Green functors, crossed G-monoids, and Hochschild constructions

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1. Introduction

Let G be a finite group, and R be a commutative ring. This note proposes a generalization to any Green functor for G over R of the construction of the Hochschild cohomology ring $HH^*(G,R)$ from the ordinary cohomology functor $H^*(-,R)$. Another special case is the construction of the crossed Burnside ring of G from the ordinary Burnside functor.

The general abstract setting is the following : let A be a Green functor for the group G. Let G^c denote the group G, on which G acts by conjugation. Suppose Γ is a crossed G-monoid, i.e. that Γ is a G-monoid over the G-group G^c. Then the Mackey functor A_Γ obtained from A by the Dress construction has a natural structure of Green functor. In particular $A_\Gamma(G)$ is a ring.

In the case where Γ is the crossed G-monoid G^c, and A is the cohomology functor (with trivial coefficients R), the ring $A_\Gamma(G)$ is the Hochschild cohomology ring of G over R. If A is the Burnside functor for G over R, then the ring $A_\Gamma(G)$ is the crossed Burnside ring of G over R.

This note presents some properties of those Green functors A_Γ, and the functorial relations between the corresponding categories of modules. In particular, it states a general formula for the product in the ring $A_\Gamma(G)$, shedding a new light on a result of S. Siegel and S. Witherspoon ([6]), which was conjectured by C. Cibils ([3]) and C. Cibils and A. Solotar ([4]).

2. Green functors and G-sets

For the various definitions of Mackey and Green functors for a finite group G over a commutative ring R, the reader is referred to [2]. The definition in use here is the one in terms of G-sets : a Mackey functor for G over R is a bivariant functor from the category of finite G-sets to the category of R-modules, which transforms disjoint unions into direct sums, and has some compatibility property with cartesian squares (see [2] 1.1.2 for details).

A Green functor A for G over R is a Mackey functor for G over R, together with product maps $A(X)\otimes_R A(Y)\to A(X\times Y)$, for any finite G-sets X and Y, which are denoted by $(a,b)\mapsto a\times b$. Those maps have to be bivariant, associative, and unital in some suitable sense (see [2] 2.2 for details).

Mackey and Green functors for G over R are naturally the objects of categories, denoted respectively by $\mathsf{Mack}_R(G)$ and $\mathsf{Green}_R(G)$. The category $\mathsf{Mack}_R(G)$ is an abelian category, whereas $\mathsf{Green}_R(G)$ should be viewed as a generalization of the category of R-algebras.

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Remark 2.1 : One can recover the usual definition of Mackey and Green functors by setting $A(H) = A(G/H)$, for a subgroup H of G. The ordinary product on $A(H)$ can be recovered by setting

$$a.b = A^*(\delta_{G/H})(a \times b)$$

for $a, b \in A(H)$, where $\delta_{G/H}$ is the diagonal inclusion from G/H to $(G/H) \times (G/H)$, and $A^*(\delta_{G/H})$ is the image of $\delta_{G/H}$ by the contravariant part of the bivariant functor A.

Example 2.2 : Let X and Y be finite G-sets, and set

$$H^{\oplus}(G, RX) = \bigoplus_{n=0}^{\infty} H^n(G, RX) \quad .$$

Then the cup-product on cohomology give maps

$$H^{\oplus}(G, RX) \times H^{\oplus}(G, RY) \to H^{\oplus}(G, RX \otimes_R RY)$$

and identifying $RX \otimes_R RY$ with $R(X \times Y)$, this gives cross product maps

$$H^{\oplus}(G, RX) \times H^{\oplus}(G, RY) \to H^{\oplus}(G, R(X \times Y)) \quad .$$

This gives $H^{\oplus}(G, -)$ a Green functor structure, and if K is a subgroup of G, the induced ring structure on

$$H^{\oplus}(G, R(G/K)) \cong H^{\oplus}(K, R)$$

coincides with the ordinary ring structure of $H^{\oplus}(K, R)$ for cup-products.

