$

Hence, since $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, it is already clear that the function f cannot have one-sided derivatives at the point x_0.

Remark 5.10. Now, by recalling that $\mathbb{Q}$ has measure zero, we can also state that f is a nowhere right or left differentiable, almost everywhere continuous function on $\mathbb{R}$.

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