Partial Flags and Parabolic Group Actions

Lutz Hille, Dieter Vossieck

May 27, 2004

Abstract

Let $V = V_t \supset V_{t-1} \supset \ldots V_1 \supset V_0 = \{0\}$ be a flag of vector spaces of dimension vector $d = (\dim V_t/V_{t-1}, \ldots, \dim V_2/V_1, \dim V_1)$. We denote its stabilizer by $P(d)$. For a fixed sequence of natural numbers $a(r) = t > a(r-1) > \ldots > a(1) > a(0) = 0$ we consider the partial flag $V = V_{a(r)} \supset V_{a(r-1)} \supset \ldots V_{a(1)} \supset V_{a(0)} = 0$ and denote its stabilizer by $\overline{P}(d)$. The parabolic group $P(d)$ acts on the Lie algebra $\overline{\mathfrak{p}}(d)_u$ of the unipotent radical of $\overline{P}(d)$ via conjugation. We determine all instances of the numbers $\underline{a} = (t; a(1), a(2), \ldots, a(r-1))$ so that $P(d)$ acts with a finite number of orbits on $\overline{\mathfrak{p}}(d)_u$ for all dimension vectors d. In particular we determine a list of critical sequences $\underline{a}$.

Actions of reductive groups are a classical subject in pure mathematics. They appear in many branches, in particular, in connection with classification problems. In contrast, not much is known for parabolic group actions, the next case which should be studied (see e .g [RRS], [HR] and references therein). Recently the instances of the action of a parabolic group on its unipotent radical (see [HR]) and the members of its descending central series (see [BH1] and [BHR]) with a finite number of orbits could be classified. For the action of a parabolic group on an arbitrary unipotent subgroup the methods already developed do not work (for further results in this direction we also refer to [BH2], [BH3] and [H]). In this note we consider a particular case of those actions which we can solve using a computer program: a parabolic subgroup in the general linear group is the stabilizer of a flag and acts on the Lie algebra of the unipotent radical of the stabilizer of a partial flag. The result is based on a more general conjecture for the action of a parabolic subgroup on an arbitrary unipotent ideal.

We fix a natural number t and consider for any given dimension vector $d = (d_1, \ldots, d_t)$ a fixed flag

$$F : V = V_t \supset V_{t-1} \supset \ldots \supset V_1 \supset V_0 = \{0\}$$

with $\dim V_i/V_{i-1} = d_i$ for $i = 1, \ldots, t$. By $P(d)$ we denote the stabilizer of this flag. It is a parabolic subgroup in GL_n, where $n = \sum_{i=1}^{t} d_i$. For fixed numbers $\underline{a} = (a(1), \ldots, a(r))$ with $a(r) = t > a(r-1) > \ldots a(1) > 0$ we consider the partial flag

$$\overline{F} : V = V_{a(r)} = \overline{V}_r \supset V_{a(r-1)} = \overline{V}_{r-1} \supset \ldots \supset V_{a(1)} = \overline{V}_1 \supset V_0 = \{0\}.$$

We denote the dimension vector of this partial flag by

$$\overline{d} = (\overline{d}_1, \ldots, \overline{d}_r) = (d_1 + \ldots + d_{a(1)}, \ldots, d_{a(r-1)+1} + d_{a(r)}).$$

