Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
Tomus 14 (1998), 25-30

ON THE CALDERON–ZYGMUND DECOMPOSITION LEMMA ON THE WALSH–PALEY GROUP

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ABSTRACT. It is well-known that the Calderon-Zygmund decomposition lemma plays an extremely prominent role in the theory of harmonic analysis on the Walsh-Paley group. However, the proof of this lemma uses the fact that the 2^nth partial sums of the Walsh-Fourier series of an integrable function converges a.e. to the function. This later proved by techniques known in the martingale theory. In this paper we give a "purely dyadic harmonic analysis" proof for the Calderon-Zygmund decomposition lemma and also for this a.e. convergence.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $I := [0, 1)$ the unit interval. Denote the Lebesgue measure of any set $E \subset I$ by $|E|$. Denote the $L^p(I)$ norm of any function f by $\|f\|_p$ $(1 \leq p \leq \infty)$.

Denote the dyadic expansion of $n \in \mathbf{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbf{N}$ choose the expansion which terminates in zeros (these numbers are the dyadic rationals)). n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} (x_j + y_j \mod 2) 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbf{P}$ and $I_0(x) := I$ are the dyadic intervalls of I. Set $e_n := (0, \dots, 0, 1, 0, \dots)$ where the nth coordinate of e_n is 1 the rest are zeros for all $n \in \mathbf{N}$. The dyadic rationals are the finite $0, 1$ combinations of the elements of the set $(e_n : n \in \mathbf{N})$ (which dense in I).

Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh-Paley system [F, GES, SWS] that is,

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}, \ n \in \mathbf{N}, x \in I.$$

Denote by $D_n := \sum_{k=0}^{n-1} \omega_k$, the Walsh-Dirichlet kernels. It is well-known that [F, GES, SWS]

$$S_n f(y) = \int_I f(x) D_n(y + x) dx = f * D_n(y)$$

1980 *Mathematics Subject Classification* (1985 *Revision*). primary: 42C10, secondary: 43A75.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA) , grant no. F020334 and by the Hungarian "Művelődési és Közoktatási Minisztérium", grant no. FKFP 0710/1997.

($y \in I, n \in \mathbf{P}$) the n-th partial sum of the Walsh-Fourier series. Moreover, ([SWS, p. 28.])

$$D_{2^n}(x) := \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise} \end{cases}.$$

Then, this gives $S_{2^n}f(y) = 2^n \int_{I_n(y)} f(x)dx$ ($n \in \mathbf{N}$). We say that an operator $T : L^1 \to L^0$ ($L^0(I)$ is the space of measurable functions on I) is of type (p,p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq c_p\|f\|_p$ for all $f \in L^p(I)$ and constant c_p depends only on p. We say that T is of weak type $(1,1)$ if $|\{|Tf| > \lambda\}| \leq c\|f\|_1/\lambda$ for all $f \in L^1(I)$ and $\lambda > 0$.

In this paper c denotes an absolute constant which may not be the same at different occurences. For more on the Walsh system see [F, GES, SWS, Tai].

Theorem 1. (The Calderon-Zygmund decomposition, see, e.g. [SWS]). Let $f \in L^1(I)$, $\lambda > \|f\|_1$. *Then there exists a decomposition* $f = \sum_{j=0}^{\infty} f_j$, $I^j := I_{k_j}(u^j)$ *disjoint intervalls for which* supp $f_j \subseteq I^j$, $\int_{I^j} f_j = 0$, $\lambda < |I^j|^{-1}\int_{I^j}|f_j| \leq c\lambda$, $(u^j \in I,\ k_j \in \mathbf{N}, j \in \mathbf{P})$, $\|f_0\|_\infty \leq c\lambda$, $|F| \leq c\|f\|_1/\lambda$, *where* $F = \cup_{j\in\mathbf{P}} I^j$.

The proof of Theorem 1 uses the fact that the 2^nth partial sums of the Walsh-Fourier series of an integrable function converges a.e. to the function. This later is proved by techniques known in the martingale theory. We give a new proof for Theorem 1, which use techniques known in the theory of dyadic harmonic analysis, only. First we prove the following lemma which is similar to Theorem 1, but differs in the conditions to be proved for f_0.

