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SECOND ORDER CONNECTIONS ON SOME FUNCTIONAL BUNDLES

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Abstract. We study the second order connections in the sense of C. Ehresmann. On a fibered manifold Y, such a connection is a section from Y into the second non-holonomic jet prolongation of Y. Our main aim is to extend the classical theory to the functional bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base. This requires several new geometric results about the second order connections on Y, which are deduced in the first part of the paper.

Having two fibered manifolds Y_1 and Y_2 over the same base M, we are interested in the functional bundle $\mathcal{F}(Y_1, Y_2)$ of all C^∞-maps from a fiber of Y_1 into the fiber of Y_2 over the same base point. In [2] we started the study of those connections on $\mathcal{F}(Y_1, Y_2)$ that represent a functional modification of a first order connection $\Gamma : Y \to J^1Y$ on an arbitrary fibered manifold $Y \to M$. This was iniciated by the idea of the Schrödinger connection on a double fibered manifold by Jadczyk and Modugno, [9]. Later we studied the iterated absolute differentiation on $\mathcal{F}(Y_1, Y_2)$, [3], [4]. On the one hand, the present paper was inspired by the fact that Ehresmann established a theory of higher order connections on a Lie groupoid, [7], which is equivalent to the theory of higher order principal connections on a principal bundle, [10]. In [11], the second author extended the Ehresmann's theory to the case of an arbitrary fibered manifold Y. On the other hand, we realized already in [2] that the analogous concept of second order connection on $\mathcal{F}(Y_1, Y_2)$ is useful for numerous geometric problems. That is why we develop a systematic theory of second order connections on $\mathcal{F}(Y_1, Y_2)$ in the present paper.

For this purpose we need some new results on the second order connections on Y. They are deduced in Section 1–3, so that these sections are of independent interest. In Section 1 we deduce a useful identification $\Delta = (\Delta_1, \Delta_2, \Sigma)$, where Δ is a second

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order non-holonomic connection on Y, Δ_1 and Δ_2 are first order connections on Y and Σ is a section of $VY \otimes \overset{2}{\bigotimes} T^*M$, VY being the vertical tangent bundle of Y. This is based on the idea of the product $\Delta_1 * \Delta_2$ by Virsik, [19], and on an original algebraic operation on the second order non-holonomic jet bundles. In Section 2 we discuss the second order principal connections on principal bundles and the induced connections on the associated bundles, which corresponds to the original theory by Ehresmann, [7]. Section 3 is devoted to the absolute differentiation with respect to second order connections on an arbitrary fibered manifold Y. In particular, in Propositions 4 and 6 we prove two important formulae characterizing the absolute differentiation in the above identification $\Delta = (\Delta_1, \Delta_2, \Sigma)$.

The study of the functional case starts with Proposition 7, which describes an element of the second non-holonomic prolongation $\widetilde{J}^2\mathcal{F}(Y_1, Y_2)$ in terms of its associated map. Section 5 deals with the basic facts on second order connections on $\mathcal{F}(Y_1, Y_2)$. The most interesting subclass is formed by those connections that are of finite order in the operator sense. These connections can be determined by means of a C^∞-map on a suitable finite order jet space. In Example 1 we show that a second order connection Δ on Y_1 and a second order connection Λ on Y_2 define a second order connection (Δ, Λ) on the functional bundle $\mathcal{F}(Y_1, Y_2)$ analogously to the first order case, [3]. In the last section we explain how the ideas of Propositions 4 and 6 enable us to introduce the absolute differentiation with respect to second order connections on $\mathcal{F}(Y_1, Y_2)$.

If we deal with two finite dimensional manifolds and a map between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. On the other hand, the idea of smoothness in the infinite dimension is taken from the theory of smooth structures by Frölicher, [8]. Unless otherwise specified, we use the terminology and notations from the book [15].

1. Second order connections on a fibered manifold

For a fibered manifold $\pi : Y \to M$, we denote by $\beta : J^1Y \to Y$ its first jet prolongation. If x^i, y^p are some local fibered coordinates on Y, the induced coordinates on J^1Y are denoted by y^p_i. On the second non-holonomic prolongation $\widetilde{J}^2Y = J^1(J^1Y \to M)$ we have the additional coordinates y^p_{0i}, y^p_{ij}. Beside the target jet projection $\beta_1 : \widetilde{J}^2Y \to J^1Y$, we have another projection $\beta_2 := J^1\beta : \widetilde{J}^2Y \to J^1Y$. In the case of the product fibered manifold $M \times N \to M$, $\widetilde{J}^2(M \times N \to M) =: \widetilde{J}^2(M, N)$ is the space of non-holonomic 2-jets of M into N. The second semiholonomic prolongation of Y is defined by

$$\overline{J}^2Y = \{X \in \widetilde{J}^2Y;\ \beta_1(X) = \beta_2(X)\}\,.$$

The coordinate characterization of $\overline{J}^2Y$ is $y^p_i = y^p_{0i}$. The space $\overline{J}^2(M, N)$ of all semiholonomic 2-jets of M into N is introduced as $\overline{J}^2(M \times N \to M)$. The second holonomic prolongation J^2Y is a subset of $\overline{J}^2Y$, which is characterized by $y^p_{ij} = y^p_{ji}$. The following definition generalizes an idea by Ehresmann, [7].

Definition 1. A second order non-holonomic connection on Y is a section $\Delta : Y \to \widetilde{J}^2 Y$.

If the values of Δ lie in $\overline{J}^2 Y$ or $J^2 Y$, we say Δ is semiholonomic or holonomic, respectively.

Using β_1 and β_2, we obtain two underlying first order connections $\Delta_1 = \beta_1 \circ \Delta : Y \to J^1 Y$ and $\Delta_2 = \beta_2 \circ \Delta : Y \to J^1 Y$.

The coordinate form of Δ is

$$y_i^p = F_i^p(x, y)\,, \;\; y_{0i}^p = G_i^p(x, y)\,, \;\; y_{ij}^p = H_{ij}^p(x, y)\,. \tag{1}$$

The first or the second expression is the coordinate form of Δ_1 or Δ_2, respectively.

For a first order connection $\Gamma : Y \to J^1 Y$, we can construct the jet prolongation $J^1\Gamma : J^1 Y \to \widetilde{J}^2 Y$. If $\overline{\Gamma} : Y \to J^1 Y$ is another first order connection, then $(J^1\Gamma) \circ \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ is a second order non-holonomic connection on Y.

Definition 2. The second order connection $\Gamma * \overline{\Gamma} := (J^1\Gamma) \circ \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ is called the product of Γ and $\overline{\Gamma}$.

If the coordinate forms of Γ and $\overline{\Gamma}$ are

$$\Gamma \equiv y_i^p = F_i^p(x, y)\,, \qquad \overline{\Gamma} \equiv y_i^p = G_i^p(x, y)\,, \tag{2}$$

the coordinate expression of $J^1\Gamma$ is $y_{0i}^p = y_i^p$, $y_{ij}^p = \frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} y_j^q$ and the coordinate form of $\Gamma * \overline{\Gamma}$ is

$$y_i^p = F_i^p\,, \quad y_{0i}^p = G_i^p\,, \quad y_{ij}^p = \frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} G_j^q\,. \tag{3}$$

Clearly, $\Gamma * \overline{\Gamma}$ is semiholonomic, if and only if $\Gamma = \overline{\Gamma}$. A well known fact is that $\Gamma * \Gamma$ is holonomic, if and only if Γ is curvature-free.