The stabilizer of the partial flag $\overline{F}$ we denote by $\overline{P}(d)$. It is isomorphic to $P(\overline{d})$. The group $P(d)$ acts via conjugation on the unipotent radical $\overline{\mathfrak{p}}(d)_u$ of $\overline{P}(d)$. We can identify $\overline{\mathfrak{p}}(d)_u$ with all endomorphisms f of V, satisfying $f(\overline{V}_i) \subseteq \overline{V}_{i-1}$. In this note we are interested in a classification of all sequences $(t; a(1), \ldots, a(r-1))$, so that $P(d)$ acts with a finite number of orbits on $\overline{\mathfrak{p}}(d)_u$ for all dimension vectors d. In this case we call the sequence of numbers $(t; a(1), \ldots, a(r-1))$ a *sequence of finite type*, otherwise a *sequence of infinite type*. Assume $(t; a(1), \ldots, a(r-1))$ is a sequence of infinite type, then any sequence $(t, b(1), \ldots, b(s-1))$ with $\{a(1), \ldots, a(r-1)\} \subset \{b(1), \ldots, b(s-1)\}$, is also of infinite type. Moreover, any sequence $(s; b(1) = a(1)+c(1), \ldots, b(r-1) = a(r-1)+c(r-1))$ is of infinite type, for $s > t$ and $0 \leq c(1) \leq \ldots \leq c(r-1) \leq s-t$. In both cases we say the first sequence $(t; a(1), \ldots, a(r-1))$ is *smaller* than the latter sequence $(-; b(1), \ldots, b(r-1))$. A sequence of infinite type is called *critical* if each smaller sequence is of finite type. For convenience we sometimes draw a picture of the corresponding Lie algebra $\overline{\mathfrak{p}}(d)_u$ (see Figure 1).

Theorem 1 *The following is the list of critical sequences (see Figure 1):*

$t = 6$, $\underline{a} = (1, 3, 5), (1, 2, 4, 5)$;
$t = 7$, $\underline{a} = (2, 5), (1, 2, 4), (1, 2, 6), (1, 4, 5), (1, 5, 6), (2, 3, 6), (3, 5, 6)$;
$t = 8$, $\underline{a} = (1, 5), (3, 5), (3, 7)$;
$t = 9$, $\underline{a} = (1, 4), (1, 7), (2, 4), (2, 8), (5, 7), (5, 8)$.

The theorem implies that any sequence for $t > 9$ and $r \geq 2$ is already infinite except the sequences $(t; 1, 2, 3), (t; t-3, t-2, t-1)$, $(t; 1, t-1)$, and $(t; i, i+1)$ for $1 \leq i \leq t-2$. Further, any sequence with $r = 1$ is obviously finite.

Proof (of the theorem)
We prove the result using a computer program [XPar] written by D. Guhe and T. Brüstle. First we show that the sequences $(t; 1, 2, 3), (t; t-3, t-2, t-1)$, $(t; 1, t-1)$, and $(t; i, i+1)$ for $1 \leq i \leq t-2$ are of finite type: this can be done explicitly using matrix reduction. Secondly we check the remaining finite number of sequences with the computer program [XPar] (see List 1 for a complete list of these sequences, for $t \geq 9$ we omitted the infinite cases for convenience). Finally, we determine the critical sequences using the list of sequences of finite and infinite type. This finishes the proof. □

Some Complements. According to the results in [BH2] and [BH3] we also determined for a certain class of quasi-hereditary algebras, whether they are of Δ-finite representation type. Moreover, the list in Figure 1 leeds to a list of minimal infinite configurations in the sense defined in [H], whereas three of these minimal infinite configurations coincide with already known ones: $(6; 1, 3, 5), (6; 1, 2, 4, 5)$, and $(7; 2, 5)$ (see [H], Figure 1). The computer program we have used uses a matrix reduction algorithm for certain particular bimodules. Finally, we should mention, that the program does *not* determine the dimension vectors of critical one-parameter families.

References

[BH1] T. Brüstle, L. Hille: *Finite, tame, and wild actions of parabolic subgroups in* GL(V) *on certain unipotent subgroups.* J. Algebra **226**, no. 1 (2000), 347–360.

[BH2] T. Brüstle, L. Hille: *Matrices over Upper Triangular Bimodules and Δ–filtered Modules over Quasi-hereditary Algebras.* Coll. Math. **83** (2000), 295–303.

[BH3] T. Brüstle, L. Hille: *Actions of Parabolic Subgroups in GL_n on Unipotent Normal Subgroups and Quasi-hereditary Algebras.* Coll. Math. **83** (2000), 281–294.

[BHR] T. Brüstle, L. Hille, R. Röhrle: *Finiteness for Parabolic Group Actions in Classical Groups.* Arch. Math. (Basel) **76**, no. 2 (2001), 81–87.