Lemma 2. Let $f \in L^1(I)$, $\lambda > \|f\|_1$. *Then there exists a decomposition* $f = \sum_{j=0}^{\infty} f_j$, $I^j := I_{k_j}(u^j)$ *disjoint intervalls for which supp* $f_j \subseteq I^j$, $\int_{I^j} f_j = 0$, $\lambda < |I^j|^{-1}\int_{I^j}|f_j| \leq c\lambda$, $(u^j \in I,\ k_j \in \mathbf{N}, j \in \mathbf{P})$, $\limsup_{n\to\infty} S_{2^n}|f_0| \leq c\lambda$, $|F| \leq c\|f\|_1/\lambda$, *where* $F = \cup_{j\in\mathbf{P}} I^j$.

Proof of Lemma 2. Construct the following decomposition of the Walsh group I.

$$\Omega_0 := \{I_0(x) : 2^0 \int_{I_0(x)} |f(y)|dy > \lambda, x \in I\} = \emptyset,$$

$$\Omega_1 := \{I_1(x) : 2^1 \int_{I_1(x)} |f(y)|dy > \lambda, \nexists J \in \Omega_0 : I_1(x) \subset J, x \in I\},$$

$$\dots$$

$$\Omega_n := \{I_n(x) : 2^n \int_{I_n(x)} |f(y)|dy > \lambda, \nexists J \in \cup_{j=0}^{n-1}\Omega_j : I_n(x) \subset J, x \in I\}$$

($n \in \mathbf{P}$). Then, the elements of Ω_n are disjoint intervalls of measure $1/2^n$ ($n \in \mathbf{N}$). Moreover, if $i \neq j$, then for all $J \in \Omega_i, K \in \Omega_j$ we have $J \cap K = \emptyset$. If $I_n(x) \in \Omega_n$, then since there is no $J \in \cup_{j=0}^{n-1}\Omega_j : I_n(x) \subset J$, then $2^j \int_{I_j(x)} |f(y)|dy \leq \lambda$ for $j = 0, 1, ..., n-1$. Thus, $\lambda < 2^n \int_{I_n(x)} |f(y)|dy \leq 2^n \int_{I_{n-1}(x)} |f(y)|dy \leq 2\lambda$. Since Ω_n has a finite number of elements, then set the notation:

$$\Omega_n = \{I_n(x^i) : i = 1, ..., k_n\}, \quad F := \cup_{n=0}^{\infty} \cup_{i=1}^{k_n} I_n(x^i).$$

Then,

$$\begin{aligned}
|F| &= \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} |I_n(x^i)| \\
&= \frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{i=1}^{k_n} \lambda|I_n(x^i)| \\
&\le \frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{i=1}^{k_n} \int_{I_n(x^i)} |f| \le \frac{1}{\lambda}\int_I |f| = \|f\|_1/\lambda.
\end{aligned}$$

Let $1_B(x) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B \end{cases}$ the characteristic function of set $B \subset I$ $(x \in I)$. Then,

$$\begin{aligned}
f &= \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} f 1_{I_n(x^i)} + f 1_{I\setminus F} \\
&= \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} \left(f - 2^n \int_{I_n(x^i)} f\right) 1_{I_n(x^i)} \\
&+ \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} \left(2^n \int_{I_n(x^i)} f\right) 1_{I_n(x^i)} + f 1_{I\setminus F} \\
&=: \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} f_n^i + f_0.
\end{aligned}$$

Discuss the functions f_n^i.
supp $f_n^i \subset I_n(x^i)$, $\int_{I_n(x^i)} f_n^i = \int_{I_n(x^i)} (f(t) - 2^n \int_{I_n(x^i)} f(y)dy)dt = 0$,

$$2^n \int_{I_n(x^i)} |f_n^i| \le 2^n \int_{I_n(x^i)} |f| + 2^n \int_{I_n(x^i)} |2^n \int_{I_n(x^i)} f| \le 2 \cdot 2^n \int_{I_n(x^i)} |f| \le 4\lambda.$$