$\beta : J^1 Y \to Y$ is an affine bundle with derived vector bundle $VY \otimes T^*M$, [15]. So $\beta_1 : \widetilde{J}^2 Y \to J^1 Y$ is also an affine bundle, whose derived vector bundle is $VJ^1Y \otimes T^*M$. We recall a well known exact sequence of vector bundles over $J^1 Y$

$$0 \to VY \otimes_{J^1Y} T^*M \hookrightarrow VJ^1Y \overset{V\beta}{\to} \beta^* VY \to 0\,, \tag{4}$$

where β^* indicates the pullback. Tensorizing with T^*M, we obtain another exact sequence

$$0 \to VY \otimes_{J^1Y} \bigotimes^2 T^*M \hookrightarrow VJ^1Y \otimes T^*M \to \beta^* VY \otimes T^*M \to 0\,. \tag{5}$$

Consider $A \in (\widetilde{J}^2 Y)_y$, $B \in (VY \otimes \bigotimes^2 T^*M)_y$, $y \in Y$. Write $B(\beta_1 A) \in VY \otimes_{J^1Y} \bigotimes^2 T^*M \subset VJ^1Y \otimes T^*M$ for the pullback of B into the fiber over $\beta_1 A \in J^1 Y$. Then we have defined

$$A + B(\beta_1 A) =: A \dot{+} B\,. \tag{6}$$

In coordinates, if $A = (x^i, y^p, a^p_i, a^p_{0i}, a^p_{ij})$, $B = (x^i, y^p, b^p_{ij})$, then $B(\beta_1 A) = (x^i, y^p, a^p_i, 0, b^p_{ij})$ and

$$A \dotplus B = (x^i, y^p, a^p_i, a^p_{0i}, a^p_{ij} + b^p_{ij})\,. \tag{7}$$

Conversely, if we have another $\overline{A} \in \widetilde{J}^2 Y$ satisfying $\beta_1 \overline{A} = \beta_1 A$, $\beta_2 \overline{A} = \beta_2 A$, there is a unique $B \in VY \otimes \overset{2}{\bigotimes} T^*M$ such that $\overline{A} = A \dotplus B$.

Definition 3. We write $B = \overline{A} \dot{-} A$ and we say that B is the strong difference of $\overline{A}$ and A.

For a second order connection $\Delta : Y \to \widetilde{J}^2 Y$, we have Δ_1, $\Delta_2 : Y \to J^1 Y$, so that we can construct $\Delta_1 * \Delta_2 : Y \to \widetilde{J}^2 Y$.

Proposition 1. *Second order non-holonomic connections on Y are in bijection with triples $(\Gamma, \overline{\Gamma}, \Sigma)$, where $\Gamma, \overline{\Gamma} : Y \to J^1 Y$ are first order connections on Y and $\Sigma : Y \to VY \otimes \overset{2}{\bigotimes} T^*M$ is a section.*

Proof. We set $\Gamma = \Delta_1$, $\overline{\Gamma} = \Delta_2$, $\Sigma = \Delta \dot{-} \Delta_1 * \Delta_2$. □

Remark 1. Such a result for principal connections was deduced by Virsik, [19]. We remark that an analogous formula for principal connections on the first principal prolongation of a principal bundle was proved in [14].

2. Principal and associated connections

Consider a principal bundle $P \to M$ with structure group G. Let s be a local section of P. The well known formula

$$(j^1_x s(v))g = j^1_x(s(v)g)\,, \qquad g \in G\,,$$

defines a right action of G on $J^1 P$. In the same way, the formula

$$(j^1_x \sigma(v))g = j^1_x(\sigma(v)g)\,, \tag{8}$$

where σ is a local section of $J^1 P$, defines a right action of G on $\widetilde{J}^2 P$.

Definition 4. A principal second order connection on P is a G-invariant section $\Delta : P \to \widetilde{J}^2 P$.

One verifies easily that the product of two first order principal connections is a second order principal connection.

In the first order case, it is well known that every principal connection $\Gamma : P \to J^1 P$ induces a connection $\Gamma[S]$ on the associated bundle $P[S]$, where S is any left G-space. We shall need the following form of this construction. Let $q : P \times S \to P[S]$ be the canonical map

$$q(u, a) = \{(u, a)\}\,, \qquad u \in P, a \in S\,.$$

If we fix $a \in S$, $q(-,a)$ is a map $P \to P[S]$ and we can construct its first jet prolongation $J^1q(-,a) : J^1P \to J^1(P[S])$. So we obtain a map

$$\mathcal{J}^1 q : J^1P \times S \to J^1(P[S])\,, \tag{9}$$

$\mathcal{J}^1 q(j^1_x s(v), a) = j^1_x q(s(v), a)$. Then we set

$$\Gamma[S](q(u,a)) = \mathcal{J}^1 q(\Gamma(u), a)\,.$$

Since Γ is principal, this formula does not depend of the representatives u and a of $q(u,a)$.

By iteration, we obtain a map

$$\widetilde{\mathcal{J}}^2 q : \widetilde{J}^2 P \times S \to \widetilde{J}^2(P[S])\,. \tag{10}$$

Then for every principal connection $\Delta : P \to \widetilde{J}^2 P$ we construct $\Delta[S] : P[S] \to \widetilde{J}^2 P[S]$ by

$$\Delta[S](q(u,a)) = \widetilde{\mathcal{J}}^2 q(\Gamma(u), a)\,. \tag{11}$$

Even this does not depend of the representatives u and a of $q(u,a)$.

Definition 5. $\Delta[S]$ is called the induced second order connection on $P[S]$.

Proposition 2. *For every two first order principal connections Γ and $\overline{\Gamma}$ on P and every left G-space S,*

$$(\Gamma[S]) * (\overline{\Gamma}[S]) = (\Gamma * \overline{\Gamma})[S]\,. \tag{12}$$

Proof. This follows directly from (10). □

Remark 2. An r-th order non-holonomic connection on a fibered manifold Y is a section $\Gamma : Y \to \widetilde{J}^r Y$, [11]. If $\overline{\Gamma} : Y \to \widetilde{J}^k Y$ is another k-th order connection, we construct by iteration $\widetilde{J}^k\Gamma : \widetilde{J}^k Y \to \widetilde{J}^{r+k} Y$. Then $\Gamma * \overline{\Gamma} := \widetilde{J}^k\Gamma \circ \overline{\Gamma}$ is an $(r+k)$-th order connection on Y. On a principal bundle P, an r-th order connection that is right invariant in the sense of (8) is called principal. Every principal r-th order connection Γ on P induces an r-th order connection $\Gamma[S]$ on $P[S]$ analogously to (11). If $\overline{\Gamma}$ is another k-th order principal connection on P, then one verifies in the same way as above that (12) holds even in the case of arbitrary r and k.

For every vector bundle $E \to M$, $J^1E \to M$ is also a vector bundle. So even $\widetilde{J}^2 E \to M$ is a vector bundle.

Definition 6. A second order connection $\Delta : E \to \widetilde{J}^2 E$ is called linear, if Δ is a vector bundle morphism.