Example 2.3 : Let B denote the Burnside functor. If X is a finite G-set, then $B(X)$ is the Grothendieck group of the category of G-sets over X. The obvious product

$$\begin{pmatrix} Z & & T \\ \downarrow & , & \downarrow \\ X & & Y \end{pmatrix} \mapsto \begin{matrix} Z \times T \\ \downarrow \\ X \times Y \end{matrix}$$

extends linearly to a cross product $B(X) \times B(Y) \to B(X \times Y)$, which gives B its structure of Green functor ([2] 2.4).

3. The Dress construction and crossed G-monoids

The Dress construction is a fundamental endo-functor of the category $\mathsf{Mack}_R(G)$, defined as follows. Let Γ be a fixed finite G-set. If M is a Mackey functor for G over R, then the Mackey functor M_Γ is the bivariant functor defined on the finite G-set Y by $M_\Gamma(Y) = M(Y \times \Gamma)$. If $f : Y \to Z$ is a map of G-sets, then $(M_\Gamma)_*(f) = M_*(f \times Id_\Gamma)$ and $(M_\Gamma)^*(f) = M^*(f \times Id_\Gamma)$. One checks easily ([2] 1.2) that M_Γ is a Mackey functor for G over R.

It follows from this definition that the evaluation of M_Γ at the trivial G-set $\bullet = G/G$ is equal to $M_\Gamma(G) = M_\Gamma(\bullet) \cong M(\Gamma)$.

When A is a Green functor for G over R, and when the G-set Γ has some additional structure (see below), then A_Γ is another Green functor for G over R.

Definition 3.1 : *Let G be a finite group. A crossed G-monoid (Γ, φ) is a pair consisting of a finite monoid Γ with a left action of G by monoid automorphisms (denoted by $(g, \gamma) \mapsto g\gamma$ or $(g, \gamma) \mapsto {}^g\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of G-monoids φ from Γ to G^c (i.e. a map φ which is both a map of monoids and a map of G-sets). A morphism of crossed G-monoids from (Γ, φ) to (Γ', φ') is a map of G-monoids $\theta : \Gamma \to \Gamma'$ such that $\varphi' \circ \theta = \varphi$.*

A crossed G-group (Γ, φ) is a crossed G-monoid for which Γ is a group.

Remark 3.2 : Generally the map $\varphi : \Gamma \to G^c$ will be clear from context, and will be understood in the notation.

Example 3.3 :

1. Let H be a normal subgroup of G, and φ be the inclusion homomorphism from H to G. Then $H^c = (H, \varphi)$ is a crossed G-group.
2. Let Γ be any G-monoid (i.e. any monoid with a left action of G by monoid automorphisms). Let u be the trivial monoid homomorphism from Γ to G. Then $\Gamma^u = (\Gamma, u)$ is a crossed G-monoid.
3. Let (Γ, φ) be a crossed G-monoid. Then $\varphi(\Gamma)$ is a normal subgroup of G, and $\varphi^{-1}(1)$ is a G-submonoid of Γ. There is a natural inclusion of crossed G-monoids from $\varphi^{-1}(1)^u$ to (Γ, φ), and a natural surjection from (Γ, φ) to $\varphi(\Gamma)^c$.
4. Let $\mathbb{E}$ be a group of cardinality 1, with trivial G-action. Let $u : \mathbb{E} \to G^c$ be the map sending the unique element of $\mathbb{E}$ to the identity of G. Then $(\mathbb{E}, u)$ is an initial object in the category of crossed G-monoids. On the other hand the crossed G-monoid $G^c = (G, Id_G)$ is a final object in the category of crossed G-monoids.