The only relation rest to prove is $\limsup_n S_{2^n}|f_0| \le c\lambda$.

$$f_0 = \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} \left(2^n \int_{I_n(x^i)} f\right) 1_{I_n(x^i)} + f 1_{I\setminus F} =: f_0^1 + f_0^2.$$

First, discuss function f_0^1.

$$|f_0^1| \le \sum_{n=0}^{\infty}\sum_{i=1}^{k_n} 2\lambda 1_{I_n(x^i)} = 2\lambda 1_{\cup_{n=0}^{\infty}\cup_{i=1}^{k_n} I_n(x^i)} \le 2\lambda.$$

Thus,

$$S_{2^n}|f_0^1(x)| = 2^n \int_{I_n(x)} |f_0^1(t)|dt \le 2\lambda$$

for all $x \in I, n \in \mathbf{N}$. Consequently, $\limsup_n S_{2^n}|f_0^1| \le 2\lambda$ everywhere.

Secondly, discuss function f_0^2. If $x \in F$, then since set F is open (the union of intervalls (intervalls are both open and closed)), then there exists a $n \in \mathbf{N}$ such as $I_n(x) \subset F$. Since $f_0^2 = f 1_{I\setminus F}$, then f_0^2 is zero on the intervall $I_n(x)$. Thus, for each $m \ge n$ we have $2^m \int_{I_m(x)} |f_0^2(t)|dt = 0$. This follows, $\limsup_n S_{2^n}|f_0^2(x)| = 0$ for $x \in F$. Finally, let $x \notin F$. Then $2^j \int_{I_j(x)} |f(y)|dy \le \lambda$ for $j = 0, 1, \ldots.$ This follows

$$S_{2^j}|f_0^2(x) = 2^j \int_{I_j(x)} |f(y) 1_{I\setminus F}(y)|dy \le 2^j \int_{I_j(x)} |f(y)|dy \le \lambda$$

for $j = 0, 1, \ldots.$ This follows that $\limsup_n S_{2^n}|f_0^2(x)| \le \lambda$ in the case of $x \notin F$. Consequently, $\limsup_n S_{2^n}|f(x)| \le \limsup_n S_{2^n}|f_0^1(x)| + \limsup_n S_{2^n}|f_0^2(x)| \le c\lambda$. This completes the proof of Lemma 2. $\square$

Set the following maximal operators

$$S^\circ f := \limsup_n |S_{2^n} f(x)|, \quad Sf := \sup_n |S_{2^n} f(x)|.$$

for $f \in L^1(I)$.

Lemma 3. *Operators S° and S are of type (∞, ∞).*

Proof.

$$\|S^\circ f\|_\infty \le \|Sf\|_\infty = \| \sup_{n\in\mathbf{N}} |2^n \int_{I_n(x)} f(t)dt|\|_\infty \le \|\|f\|_\infty \sup_{n\in\mathbf{N}} 2^n \int_{I_n(x)} 1dt\|_\infty = \|f\|_\infty.$$

$\square$

Lemma 4. *Operator S° is of weak type $(1,1)$.*

Proof. $\lambda > \|f\|_1$ can be supposed. Apply Lemma 2.

$$|S^\circ f > 2c\lambda| \le |S^\circ f_0 > c\lambda| + |S^\circ(\sum_{n,i} f_n^i) > c\lambda| =: l_1 + l_2.$$

Since $|S^\circ f_0| \le c\lambda$ a.e., then $l_1 = 0$. On the other hand, by the σ-sublinearity of operator S

$$\begin{aligned} l_2 &\le |F| + |x \in I \setminus F : S(\sum_{n,i} f_n^i) > c\lambda| \\ &\le |F| + \frac{c}{\lambda} \int_{I\setminus F} S(\sum_{n,i} f_n^i) \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \int_{I\setminus F} \sum_{n,i} S(f_n^i) \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{n,i} \int_{I\setminus F} S(f_n^i) \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{n,i} \int_{I\setminus I_n(x^i)} S(f_n^i). \end{aligned}$$

We prove that $\int_{I\setminus I_n(x^i)} S(f_n^i) = 0$ for all $i = 1, ..., k_n$, $n \in \mathbf{P}$.