If the fiber coordinates y^p on E are linear, then the coordinate form of a linear connection is

$$y^p_i = F^p_{qi}(x) y^q\,, \quad y^p_{0i} = G^p_{qi}(x) y^q\,, \quad y^p_{ij} = H^p_{qij}(x) y^q\,. \tag{13}$$

One verifies directly that the product of two linear first order connections is a linear second order connection.

Consider the frame bundle PE of E, i.e. the bundle of all linear frames in the individual fibers of E. This is a principal fiber bundle with structure group $GL(n,\mathbb{R})$, where n is the fiber dimension of E. The local coordinates on PE corresponding to x^i, y^p are x^i, a^p_q, $\det(a^p_q)\neq 0$. Then the induced coordinates on $\widetilde{J}^2(PE)$ are a^p_{qi}, a^p_{q0i}, a^p_{qij}. The action of $(b^p_q)\in GL(n,\mathbb{R})$ on $\widetilde{J}^2(PE)$ has the following form

$$(a^p_q, a^p_{qi}, a^p_{q0i}, a^p_{qij})\,(b^p_q) = (a^p_r b^r_q, a^p_{ri} b^r_q, a^p_{r0i} b^r_q, a^p_{rij} b^r_q)\,.$$

Hence the coordinate form of a second order principal connection Δ on PE is

$$a^p_{qi} = F^p_{ri}(x)a^r_q,\ a^p_{q0i} = G^p_{ri}(x)a^r_q\,,\ a^p_{qij} = H^p_{rij}(x)a^r_q\,. \tag{14}$$

Clearly, E coincides with the associated fiber bundle $PE[\mathbb{R}^n]$ and one evaluates easily that the equations of the induced connection $\Delta[\mathbb{R}^n]$ are (13). This implies the following assertion, which is quite similar to the first order case.

Proposition 3. *The construction of induced connections establishes a bijection between second order principal connections on PE and second order linear connections on E.*

3. The absolute differentiation on fibered manifolds

The Ehresmann's idea of the absolute differentiation with respect to principal higher order connections can be extended to connections on arbitrary fibered manifolds, [11]. We are going to discuss the second order case in detail, for we are looking for an approach that can be generalized to the functional bundles mentioned in the introduction.

In the first order case, consider a section $C : Y_x \to J^1_x Y$, $x\in M$. This defines a bijection $\mu C : J^1_x(M, Y_x) \to J^1_x Y$ as follows. Take locally a smoothly parametrized family of maps $\psi(v,y) : Y_x \to Y_v$, $v\in M$, such that $\psi(x,y) = \mathrm{id}_{Y_x}$ and $j^1_x\psi(v,y) = C(y)$. Let $X = j^1_x\varphi(v) \in J^1_x(M, Y_x)$. Then $\psi(v,\varphi(v))$ is a local section of Y and we define

$$\mu C(X) = j^1_x\psi(v,\varphi(v))\,. \tag{15}$$

In coordinates, if Y^p_i are the induced jet coordinates on $J^1_x(M,Y_x)$ and $y^p_i = F^p_i(x,y)$ is the coordinate form of C, then (15) yields the following coordinate form of μC

$$y^p_i = Y^p_i + F^p_i(x,y)\,. \tag{16}$$

Hence the inverse map $(\mu C)^{-1}$ is the standard absolute differentiation of the theory of first order connections on Y.

In the second order case, we consider a section $D : Y_x \to \widetilde{J}^2_x Y$. We can write locally $D(y) = j^1_x \Psi(v,y)$, where $\Psi(v,y) \in J^1_v Y$ with target $\psi(v,y) \in Y_v$ and $\psi(x,y) = y$. An element $X \in \widetilde{J}^2_x(M, Y_x)$ is of the form $j^1_x \Phi(v)$, $\Phi(v) \in J^1_v(M, Y_x)$ with target $\varphi(v) \in Y_x$. The first jet prolongation of the map $\psi_v := \psi(v, -) : Y_x \to Y_v$ is

$$J^1(\mathrm{id}_M, \psi_v) : J^1(M, Y_x) \to J^1(M, Y_v)\,.$$

Hence $J^1(\mathrm{id}_M, \psi_v)\ (\Phi(v)) \in J^1_v(M, Y_v)$ with target $\psi(v, \varphi(v))$. Since $J^1 Y$ is an affine bundle with the derived vector bundle $VY \otimes T^*M = \bigcup_{x \in M} J^1_x(M, Y_x)$, we have defined

$$\Psi(v, \varphi(v)) + J^1(\mathrm{id}_M, \psi_v)(\Phi(v)) \in J^1_v Y \tag{17}$$

and we can set

$$\mu D(X) = j^1_x[\Psi(v, \varphi(v)) + J^1(\mathrm{id}_M, \psi_v)(\Phi(v))] \in \widetilde{J}^2_x Y\,. \tag{18}$$

Lemma 1. *$\mu D : \widetilde{J}^2_x(M, Y_x) \to \widetilde{J}^2_x Y$ is a diffeomorphism.*

Proof. Let (1) be the coordinate form of D. If the coordinate form of Ψ is $\psi^p(v,y)$, $\psi^p_i(v,y)$, then

$$F^p_i(x,y) = \psi^p_i(x,y), \quad G^p_i(x,y) = \frac{\partial \psi^p(x,y)}{\partial x^i}\,, \quad H^p_{ij}(x,y) = \frac{\partial \psi^p_i(x,y)}{\partial x^j}\,.$$

Let Y^p_{0i}, Y^p_{ij} be the additional coordinates on $\widetilde{J}^2_x(M, Y_x)$. If $\varphi^p(v)$, $\varphi^p_i(v)$ is the coordinate expression form of $\Phi(v)$, the coordinates of X are

$$Y^p_i = \varphi^p_i(x)\,, \quad Y^p_{0i} = \frac{\partial \varphi^p(x)}{\partial x^i}\,, \quad Y^p_{ij} = \frac{\partial \varphi^p_i(x)}{\partial x^j}\,.$$

The coordinate form of (17) is $\psi^p(v, \varphi(v))$ and

$$\psi^p_i(v, \varphi(v)) + \frac{\partial \psi^p(v,y)}{\partial y^q} \varphi^q_i(v)\,.$$

Passing to 1-jets, we obtain the following coordinate expression of μD

$$\begin{cases} y^p_i = F^p_i(x,y) + Y^p_i\,, \qquad y^p_{0i} = G^p_i(x,y) + Y^p_{0i}\,, \\ y^p_{0i} = H^p_{ij}(x,y) + \dfrac{\partial F^p_i(x,y)}{\partial y^q} Y^q_{0j} + \dfrac{\partial G^p_j(x,y)}{\partial y^q} Y^q_i + Y^p_{ij}\,. \end{cases} \tag{19}$$

□

Remark 3. If the values of D lie in $J^2_x Y$ and $X \in J^2_x(M, Y_x)$, then $\mu D(X) \in J^2_x Y$. In this case one can construct $\mu D(X)$ by the second order analogy of formula (15).

Definition 7. The map $\nabla_\Delta := (\mu D)^{-1} : \widetilde{J}^2_x Y \to \widetilde{J}^2_x(M, Y_x)$ will be called the absolute differentiation with respect to D.