Notation 3.4 : *Let (Γ, φ) be a crossed G-monoid. If X is any G-set, there is a natural monoid action of Γ on X, denoted by $(\gamma, x) \in \Gamma \times X \mapsto \gamma.x \in X$ and defined by $\gamma.x = \varphi(\gamma)x$.*

4. The Green functor structure on A_Γ

Let R be a commutative ring, and Γ be a crossed G-monoid. If A is a Green functor for G over R, then the Dress construction gives a Mackey functor A_Γ, whose evaluation at the G-set X is $A_\Gamma(X) = A(X \times \Gamma)$. If X and Y are finite G-sets, define maps

$$A_\Gamma(X) \otimes_R A_\Gamma(Y) \to A_\Gamma(X \times Y) : a \otimes b \mapsto a \times_\Gamma b = A_* \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix} (a \times b) \quad .$$

The notation $A_* \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}$ means $A_*(f)$, where $f = \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}$ is the map from $X \times \Gamma \times Y \times \Gamma$ to $X \times Y \times \Gamma$ sending $(x, \gamma_1, y, \gamma_2)$ to $(x, \gamma_1.y, \gamma_1\gamma_2)$, and A_* is the covariant part of A.

This definition makes sense, since the map f is a map of G-sets if Γ is a crossed G-monoid. Moreover if $a \in A(X \times \Gamma)$ and $b \in A(Y \times \Gamma)$, then $a \times b \in A(X \times \Gamma \times Y \times \Gamma)$, hence $a \times_\Gamma b \in A(X \times Y \times \Gamma) = A_\Gamma(X \times Y)$.

Let moreover ε_{A_Γ} denote the element $A_* \begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix} (\varepsilon_A)$ of $A(\Gamma) = A_\Gamma(\bullet)$, where $\begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix}$ is the map sending the unique element of $\bullet$ to the identity of Γ, and $\varepsilon_A \in A(\bullet)$ is the unit element of A.

Theorem 4.1 : *The functor A_Γ is a Green functor for G over R, with unit ε_{A_Γ}. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category $\mathsf{Green}_R(G)$.*

Proof : The proof is a series of straightforward verifications. □

Remark 4.2 : The evaluation at the trivial G-set of the Green functor A_Γ is $A_\Gamma(\bullet) \cong A(\Gamma)$, and with this identification the product on $A(\Gamma)$ is given by

$$(a,b) \in A(\Gamma) \times A(\Gamma) \mapsto A_*\begin{pmatrix} \gamma_1,\gamma_2 \\ \downarrow \\ \gamma_1\gamma_2 \end{pmatrix}(a \times b) \quad .$$

An explicit version of this product formula will be given in theorem 5.1.

Proposition 4.3 : *Let $f : (\Gamma', \varphi') \to (\Gamma, \varphi)$ be a morphism of crossed G-monoids.*

1. *For any G-set X, denote by $A_{f,X}$ the map $A_*(Id_X \times f)$ from $A_{\Gamma'}(X)$ to $A_\Gamma(X)$. Then these maps $A_{f,X}$ define a morphism of Green functors A_f from $A_{\Gamma'}$ to A_Γ.*
2. *Moreover, if f in injective, then A_f is a split injection of Mackey functors.*
3. *In particular, the inclusion $(\mathbb{E}, u) \to (\Gamma, \varphi)$ induces a morphism of Green functors $\iota : A \to A_\Gamma$, which is a split injection of Mackey functors.*

Proof : Here again, the proof is straightforward, except maybe for the last assertion : if f is injective, then the maps $A^f_X = A^*(Id_X \times f)$ define a morphism of Mackey functors A^f from A_Γ to $A_{\Gamma'}$ ([2] 1.2), which is a section to A_f. □

5. The product formula

The following product formula for the ring $A_\Gamma(G)$ is is just a reformulation of the definition, using the translation between the different definitions of Green functors.