[H] L. Hille: *Minimal Infinite Configurations for Strictly Filtered $k[T]/T^n$–modules and Parabolic Group Actions.* in "Representations of Algebras", Proceedings of the Ninth International Conference, Beijing, 2000, 214–224.

[HR] L. Hille, G. Röhrle: *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical.* Transform. Groups **4**, no. 1 (1999), 35–52.

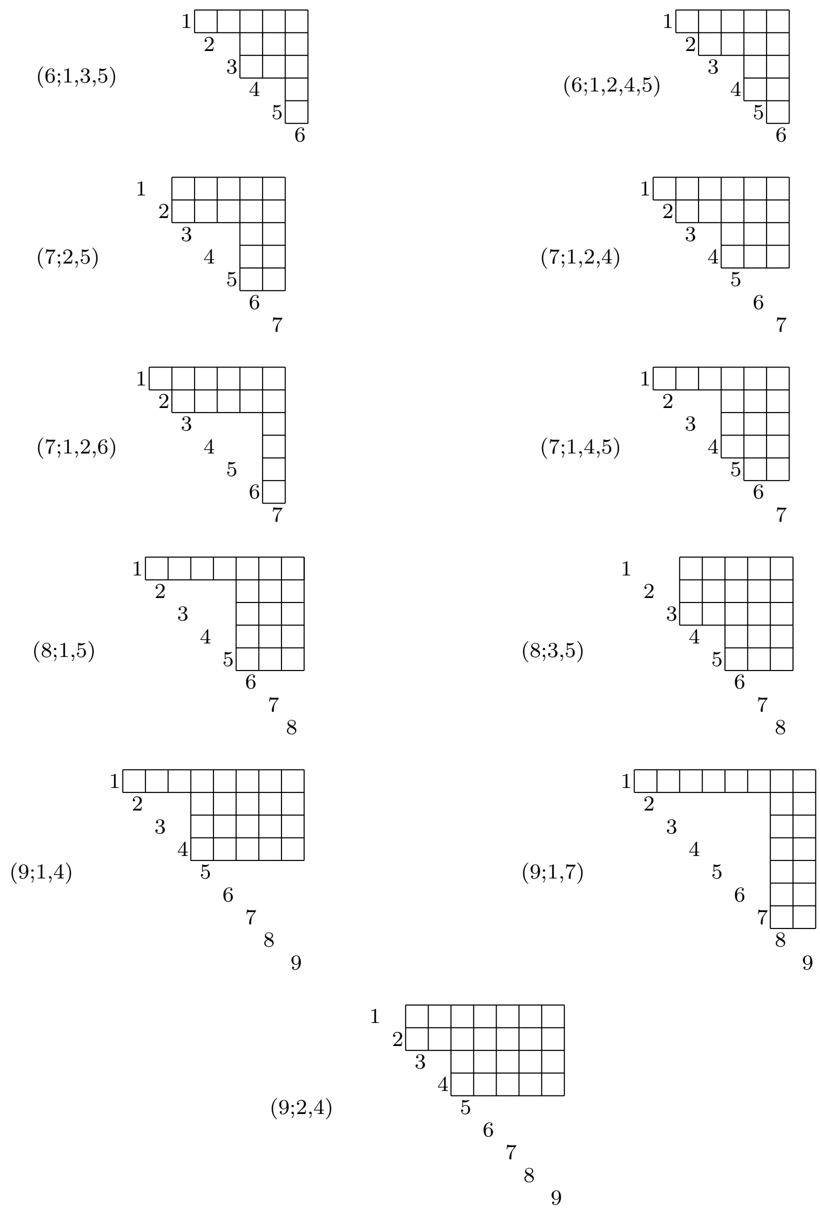
[RRS] R. Richardson, G. Röhrle, R. Steinberg: *Parabolic Subgroups with Abelian Unipotent Radical.* Inv. Math. **110**, no. 3 (1992), 649–671.