If Δ is a second order connection on Y, then the restricted map over $x \in M$ is a section $\Delta(x) : Y_x \to \widetilde{J}^2_x Y$. Hence we have $\nabla_{\Delta(x)} : \widetilde{J}^2_x Y \to \widetilde{J}^2_x(M, Y_x)$. If s is a section of Y, we construct its second order jet prolongation $j^2 s : M \to J^2 Y$.

Definition 8. The map

$$\big(\nabla_\Delta s\big)(x) = \nabla_{\Delta(x)}(j^2_x s) : M \to \bigcup_{x \in M} \widetilde{J}^2_x(M, Y_x)$$

will be called the absolute differential of s with respect to Δ.

In Section 1, we deduced the formula

$$\Delta = \Delta_1 * \Delta_2 \dotplus \Sigma\,. \tag{20}$$

We are going to describe the effect of (20) at the level of the absolute differentiation. If we consider the product fibered manifold $M \times N \to M$, we have $\widetilde{J}^2(M \times N \to M) = \widetilde{J}^2(M, N)$ and $VY \otimes \bigotimes^2 T^*M = TN \otimes \bigotimes^2 T^*M$. Hence every $A \in \widetilde{J}^2_x(M, Y)_y$ and every $B \in T_y N \otimes \bigotimes^2 T^*_x M$ determine

$$A \dotplus B \in \widetilde{J}^2_x(M, N)_y\,. \tag{21}$$

In particular, for $A \in \widetilde{J}^2_x(M, Y_x)_y$ and $B \in V_y Y \otimes T^*_x M$, we have $A \dotplus B \in \widetilde{J}^2_x(M, Y_x)_y$. If we compare (3), (7) and (19), we obtain

Proposition 4. *For every $X \in \widetilde{J}^2_y Y$,*

$$\nabla_{\Delta(x)}(X) = \nabla_{(\Delta_1 * \Delta_2)(x)}(X) \dotplus \Sigma(y)\,. \tag{22}$$

For a section s of $Y, \nabla_{\Delta_1 * \Delta_2} s$ can be expressed as the absolute differential of $\nabla_{\Delta_1} s$ with respect to a connection derived from Δ_2, [4], [11]. However, we are going to construct the latter connection in a new way, that can be applied to the functional case as well.

Write

$$J^1_{\text{fib}} Y = \bigcup_{x \in M} J^1(M, Y_x)\,.$$

This sum defines a fibered manifold over M. The local coordinates on $J^1_{\text{fib}} Y$ induced by x^i, y^p are x^i, u^i, y^p, Y^p_i, where u^i are the coordinates of the source of an element from $J^1(M, Y_x)$ and Y^p_i are its jet coordinates. We extend J^1_{fib} into a functor on the category $\mathcal{FM}_M$ of fibered manifolds over M and their base-preserving morphisms as follows. For another fibered manifold $Z \to M$ and a base-preserving morphism $f : Y \to Z$ with the restrictions $f_x : Y_x \to Z_x$, $x \in M$, we set

$$J^1_{\text{fib}} f = \bigcup_{x \in M} J^1(\mathrm{id}_M, f_x)\,,$$

where $J^1(\mathrm{id}_M, f_x) : J^1(M, Y_x) \to J^1(M, Z_x)$.

We are going to introduce a canonical map

$$i_Y : J^1_{\text{fib}}(J^1 Y) \to J^1(J^1_{\text{fib}} Y)\,.$$

Let $X \in J^1_{\text{fib}}(J^1 Y)$, $X = j^1_z \sigma(v)$, $\sigma(v) \in J^1_x Y$, $v \in M$. We have $\sigma(v) = j^1_x s(v, w)$, $w \in M$. Hence $X = \underset{1}{j^1_z}(\underset{2}{j^1_x} s(v, w))$, where the subscript 1 or 2 means the partial jet with respect to the first or second factor, respectively. Then we apply exchange and define

$$i_Y(X) = \underset{2}{j^1_x}(\underset{1}{j^1_z} s(v, w))\,. \tag{23}$$

Since $\sigma(v) \in J^1 Y$, we have $s(v, w) \in Y_w$. Hence $\underset{1}{j^1_z} s(v, w) \in J^1_z(M, Y_w)$ and this is a section of $J^1_{\text{fib}} Y$.

Every connection $\Gamma : Y \to J^1 Y$ is a base-preserving morphism, so that we can construct

$$J^1_{\text{fib}} \Gamma : J^1_{\text{fib}} Y \to J^1_{\text{fib}}(J^1 Y)\,. \tag{24}$$

Definition 9. The map $\Gamma^1 = i_Y \circ J^1_{\text{fib}} \Gamma : J^1_{\text{fib}} Y \to J^1(J^1_{\text{fib}} Y)$ will be called the connection induced by Γ on $J^1_{\text{fib}} Y$.

Proposition 5. *If* (2) *is the coordinate expression of* Γ*, then the coordinate form of* Γ^1 *is*

$$u^i_j = 0\,, \quad y^p_i = F^p_i(x, y)\,, \quad Y^p_{ij} = \frac{\partial F^p_j(x, y)}{\partial y^q} Y^q_i\,. \tag{25}$$

Proof. The proof consists in direct evaluation. □

Hence Γ^1 coincides with the connection (21) from [4].

We recall that the absolute differential of a section of Y with respect to a first order connection on Y is a section of $J^1_{\text{fib}} Y$.

Proposition 6. *For every section* s *of* Y *and every two first order connections* Γ *and* $\overline{\Gamma}$ *on* Y,

$$\nabla_{\overline{\Gamma} * \Gamma} s = \nabla_{\Gamma^1}(\nabla_{\overline{\Gamma}} s)\,. \tag{26}$$

Proof. Let (2) be the coordinate expressions of Γ and $\overline{\Gamma}$ and let s be given by $s^p(x)$. By (25), the coordinate form of the main term of $\nabla_{\Gamma^1}(\nabla_{\overline{\Gamma}} s)$ is

$$\frac{\partial^2 s^2}{\partial x^i \partial x^j} - \frac{\partial G^p_i}{\partial x^j} - \frac{\partial G^p_i}{\partial y^q}\frac{\partial s^q}{\partial x^j} - \frac{\partial F^p_j}{\partial y^q}\left(\frac{\partial s^q}{\partial x^i} - G^q_i\right)\,.$$

Comparing with (3) and (19), we prove our claim. □

4. Representations of some functional jets

We shall need the composition of non-holonomic 2-jets, [6]. Consider three manifolds Q_1, Q_2, Q_3. Let $A = j^1_x\sigma \in \widetilde{J}^2_x(Q_1,Q_2)_y$ and $B = j^1_y\varrho \in \widetilde{J}^2_y(Q_2,Q_3)_z$. Hence σ is a map $Q_1 \to J_1(Q_1,Q_2)$ satisfying $\alpha\circ\sigma = \mathrm{id}_{Q_1}$ and ϱ is a map $Q_2 \to J^1(Q_2,Q_3)$ satisfying $\alpha\circ\varrho = \mathrm{id}_{Q_2}$. Then $\sigma(u) \in J^1(Q_1,Q_2)$ and $\varrho(\beta\sigma(u)) \in J^1(Q_2,Q_3)$ are two composable 1-jets and one defines

$$B\circ A = j^1_x(\varrho(\beta\sigma(u))\circ\sigma(u)) \in \widetilde{J}^2_x(Q_1,Q_3)_z$$

with the composition of 1-jets on the right-hand side, [6]. In coordinates, if $A = (x^i, y^p, a^p_i, a^p_{0i}, a^p_{ij})$, $B = (y^p, z^a, b^a_p, b^a_{0p}, b^a_{pq})$, then

$$B\circ A = (x^i, z^a, b^a_p a^p_i, b^a_{0p}a^p_{0i}, b^a_{pq}a^p_i a^q_{0j} + b^a_p a^p_{ij})\,. \tag{27}$$