Theorem 5.1 : *Let A be a Green functor for G over R, and Γ be a crossed G-monoid. Then*

$$A_\Gamma(G) = A(\Gamma) = \left(\bigoplus_{\gamma\in\Gamma} A(G_\gamma)\right)^G$$

and for $\gamma \in \Gamma$, the γ-component of the product of the elements a and b of $A(\Gamma)$ is given by

$$(a \times_\Gamma b)_\gamma = \sum_{\substack{(\alpha,\beta)\in G_\gamma\backslash(\Gamma\times\Gamma) \\ \alpha\beta=\gamma}} t^{G_\gamma}_{G_{(\alpha,\beta)}}\left(r^{G_\alpha}_{G_{(\alpha,\beta)}} a_\alpha . r^{G_\beta}_{G_{(\alpha,\beta)}} b_\beta\right) \quad .$$

Remark 5.2 : One can write this formula after taking sets of representatives for the action of G on Γ. In this form, when A is the ordinary cohomology functor, and $\Gamma = G^c$, it was the conjecture of Cibils and Solotar mentioned in the introduction. Theorem 5.1 shows that in the proof of this conjecture by Siegel and Witherspoon ([6] Theorem 5.1), the essential point is that cup products for Hochschild cohomology and for ordinary cohomology are the same. The rest of the proof appears as a formal consequence of the underlying Green functor structure.

An easy corollary of the product formula is the following :

Corollary 5.3 : *Let H be a normal subgroup of G. Suppose that A is a (graded) commutative Green functor. If for any subgroup K of G, the group $H \cap C_G(K)$ acts trivially on $A(K)$, then the ring $A(H^c)$ is (graded) commutative.*

Remark 5.4 : Corollary 5.3 shows in particular that the crossed Burnside ring of G is commutative. Similarly, the Hochschild cohomology ring of G is graded commutative. This was first proved by Gerstenhaber ([5]).

6. Semi-direct products of crossed G-monoids

Theorem 4.1 shows that the correspondence $A \mapsto A_\Gamma$ is an endo-functor of $\mathsf{Green}_R(G)$. It is natural to compose those endo-functors, and this leads to the notion of semi-direct product of crossed G-monoids. All the following results are straightforward :

Proposition 6.1 : *Let (Γ, φ) and (Γ', φ') be crossed G-monoids. Let $\Gamma"$ denote the direct product $\Gamma' \times \Gamma$, with diagonal G-action. Define the following multiplication on $\Gamma"$:*

$$(\gamma'_1, \gamma_1)(\gamma'_2, \gamma_2) = \Big(\gamma'_1(\gamma_1.\gamma'_2), \gamma_1\gamma_2\Big) \quad \forall \gamma_1, \gamma_2 \in \Gamma,\ \forall \gamma'_1, \gamma'_2 \in \Gamma' \quad .$$

Define $\varphi" : \Gamma" \to G^c$ by $\varphi"(\gamma', \gamma) = \varphi'(\gamma')\varphi(\gamma)$ for all $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$.

Then $(\Gamma", \varphi")$ is a crossed G-monoid, with identity $(1_{\Gamma'}, 1_\Gamma)$.

Definition 6.2 : *The crossed G-monoid $(\Gamma", \varphi")$ of proposition 6.1 is called the semi-direct product of the crossed G-monoids (Γ', φ') and (Γ, φ), and it is denoted by $(\Gamma', \varphi') \rtimes (\Gamma, \varphi)$, or $\Gamma' \rtimes \Gamma$ for short.*

Proposition 6.3 : *Let A be a Green functor for G over R. If Γ and Γ' are crossed G-monoids, then the Green functor $(A_\Gamma)_{\Gamma'}$ is naturally isomorphic to $A_{\Gamma' \rtimes \Gamma}$.*

7. From A-modules to A_Γ-modules

There is a natural notion of module over a Green functor (see [2] 2.2), and it follows in particular from proposition 4.3 that there is a functor of restriction r_Γ along the Green functor homomorphism $\iota : A \to A_\Gamma$, from the category A_Γ-Mod of A_Γ-modules to the category A-Mod. This section describes a functor i_Γ from A-Mod to A_Γ-Mod.