[XPar] T. Brüstle, D. Guhe: *XParabol.* a computer program which solves matrix problems associated to parabolic group actions.

Lutz Hille
Mathematisches Seminar
Universität Hamburg
D-20 146 Hamburg
Germany
E-mail: hille@math.uni-hamburg.de
http://www.math.uni-hamburg.de/home/hille/

Dieter Vossieck
Am Frerks Hof 20
D-33647 Bielefeld

Germany
E-mail: dvossieck@web.de



the critical sequences (up to reflection)
Figure 1

$t=6, r=2$
finite: $\{i,j\}$ for $0 \geq i < j \leq 5$,
$t=6, r=3$
finite: $\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}$
infinite: $\{1,3,5\}$
critical: $\{1,3,5\}$
$t=6, r=4$
finite: $\{1,2,3,4\},\{2,3,4,5\}$
infinite: $\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}$
critical: $\{1,2,4,5\}$

$t=7, r=2$
finite: $\{i,j\}$ for $0 \geq i < j \leq 5$ and $(i,j) \neq (2,5)$
infinite: $\{2,5\}$
critical: $\{2,5\}$
$t=7, r=3$
finite: $\{1,2,3\},\{1,3,4\},\{2,3,4\},\{3,4,5\},\{3,5,6\}$, $\{4,5,6\}$
infinite: $\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,5\},\{1,3,6\}$, $\{1,4,5\},\{1,4,6\},\{1,5,6\}$,
$\{2,3,5\},\{2,3,6\},\{2,4,5\}$, $\{2,4,6\},\{3,5,6\}$
critical: $\{1,2,4\},\{1,2,6\},\{1,4,5\},\{1,5,6\},\{2,3,6\}$, $\{3,5,6\}$
$t=7, r=4$
all are infinite

$t=8, r=2$
finite: $\{1,2\},\{1,3\},\{1,4\},\{1,6\},\{1,7\},\{2,3\}$, $\{2,4\},\{2,7\},\{3,4\},\{4,5\}$,
$\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}$
infinite: $\{1,5\},\{2,5\},\{2,6\},\{3,5\},\{3,6\},\{3,7\}$
critical: $\{1,5\},\{3,5\},\{3,7\}$
$t=8, r=3$
finite: $\{1,2,3\},\{1,2,4\},\{1,2,6\},\{1,2,7\},\{1,3,4\},\{1,3,6\},\{1,3,7\},\{1,4,6\},\{1,4,7\},\{1,6,7\}$,
$\{2,3,4\},\{2,3,6\},\{2,3,7\},\{2,4,6\},\{2,4,7\},\{2,6,7\},\{4,5,6\},\{4,5,7\},\{4,6,7\},\{5,6,7\}$
infinite: $\{1,2,5\},\{1,3,5\},\{1,4,5\},\{1,5,6\},\{1,5,7\}$, $\{2,3,5\},\{2,4,5\},\{2,5,6\}$,
$\{2,5,7\},\{3,4,5\},\{3,4,6\}$, $\{3,4,7\},\{3,5,6\},\{3,5,7\},\{3,6,7\}$
$t=8, r \geq 4$
all are infinite

$t=9, r=2$
finite: $\{1,2\},\{1,3\},\{1,8\},\{2,3\},\{3,4\},\{4,5\},\{5,6\}$, $\{6,7\},\{6,8\},\{7,8\}$
critical: $\{1,4\},\{1,7\},\{2,4\},\{2,8\},\{5,7\},\{5,8\}$
$t=9, r=3$
finite: $\{1,2,3\},\{6,7,8\}$
$t=9,\ r=4$
all are infinite

$t \geq 10, r=2$
finite: $\{i,i+1\}$ for $i=1,\ldots,t-2$, $\{1,3\}$, $\{t-3,t-1\}$, $\{1,t-1\}$
$t \geq 10, r=3$
finite: $\{1,2,3\}$, $\{t-3,t-2,t-1\}$
$t \geq 10, r \geq 4$
all are infinite

the list of all sequences with $t \leq 9$

List 1