Let $p_1 : Y_1 \to M$ and $p_2 : Y_2 \to M$ be two fibered manifolds over the same base. Consider the set of all fiber maps

$$\mathcal{F}(Y_1,Y_2) = \bigcup_{x\in M} C^\infty(Y_{1x}, Y_{2x})$$

and denote by $p : \mathcal{F}(Y_1,Y_2) \to M$ the canonical projection. The set $\mathcal{F}(Y_1,Y_2)$ is a smooth space in the sense of Frölicher, [8]. We shall use some local coordinates x^i on M and some additional local coordinates y^p or z^a on Y_1 or Y_2, respectively.

Each section $s : M \to \mathcal{F}(Y_1,Y_2)$ is interpreted as a base-preserving morphism $\widetilde{s} : Y_1 \to Y_2$, $\widetilde{s}(y) = s(p_1 y)(y)$. Two sections $s_1, s_2 : M \to \mathcal{F}(Y_1,Y_2)$ determine the same element $j^1_x s_1 = j^1_x s_2$ of $J^1\mathcal{F}(Y_1,Y_2)$ at $x \in M$, if $j^1_y\widetilde{s}_1 = j^1_y\widetilde{s}_2$ for all $y \in Y_{1x}$, [2]. Let $X = j^1_x s \in J^1_x\mathcal{F}(Y_1,Y_2)$ and $\varphi = s(x) \in \mathcal{F}(Y_1,Y_2)$ be its target. The map

$$\widetilde{X} : J^1_x Y_1 \to J^1_x Y_2\,, \qquad \widetilde{X}(j^1_x\sigma) = j^1_x(\widetilde{s}\circ\sigma)$$

is called the associated map of X. If $z^a = \varphi^q(x,y)$ is the coordinate expression of $\widetilde{s}$, then $\widetilde{X}$ is of the form

$$z^a_i = \frac{\partial\varphi^a(x,y)}{\partial y^p}\, y^p_i + \frac{\partial\varphi^a(x,y)}{\partial x^i}\,.$$

This is an affine bundle morphism $J^1_x Y_1 \to J^1_x Y_2$ over $\varphi : Y_{1x} \to Y_{2x}$ whose derived linear map $V_x Y_1 \otimes T^*M \to V_x Y_2 \otimes T^*M$ is $T\varphi\otimes\mathrm{id}$. Conversely, in [2] we deduced that for every map $\Phi : J^1_x Y_1 \to J^1_x Y_2$ with these properties there exists a unique $X \in J^1_x\mathcal{F}(Y_1,Y_2)$ such that $\Phi = \widetilde{X}$. Hence the coordinate form of an element of $J^1_x\mathcal{F}(Y_1,Y_2)$ is $z^a = \varphi^a(y)$ and

$$z^a_i = \frac{\partial\varphi^a(y)}{\partial y^p}\, y^p_i + \varphi^a_i(y)\,. \tag{28}$$

In the second order we must proceed systematically in the jet way. We shall need a general concept. Let M_1, M_2, N_1, N_2 be four manifolds and $f : N_1 \to M_1$, $g : N_2 \to M_2$ be two maps.

Definition 10. Two r-jets $Z \in J^r_{z_1}(N_1, N_2)_{z_2}$ and $X \in J^r_{f(z_1)}(M_1, M_2)_{g(z_2)}$ are called (f, g)-related, if

$$X \circ j^r_{z_1} f = j^r_{z_2} g \circ Z \,. \tag{29}$$

This concept can be applied to non-holonomic jets as well. Let $Z \in \widetilde{J}^r_{z_1}(N_1, N_2)_{z_2}$ and $X \in \widetilde{J}^r_{f(z_1)}(M_1, M_2)_{g(z_2)}$ (the reader may assume $r = 1, 2$). They are said to be (f, g)-related, if (29) holds with the composition of non-holonomic r-jets on both sides.

Consider the case $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$.

Definition 11. $\widetilde{J}^r_M(Y_1, Y_2)$ is the subset of all $Z \in \widetilde{J}^r(Y_1, Y_2)$ that satisfy $p_1(\alpha Z) = p_2(\beta Z) =: x$ and are (p_1, p_2)-related with $j^r_x \,\mathrm{id}_M$.

Analogously to [2] we define

$$\mathcal{F}\widetilde{J}^r(Y_1, Y_2) = \bigcup_{x \in M} \widetilde{J}^r(Y_{1x}, Y_{2x}) \,.$$

A canonical map

$$D : \widetilde{J}^r_M(Y_1, Y_2) \to \mathcal{F}\widetilde{J}^r(Y_1, Y_2) \tag{30}$$

is induced by the injection of individual fibers. Namely, if $Z \in \widetilde{J}_M(Y_1, Y_2)$, $\alpha Z = y$, $py = x$ and $i_x : Y_{1x} \to Y_1$ is the injection, then $Z \circ (j^r_y i_x)$ is identified with an element $D(Z)$ of $\widetilde{J}^r(Y_{1x}, Y_{2x})$ with target βZ.

It is well known that $\widetilde{J}^r_x Y_i$ can be defined as the subset of all elements $A \in \widetilde{J}^r(M, Y_i)$ satisfying

$$(j^r_{\beta A} p_i) \circ A = j^r_x \,\mathrm{id}_M \,, \qquad i = 1, 2 \,.$$

Hence every $Z \in \widetilde{J}^r_M(Y_1, Y_2)$, $\alpha Z = y_1$, $\beta Z = y_2$, $x = p_1 y_1$ defines a map

$$\mu Z : \widetilde{J}^r_{y_1} Y_1 \to \widetilde{J}^r_{y_2} Y_2$$

by the jet composition $\mu Z(A) = Z \circ A$. Indeed, $(j^r_{y_2} p_2) \circ Z \circ A = (j^r_x \,\mathrm{id}_M) \circ j^r_{y_1} p_1 \circ A$ by projectability and $j^r_{y_1} p_1 \circ A = j^r_x \,\mathrm{id}_M$ by $A \in \widetilde{J}^r_x Y_1 \subset \widetilde{J}^r_x(M, Y_1)$, so that $Z \circ A \in \widetilde{J}^r_{y_2} Y_2$.

In the first order, the local coordinates of an element $X \in J^1_M(Y_1, Y_2)$ are x^i, y^p, z^a, z^a_p, z^a_i and $D(X)$ is determined by z^a_p. Hence the result preceeding (28) can be reformulated by saying that the elements of $J^1_x \mathcal{F}(Y_1, Y_2)$ with target φ are in bijection with sections $S : Y_{1x} \to J^1_M(Y_1, Y_2)$ satisfying $D \circ S = j^1 \varphi : Y_{1x} \to J^1(Y_{1x}, Y_{2x})$.