Notation 7.1 : *Let A be a Green functor for G over R, and M be an A-module. If X and Y are finite G-sets, if $a \in A_\Gamma(X)$ and $m \in M(Y)$, denote by $a \times_\Gamma m$ the element of $M(X \times Y)$ defined by $a \times_\Gamma m = M_* \begin{pmatrix} x,\gamma,y \\ \downarrow \\ x,\gamma.y \end{pmatrix} (a \times m) \in M(X \times Y)$.*

Theorem 7.2 : *Let Γ be a crossed G-monoid, and let A be a Green functor for G over R.*

1. *If M is an A-module, then the product $(a, m) \in A_\Gamma(X) \times M(Y) \mapsto a \times_\Gamma m \in M(X \times Y)$ endows M with a structure of A_Γ-module, denoted by $\mathsf{i}_\Gamma(M)$.*
2. *If $f : M \to N$ is a morphism of A-modules, then the maps $f_X : M(X) \to N(X)$ define a morphism $\mathsf{i}_\Gamma(f)$ of A_Γ-modules from $\mathsf{i}_\Gamma(M)$ to $\mathsf{i}_\Gamma(N)$.*
3. *This defines a functor i_Γ from A-Mod into A_Γ-Mod, which is is a full embedding.*

8. Centres and centralizers

Let A be a Green functor for G over R. If M is a Mackey subfunctor of A, one can define the commutant $C_A(M)$ of M in A. It is is a Green subfunctor of A ([2] 6.5.3).

If X is a finite G-set, define $\zeta_A(X)$ as the set of natural transformations from the identity functor $\mathcal{I}$ of A-Mod to the endo-functor $\mathcal{I}_X$ of A-Mod given by the Dress construction associated to X. In section 12.2 of [2], it is shown that ζ_A has a natural structure of Green functor. Its evaluation at the trivial G-set is the center of the category A-Mod, i.e. the set of natural transformations from the identity functor of A-Mod to itself.

Theorem 8.1 : *Let Γ be a crossed G-monoid, and A be a Green functor. Let $C(A,\Gamma)$ denote the commutant of $\iota(A)$ in A_Γ. If X and Y are finite G-sets, if M is an A-module, and if $\alpha \in C(A,\Gamma)(X)$, define a map $z_X(\alpha)_{M,Y} : M(Y) \to M(Y \times X)$ by $z_X(\alpha)_{M,Y}(m) = M_* \begin{pmatrix} x,y \\ \downarrow \\ y,x \end{pmatrix} (\alpha \times_\Gamma m)$. Then :*

1. *For given X, α and M, the maps $z_X(\alpha)_{M,Y}$ define a morphism of A-modules $z_X(\alpha)_M$ from M to M_X.*
2. *For given X and α, these morphisms $z_X(\alpha)_M$ define an element $z_X(\alpha)$ of $\zeta_A(X)$.*
3. *The maps z_X define a morphism of Green functors z from $C(A,\Gamma)$ to ζ_A.*

Remark 8.2 : Theorem 8.1 provides in particular a natural ring homomorphism from $C(A,\Gamma)(\bullet)$ to the center of the category A-Mod. If A is the Burnside functor B, and $\Gamma = G^c$, then actually $C(A,\Gamma) = A_\Gamma$, and the previous ring homomorphism is the natural morphism from the crossed Burnside ring of G over R to the center of the Mackey algebra of G over R. This morphism leads in particular to a description of the block idempotents of the Mackey algebra ([1]).

References

[1] S. Bouc. The p-blocks of the Mackey algebra. Preprint (2001), to appear in "Algebra and Representation Theory".

[2] S. Bouc. *Green-functors and G-sets*, volume 1671 of *Lecture Notes in Mathematics*. Springer, October 1997.

[3] C. Cibils. Tensor product of Hopf bimodules over a group algebra. *Proc. Amer. Math. Soc.*, 125:1315–1321, 1997.

[4] C. Cibils and A. Solotar. Hochschild cohomology algebra of abelian groups. *Arch. Math.*, 68:17–21, 1997.

[5] M. Gerstenhaber. The cohomology structure of an associative ring. *Ann. Math.*, 78:267–288, 1963.

[6] S. F. Siegel and S. J. Witherspoon. The Hochschild cohomology ring of a group algebra. *Proc. London Math. Soc.*, 79:131–157, 1999.

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