If $y \in I \setminus I_n(x^i)$, then let $y \in I_a(x^i) \setminus I_{a+1}(x^i)$ for some $a = 0, ..., n-1$. If $N \geq n$, then $S_{2^N} f_n^i(y) = 2^N \int_{I_N(y)} f_n^i = 0$, because $I_N(y) \cap I_n(x^i) = \emptyset$.

If $N < n$, then either $I_N(y) \cap I_n(x^i) = \emptyset$ or $I_N(y) \supset I_n(x^i)$ and in this case $2^N \int_{I_N(y)} f_n^i = 2^N \int_{I_n(x^i)} f_n^i = 0$, that is, in all case for all $N \in \mathbf{N}$ we have $S_{2^N} f_n^i(y) = 0$, thus $Sf_n^i(y) = 0$ for all $y \in I \setminus I_n(x^i)$. Consequently, $l_2 \leq c\|f\|_1/\lambda$. The proof of Lemma 4 is complete. □

The proof of the following theorem known till now is based on the martingale theory (see e.g. [Sto]). We give a "pure dyadic analysis" proof for it.

Theorem 5. *Let $f \in L^1(I)$. Then $S_{2^n} f \to f$ a.e.*

Proof. Let $\epsilon > 0$. Then let P be a Walsh polynomial , that means $P = \sum_{i=0}^{k-1} d_i \omega_i$ for some $d_0, ..., d_{k-1} \in \mathbf{C}$, $k \in \mathbf{P}$. Since $S_{2^n} P(x) \to P$ everywhere (moreover, $S_{2^n} P = P$ for $2^n \geq k$), then we have

$$\begin{aligned}
&|\{x \in I : \limsup_n |S_{2^n} f(x) - f(x)| > \epsilon\}| \\
&\leq |\{x \in I : \limsup_n |S_{2^n} f(x) - S_{2^n} P(x)| > \epsilon/3\}| \\
&+ |\{x \in I : \limsup_n |S_{2^n} P(x) - P(x)| > \epsilon/3\}| + |\{x \in I : \limsup_n |P(x) - f(x)| > \epsilon/3\}| \\
&\leq |\{x \in I : \sup_n |S_{2^n}(f(x) - P(x))| > \epsilon/3\}| + 0 + \|P - f\|_1 \frac{3}{\epsilon} \\
&\leq c\|P - f\|_1/\epsilon =: \delta.
\end{aligned}$$

Since the set of Walsh polynomial is dense in $L^1(I)$ (see e.g. [SWS]), then δ can be less than an arbitrary small positive real number. This follows $|\{x \in I : \limsup_n |S_{2^n} f(x) - f(x)| > \epsilon\}| = 0$ for all $\epsilon > 0$. This gives the relation $S_{2^n} f \to f$ almost everywhere. □

The proof of Theorem 1. We apply Lemma 2 and Theorem 5. The proof follows the proof of Lemma 2. The only difference is that we have to prove $\|f_0\|_\infty \leq c\lambda$ instead of $\limsup_{n\to\infty} S_{2^n}|f_0| \leq c\lambda$. By Theorem 5 we have $S_{2^n} f_0 \to f_0$ a.e. Thus, we have the a.e. inequality

$$|f_0| = \limsup_n |S_{2^n} f_0| \leq \limsup_n S_{2^n}|f_0| \leq c\lambda.$$

That is, the proof is complete. □

Corollary 6. The operator S is of type (p, p) for each $1 < p$.

Proof. Since we have proved that operator S is of type (∞, ∞) and of weak type $(1, 1)$, then by the interpolation theorem of Marczikiewicz (see e.g. [SWS]) the proof of Corollary 6 is complete. □

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(Received October 6, 1997)