The second non-holonomic prolongation $\widetilde{J}^2\mathcal{F}(Y_1,Y_2)$ is defined by the iteration $J^1(J^1\mathcal{F}(Y_1,Y_2))$, [2]. In formula (12) of [2] we deduced that the associated map $\widetilde{X} : \widetilde{J}^2_x Y_1 \to \widetilde{J}^2_x Y_2$ of an element $X \in \widetilde{J}^2_x\mathcal{F}(Y_1,Y_2)$ is of the form

$$\left\{\begin{aligned} z^a_i &= \tfrac{\partial\varphi^a}{\partial y^p}\, y^p_i + \varphi^a_i\,, \qquad z^a_{0i} = \tfrac{\partial\varphi^a}{\partial y^p}\, y^p_{0i} + \varphi^a_{0i}\,, \\ z^a_{ij} &= \varphi^a_{ij} + \tfrac{\partial\varphi^a_i}{\partial y^p}\, y^p_{0j} + \tfrac{\partial\varphi^a_{0j}}{\partial y^p}\, y^p_i + \tfrac{\partial^2\varphi^a}{\partial y^p\partial y^q}\, y^p_i\, y^q_{0j} + \tfrac{\partial\varphi^a}{\partial y^q}\, y^q_{ij} \end{aligned}\right. \tag{31}$$

where $\varphi^a(y)$ is the target of X and $\varphi^a_i(y)$, $\varphi^a_{0i}(y)$, $\varphi^a_{ij}(y)$ are arbitrary functions on Y_{1x}.

The local coordinates of an element $X \in \widetilde{J}^2_M(Y_1,Y_2)$ are x^i, y^p, z^a, z^a_p, z^a_i, z^a_{0p}, z^a_{0i}, z^a_{pq}, z^a_{ip}, z^a_{pi}, z^a_{ij} and $D(X)$ is determined by z^a_p, z^a_{0p}, z^a_{pq}. The induced map $\mu X : \widetilde{J}^2_{Y_1}Y_1 \to \widetilde{J}^2_{Y_2}Y_2$, which is defined by the composition of non-holonomic 2-jets, is of the form

$$\left\{\begin{aligned} z^a_i &= z^a_p y^p_i + z^a_i\,, \qquad z^a_{0i} = z^a_{0p} y^p_{0i} + z^a_{0i}\,, \\ z^a_{ij} &= z^a_p y^p_{ij} + z^a_{pq} y^p_i\, y^q_{0j} + z^a_{ip} y^p_{0j} + z^a_{pj} y^p_i + z^a_{ij}\,. \end{aligned}\right. \tag{32}$$

Hence every section $S : Y_{1x} \to \widetilde{J}^2_M(Y_1,Y_2)$ over $\varphi : Y_{1x} \to Y_{2x}$ satisfying $D \circ S = j^2\varphi : Y_{1x} \to J^2(Y_{1x},Y_{2x})$ is determined by arbitrary functions $\varphi^a(y)$, $\varphi^a_i(y)$, $\varphi^a_{0i}(y)$, $\varphi^a_{ip}(y)$, $\varphi^a_{pi}(y)$, $\varphi^a_{ij}(y)$ on Y_{1x}. To obtain (31), the following compatibility conditions

$$\frac{\partial\varphi^a_i}{\partial y^p} = \varphi^a_{ip}\,, \qquad \frac{\partial\varphi^a_{0i}}{\partial y^p} = \varphi^a_{pi} \tag{33}$$

are to be satisfied. We have found a global geometrical interpretation of these conditions in terms of the representation of non-holonomic 2-jets by means of the induced maps between second iterated tangent bundles by Pradines, [17]. This is a straightforward procedure, but it is too long to be discussed here. Thus, we can summarized by

Proposition 7. *Let $S : Y_{1x} \to \widetilde{J}^2_M(Y_1,Y_2)$ be a section over $\varphi : Y_{1x} \to Y_{2x}$ satisfying $D \circ S = j^2\varphi$ and the compatibility conditions (33). Then there is a unique element $X \in \widetilde{J}^2_x\mathcal{F}(Y_1,Y_2)$ such that $\widetilde{X}(y) = \mu S(y)$ for all $y \in Y_{1x}$.*

We have two jet projections

$$\beta_1, \beta_2 : \widetilde{J}^2\mathcal{F}(Y_1,Y_2) \to J^1\mathcal{F}(Y_1,Y_2)$$

defined analogously to Section 1. On the other hand, we clarified in [2] that the vertical tangent bundle $V\mathcal{F}(Y_1,Y_2)$ of $\mathcal{F}(Y_1,Y_2)$ coincides with $\mathcal{F}(Y_1,VY_2)$. Since $T^*M \to M$ is a finite dimensional vector bundle, the tensor product $V\mathcal{F}(Y_1,Y_2) \otimes \bigotimes^2 T^*M$ is defined fiberwise by means of the linear maps from the vector bundle dual to $\bigotimes^2 T^*M$. Every $X \in V\mathcal{F}(Y_1,Y_2) \otimes \bigotimes^2 T^*M$ over φ is characterized by the associated map

$$\widetilde{X} : V_xY_1 \otimes \bigotimes^2 T^*_xM \to V_xY_2 \otimes \bigotimes^2 T^*_xM\,.$$

If Y^p_{ij} or Z^a_{ij} are the induced coordinates on the first or the second bundle, respectively, then the coordinate form of $\widetilde{X}$ is

$$Z^a_{ij} = \frac{\partial \varphi^a(y)}{\partial y^p} Y^p_{ij} + \xi^a_{ij}(y)\,. \tag{34}$$

Let $U, V \in \widetilde{J}^2\mathcal{F}(Y_1, Y_2)$ over the same φ satisfy $\beta_1 U = \beta_1 V$, $\beta_2 U = \beta_2 V$. For an element $B \in V_y Y_1 \otimes \bigotimes^2 T^*_x M$ we take $A, \overline{A} \in \widetilde{J}^2_y Y_1$ satisfying $\beta_1 A = \beta_1 \overline{A}$, $\beta_2 A = \beta_2 \overline{A}$ and $B = \overline{A} \dot{-} A$. By (31), $\widetilde{U}(\overline{A})$ and $\widetilde{V}(A)$ satisfy the same conditions with respect to β_1 and β_2, so that we have defined

$$\widetilde{U}(\overline{A}) \dot{-} \widetilde{V}(A) \in V_{\varphi(y)} Y_2 \otimes \bigotimes^2 T^*_x M\,. \tag{35}$$

If b^p_{ij} are the coordinates of B and $\mu^a_{ij}(y)$ or $\nu^a_{ij}(y)$ are the second order coordinate functions of U or V, respectively, then the coordinate form of (35) is

$$\frac{\partial \varphi^a(y)}{\partial y^p} b^p_{ij} + \mu^a_{ij}(y) - \nu^a_{ij}(y)\,. \tag{36}$$

In this way U and V define an element $U \dot{-} V \in V\mathcal{F}(Y_1, Y_2) \otimes \bigotimes^2 T^*M$.

