Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
Tomus 14 (1998), 19-23

A HARDY–LITTLEWOOD–LIKE INEQUALITY ON COMPACT TOTALLY DISCONNECTED SPACES

I. Blahota

Abstract. In this paper we deal with a new system was introduced by Gát (see [Gát1]). This is a common generalization of several well-known systems (see the follows). We prove an inequality of type Hardy-Littlewood with respect to this system.

Introduction, Examples

Let $\mathbb{P} = \mathbb{N}\backslash\{0\}$, and let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. Denote by G_{m_j} a set, where the number of the elements of G_{m_j} is $m_j (j \in \mathbb{P})$. Define the measure on G_{m_j} as follows

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in G_{m_k}, k \in \mathbb{N}).$$

Define the set G_m as the complete direct product of the sets G_{m_j}, with the product of the topologies and measures (denoted by μ). This product measure is a regular Borel one on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called bounded Vilenkin space, else its name is unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, ...)$ $(x_j \in G_{m_j})$. It easy to give a base the neighborhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$$

for $x \in G_m,\ n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{P}$. If $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum\limits_{j=0}^{\infty} n_j M_j$, where $n_j \in G_{m_j}$ $(j \in \mathbb{P})$ and only a finite number of n_j's differ from zero. We use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_k M_k$. Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \leq p \leq \infty)$, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

From now the **bounded**ness of the **Vilenkin space** G_m is supposed.

1980 *Mathematics Subject Classification* (1985 *Revision*). primary: 42C10, secondary: 42C15.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA) , grant no. F020334 and by the Hungarian "Művelődési és Közoktatási Minisztérium", grant no. FKFP 0710/1997.

Typeset by $\mathcal{AMS}$-TEX

The concept of the maximal Hardy space $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm

$$\|f\|_{H^1} := \|f^*\|_1$$

This definiton is suitable if the sequence m is bounded. In this case a good propery of the space $H^1(G_m)$ is the atomic structure [SWS].

A function a is said to be atom if $a = 1$ or $a : G_m \to \mathbb{C}$, $|a(x)| \leq |I_n|^{-1}$, supp $a(x) \subset I_n$ and $\int_{I_n} a(x) = 0$. We say that f element is of the Hardy space $H(G_m)$ (or in brief H), if there exists $\lambda_j \in$C $(j \in \mathbb{P})$ that $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, and if exists a_j $(j \in \mathbb{P})$ atoms, that $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover, H is Banach space with the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$. If the sequence m is bounded (in this paper this is supposed), then $H = H^1$, moreover, the two norms are equivalent. (If the sequence m is not bounded , then the situation changes.)

Next we introduce on G_m an orthonormal system (see [Gát1]) we call Vilenkin-like system. The complex valued functions which we call the generalized Rademacher functions $r_k^n : G_m \to \mathbb{C}$ have these properties:

i. r_k^n is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0, ..., x_k$ $(x \in G_m)$), $r_k^0 = 1$ for all $k, n \in \mathbb{N}$.

ii. If M_k is a divisor of n, l and $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbb{N})$, then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 \text{ if } n_k = l_k, \\ 0 \text{ if } n_k \neq l_k \end{cases}$$

($\bar{z}$ is the complex conjugate of z).

iii. If M_k is a divisor of n (that is, $n = n_k M_k + n_{k+1} M_{k+1} + ... + n_{|n|} M_{|n|}$). Then

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$$

for all $x \in G_m$.

iv. There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$.

Define the Vilenkin-like system $\psi := (\psi_n : n \in \mathbb{N})$ as follows.

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}, \quad n \in \mathbb{N}.$$

(Since $r_k^0 = 1$, then $\psi_n := \prod_{k=0}^{|n|} r_k^{n^{(k)}}$). The Vilenkin-like system ψ is orthonormal (see [Gát2]).

And now let us list some well-known examples to this system.

1. The Vilenkin and the Walsh system. For more on these see e.g. [SWS, AVD]

2. The group of 2-adic (m-adic) integers (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic). [HR, SW2, Tai]

3. Noncommutative Vilenkin groups (In this case the group is the cartesian product of common finite groups.) [GT, Gát2]

4. A system in the field of number theory. This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. [Mau]

5. The UDMD product system (is introduced by F. Schipp on the Walsh-Paley group). [SW2, SW]

6. The universal contractive projections system (UCP) (is introduced by F. Schipp). [Sch4])

For more on these examples and their proves see [Gát1].