5. 2-connections on $\mathcal{F}(Y_1, Y_2)$

In [2] we introduced a connection Γ on $\mathcal{F}(Y_1, Y_2)$ as a section $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$ that is smooth in the Frölicher sense. Such connection is said to be of order $r \geq 1$ if the condition $j^r_y \varphi = j^r_y \psi$, $\varphi, \psi \in C^\infty(Y_{1x}, Y_{2x})$, $y \in Y_{1x}$, implies

$$\widetilde{\Gamma(\varphi)} | J^1_y Y_1 = \widetilde{\Gamma(\psi)} | J^1_y Y_1\,.$$

To distinguish the order in this operator sense from the order of the jet prolongation of $\mathcal{F}(Y_1, Y_2)$ in question, we shall say that Γ is a 1-connection of order r. Using Section 4, we can reformulate the concept of the associated map $\mathcal{G}$ of Γ, [2]. This is a C^∞-map of

$$\mathcal{F}J^r(Y_1, Y_2) = \bigcup_{x \in M} J^r(Y_{1x}, Y_{2x})$$

into $J^1_M(Y_1, Y_2)$ over the identity of $Y_1 \times_M Y_2$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F}J^r(Y_1, Y_2) \arrow[rr, "\mathcal{G}"] \arrow[dr, "\beta^1_r"'] & & J^1_M(Y_1, Y_2) \arrow[dl, "D"] \\ & \mathcal{F}J^1(Y_1, Y_2) & \end{tikzcd} \tag{37}$$

where β^1_r is the jet projection. Conversely, every map $\mathcal{G}$ such that (37) commutes determines a 1-connection Γ on $\mathcal{F}(Y_1, Y_2)$ by

$$\widetilde{\Gamma(\varphi)} = \bigcup_{x \in Y_{1x}} \mathcal{G}(j^r_y \varphi)\,, \qquad \varphi \in C^\infty(Y_{1x}, Y_{2x})\,.$$

Definition 12. A non-holonomic 2-connection on $\mathcal{F}(Y_1, Y_2)$ is a smooth section $\Delta : \mathcal{F}(Y_1, Y_2) \to \widetilde{J}^2\mathcal{F}(Y_1, Y_2)$.

Clearly, Δ determines two 1-connections Δ_1 and Δ_2 by means of the two jet projections β_1, $\beta_2 : \widetilde{J}^2\mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$ analogously to Section 1.

Every $\Delta(\varphi)$, $\varphi \in C^\infty(Y_{1x}, Y_{2x})$ defines the associated map

$$\widetilde{\Delta(\varphi)} : \widetilde{J}^2_x Y_1 \to \widetilde{J}^2_x Y_2\,. \tag{38}$$

Definition 13. A 2-connection Δ on $\mathcal{F}(Y_1, Y_2)$ is said to be of order $r \geq 2$, if the condition $j^r_y \varphi = j^r_y \psi$, $\varphi, \psi \in C^\infty(Y_{1x}, Y_{2x})$, $y \in Y_{1x}$, implies

$$\widetilde{\Delta(\varphi)} | \widetilde{J}^2_y Y_1 = \widetilde{\Delta(\psi)} | \widetilde{J}^2_y Y_1\,. \tag{39}$$

By Section 4, (39) is identified with an element of $\widetilde{J}^2_M(Y_1, Y_2)$. Hence Δ defines a map

$$\mathcal{D} : \mathcal{F}J^r(Y_1, Y_2) \to \widetilde{J}^2_M(Y_1, Y_2)\,,$$

which is called the associated map of Δ. Analogously to [2] we deduce that $\mathcal{D}$ is a C^∞-map. By Section 4, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{F}J^r(Y_1 Y_2) & \xrightarrow{\quad\mathcal{D}\quad} & \widetilde{J}^2_M(Y_1, Y_2) \\
\Big\downarrow{\scriptstyle \beta^2_r} & & \Big\downarrow{\scriptstyle D} \\
\mathcal{F}J^2(Y_1, Y_2) & \hookrightarrow & \mathcal{F}\widetilde{J}^2(Y_1, Y_2)
\end{array} \tag{40}$$

where β^2_r is the jet projection. Moreover, for every $\varphi \in C^\infty(Y_{1x}, Y_{2x})$ the compatibility conditions (33) hold. Thus, in the coordinate expression of $\mathcal{D}$ we can prescribe $z^a_i = \Phi^a_i$, $z^a_{0i} = \Phi^a_{0i}$, $z^a_{ij} = \Phi^a_{ij}$ arbitrarily and we have

$$z^a_{ip} = D_p \Phi^a_i\,, \quad z^p_{pi} = D_p \Phi^a_{0i}\,, \tag{41}$$

where D_p is the formal derivative from [2], whose coordinate form is

$$D_p = \frac{\partial}{\partial y^p} + \frac{\partial}{\partial z^a}\, z^a_p + \cdots + \frac{\partial}{\partial z^a_\alpha}\, z^a_{\alpha+p}\,.$$

Since D_p increases by one the order of the jet prolongation, (41) implies that Φ^a_i and Φ^a_{0i} are projectable to $\mathcal{F}J^{r-1}(Y_1, Y_2)$. By Proposition 7, we obtain immediately

Proposition 8. *For $r \geq 2$, let $\mathcal{D} : \mathcal{F}J^r(Y_1, Y_2) \to \widetilde{J}^2_M(Y_1, Y_2)$ be a C^∞-map over the identity of $Y_1 \times_M Y_2$ such that (40) commutes and the compatibility conditions (41) are satisfied. Then there is a unique 2-connection of order r on $\mathcal{F}(Y_1, Y_2)$ such that $\mathcal{D}$ is its associated map.*

Clearly, Δ is defined by

$$\widetilde{\Delta(\varphi)} = \bigcup_{y \in Y_{1x}} \mathcal{D}(j^r_y \varphi)\,, \qquad \varphi \in C^\infty(Y_{1x}, Y_{2x})\,.$$

We shall use the following explicit coordinate expression of $\mathcal{D}$

$$\left\{ \begin{array}{l} z^a_i = z^a_p y^p_i + \Phi^p_i\,, \qquad z^a_{0i} = z^q_p y^p_{0i} + \Phi^p_{0i}\,, \\ z^a_{ij} = z^a_p y^p_{ij} + z^a_{pq} y^p_i y^q_{0j} + D_p \Phi^a_i y^p_{0j} + D_p \Phi^a_{0j} y^p_i + \Phi^p_{ij} \end{array} \right. \tag{42}$$

where locally Φ^a_i and Φ^a_{0i} are arbitrary functions on $\mathcal{F}J^{r-1}(Y_1, Y_2)$ and Φ^a_{ij} are arbitrary functions on $\mathcal{F}J^r(Y_1, Y_2)$.