Finally, let us introduce the usual definitions of the Fourier-analysis. With notation already adopted for $f \in L^1(G_m)$ we define the Fourier coefficients and partial sums by

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k d\mu \qquad (k \in \mathbb{N})$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k \qquad (n \in \mathbb{P},\ S_0 f := 0).$$

The Dirichlet kernels:

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x) \qquad (n \in \mathbb{P},\ D_0 := 0).$$

It is clear that

$$S_n f(x) = \int_{G_m} f(x) D_n(y,x) d\mu(x).$$

Result and Proof

Theorem. *There exists a $C > 0$ absolute constant that if $f \in H(G_m)$, then*

$$\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)| \le C\|f\|_H.$$

Proof of the theorem. Since $f \in H(G_m)$, let us form $f := \sum_{k=1}^{\infty} \lambda_k a_k(x)$, where $a_k(x)$ are atoms, and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$.

$$\sum_{k=1}^{\infty} k^{-1}|\hat{f}(k)| = \sum_{k=1}^{\infty} k^{-1}|\sum_{j=1}^{\infty} \lambda_j \hat{a}_i(k)| \le \sum_{j=1}^{\infty} |\lambda_j| \sum_{k=1}^{\infty} k^{-1}|\hat{a}_j(k)|,$$

that is why it will be sufficient to show that there exists $C>0$ absolute constant that for all $a(x)$ atoms

$$\sum_{k=1}^{\infty} k^{-1}|\hat{a}(k)| \leq C.$$

Let $a(x) \in H(G_m)$ be an atom. If $a \equiv 1$ then

$$\hat{a}(k) = \int_{G_m} \overline{\psi}_k = E_0(\overline{\psi}_k) = E_0(\prod_{j=1}^{|k|} \overline{r}_j^{k^{(j)}}) = E_0(E_{|k|}(\prod_{j=1}^{|k|} \overline{r}_j^{k^{(j)}})) =$$

$$E_0(\prod_{j=1}^{|k|-1} \overline{r}_j^{k^{(j)}} E_{|k|}(r_{|k|}^0 \overline{r}_{|k|}^{k^{(|k|)}})) = 0$$

because $k^{(|k|)} = k_{|k|}M_{|k|} \neq 0$ if $k \in \mathbb{P}$ and $E_k(r_k^n \overline{r}_k^l) = 0$ if $n_k \neq l_k$. In this case the statement of the theorem is trivial.

So, assume that $a \not\equiv 1$. In this case let I_n be an interval for which $|a(x)| \leq |I_n|^{-1}$, supp $a(x) \subset I_n$ and $\int_{I_n} a(x) = 0$.

Since supp $a(x) \subset I_n$ than

$$\hat{a}(k) = \int_{G_m} a(x)\overline{\psi_k(x)} = \int_{I_n} a(x)\overline{\psi_k(x)}.$$

If $k = 0, ..., M_n - 1$ than $\psi_k(x)$ depend only on the first n coordinate of x, hence the function $\psi_k(x)$ on the set I_n is invariable

$$\hat{a}(k) = \int_{I_n} a(x)\overline{\psi_k(x)} = c\int_{I_n} a(x) = 0$$

$$\Longrightarrow \sum_{k=1}^{\infty} k^{-1}|\hat{a}(k)| = \sum_{k=M_n}^{\infty} k^{-1}|\hat{a}(k)|.$$

Using the Cauchy–Buniakovski–Schwarz inequality

$$\sum_{k=M_n}^{\infty} k^{-1}|\hat{a}(k)| \leq \sqrt{\sum_{k=M_n}^{\infty} |\hat{a}(k)|^2}\sqrt{\sum_{k=M_n}^{\infty} k^{-2}},$$

and from Bessel's inequality

$$\sqrt{\sum_{k=M_n}^{\infty} |\hat{a}(k)|^2} \leq \|a(x)\|_2,$$

and estimate the approximate sum of the Riemann integral of function $\frac{1}{x^2}$

$$\sqrt{\sum_{k=M_n}^{\infty} k^{-2}} \leq \frac{C}{\sqrt{M_n}}.$$

These gives

$$\sum_{k=M_n}^{\infty} k^{-1}|\hat{a}(k)| \leq C,$$

by

$$\|a(x)\|_2^2 = \int_{I_n} |a|^2 \leq |I_n|^{-2}|I_n| = |I_n|^{-1} \leq M_{n+1} = m_{n+1}M_n \leq CM_n,$$

because of the boundedness of the sequence m.

This completes the proof of Theorem.

Acknowledgement

I would like to thank Professor G. Gát for setting the problem.

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Bessenyei College, Dept. of Math., Nyíregyháza, P.O.Box 166., H–4400,Hungary

E-mail: blahota@ny1.bgytf.hu

(Received October 1, 1997)