Example 1. In [3] we pointed out that two connections $\Gamma_1 : Y_1 \to J^1 Y_1$ and $\Gamma_2 : Y_2 \to J^1 Y_2$ determine, in a simple way, a 1-connection (Γ_1, Γ_2) on $\mathcal{F}(Y_1, Y_2)$ with interesting properties. We are going to demonstrate that a similar construction can be performed in the second order as well. Consider a second order connection $\Delta : Y_1 \to \widetilde{J}^2 Y_1$ with the coordinate expression (1) and a second order connection $\Lambda : Y_2 \to \widetilde{J}^2 Y_2$ of the coordinate form

$$z^a_i = A^a_i(x, z)\,,\; z^a_{0i} = B^a_i(x, z)\,,\; z^a_{ij} = C^a_{ij}(x, z)\,. \tag{43}$$

By Section 3, we have $\Delta^{-1}(x) : \widetilde{J}^2_x Y_1 \to \widetilde{J}^2_x(M, Y_{1x})$ and $\Lambda(x) : \widetilde{J}^2_x(M, Y_{2x}) \to \widetilde{J}^2_x Y_2$ for every $x \in M$. Every map $\varphi : Y_{1x} \to Y_{2x}$ induces

$$\varphi^2 : \widetilde{J}^2(M, Y_{1x}) \to \widetilde{J}^2(M, Y_{2x})$$

by the composition of non-holonomic jets

$$\varphi^2(X) = (j^2_y \varphi) \circ X\,, \qquad X \in \widetilde{J}^2(M, Y_{1x})_y\,.$$

Then we construct the composition

$$\Lambda(x) \circ \varphi^2 \circ \Delta^{-1}(x) : \widetilde{J}^2_x Y_1 \to \widetilde{J}^2_x Y_2\,. \tag{44}$$

We are going to deduce that there is a unique 2-connection (Δ, Λ) on $\mathcal{F}(Y_1, Y_2)$ of second order such that (44) is the associated map $\widetilde{(\Delta, \Lambda)}(\varphi)$. The simpliest proof consists in evaluating (44) and comparing with (42). Let Z^a_i, Z^a_{0i}, Z^a_{ij} be the jet coordinates on $\widetilde{J}^2_x(M, Y_{2x})$. By (27), the coordinate form of φ^2 is

$$\left\{ \begin{array}{l} Z^a_i = \dfrac{\partial \varphi^a}{\partial y^p} Y^p_i\,, \quad Z^a_{0i} = \dfrac{\partial \varphi^a}{\partial y^p} Y^p_{0i}\,, \\ Z^a_{ij} = \dfrac{\partial \varphi^a}{\partial y^p} Y^p_{ij} + \dfrac{\partial^2 \varphi^a}{\partial y^p \partial y^q} Y^p_i Y^q_{0j}\,. \end{array} \right. \tag{45}$$

Evaluating (44), we obtain a map of $\mathcal{F}J^2(Y_1, Y_2)$ into

$$\bigcup_{x\in M} C^\infty(\widetilde{J}^2_{1x}, \widetilde{J}^2_{2x})$$

with the following coordinate expression

$$z^a_i = z^a_p y^p_i + (A^a_i - z^a_p F^p_i)\,, \quad z^a_{0i} = z^a_p y^p_{0i} + (B^a_i - z^a_p G^p_i)\,,$$
$$z^a_{ij} = z^a_p y^p_{ij} + z^a_{pq} y^p_i y^q_{0j} + D_p(A^a_i - z^a_q F^q_i) y^p_{0j} + D_p(B^a_j - z^a_q G^q_j) y^p_i +$$
$$C^a_{ij} - z^a_p\left(\frac{\partial F^a_i}{\partial y^q} G^q_j + H^p_{ij}\right) - \frac{\partial A^a_i}{\partial z^b}\, z^b_p G^p_j - \frac{\partial B^a_j}{\partial z^b}\, z^b_p F^p_i + z^a_{pq} F^p_i G^q_j\,.$$

This is of the form (42).

A smooth section $B : \mathcal{F}(Y_1, Y_2) \to V\mathcal{F}(Y_1, Y_2) \otimes \bigotimes^2 T^*M$ is called a tensor field of type $V\mathcal{F}(Y_1, Y_2) \otimes \bigotimes^2 T^*M$. Such a tensor field will be said to be of order $s \geq 1$, if the condition $j^s_y\varphi = j^s_y\psi$, $\varphi, \psi \in C^\infty(Y_{1x}, Y_{2x})$, $y \in Y_{1x}$, implies

$$\widetilde{B(\varphi)}|V_yY \otimes \bigotimes^2 T^*_xM = \widetilde{B(\psi)}|VY \otimes \bigotimes^2 T^*_xM\,.$$

By [2], its associated map $\mathcal{B}$ is of the form

$$Z^a_{ij} = z^a_p Y^p_{ij} + B^p_{ij}\,, \tag{46}$$

where B^p_{ij} are locally arbitrary functions on $\mathcal{F}J^s(Y_1, Y_2)$.

From now on we assume that all 1-connections, 2-connections and tensor fields on $\mathcal{F}(Y_1, Y_2)$ are of finite order, which is the most interesting case. Let Γ be an r-th order 1-connection with the associated map

$$z^a_i = z^a_p y^p_i + \Phi^a_i\,, \tag{47}$$

where Φ^a_i are locally some functions on $\mathcal{F}J^r(Y_1, Y_2)$, and $\overline{\Gamma}$ be an k-th order connection with the associated map

$$z^a_i = z^a_p y^p_i + \Psi^a_i\,, \tag{48}$$

where Ψ^a_i are locally some functions on $\mathcal{F}J^k(Y_1, Y_2)$. In a standard way, see [2], one defines the first jet prolongation

$$J^1\Gamma : J^1\mathcal{F}(Y_1, Y_2) \to \widetilde{J}^2\mathcal{F}(Y_1, Y_2)$$

and deduces that $\Gamma * \overline{\Gamma} := J^1\Gamma \circ \overline{\Gamma}$ is a 2-connection of order $r+k$, whose associated map is given by Φ^a_i, $\Phi^a_{0i} = \Psi^a_i$ and

$$\frac{\partial \Phi^a_i}{\partial x^j} + \frac{\partial \Phi^a_i}{\partial z^b}\,\Psi^b_j + \frac{\partial \Phi^a_i}{\partial z^b_p}\,D_p\Psi^b_j + \dots + \frac{\partial \Phi^a_i}{\partial z^b_\alpha}\,D_\alpha\Psi^b_j\,, \tag{49}$$

where α is a multiindex satisfying $|\alpha| = r$.

If Δ is an r-th order 2-connection, then the underlying 1-connections Δ_1, Δ_2 are of order $r-1$ and $\Delta_1 * \Delta_2$ is another 2-connection of order $2r-2$. Since the underlying 1-connections of $\Delta_1 * \Delta_2$ are Δ_1 and Δ_2, the strong difference $\Delta \dot{-} \Delta_1 * \Delta_2 =: \Sigma$ defines a tensor field of type $V\mathcal{F}(Y_1, Y_2)$ of order $2r-2$. The coordinate form of the essential component of the associated map $\mathcal{S}$ of Σ is the difference Φ^a_{ij} – (49), where Φ^a_{ij} is the second order component of the associated map of Δ. Conversely, let Γ and $\overline{\Gamma}$ be finite order 1-connections and Σ a finite order tensor field of type $V\mathcal{F}(Y_1, Y_2) \otimes \bigotimes^2 T^*M$. Then $\Gamma * \overline{\Gamma} \dot{+} \Sigma$ is a finite order 2-connection, for the compatibility conditions (41) are fulfilled by construction. Thus, we have proved

Proposition 9. *The formula*

$$\Delta = \Gamma * \overline{\Gamma} \dot{+} \Sigma \tag{50}$$

establishes a bijection between the finite order 2-connections on $\mathcal{F}(Y_1, Y_2)$ and the triples of two finite order 1-connections on $\mathcal{F}(Y_1, Y_2)$ and a finite order tensor field of type $V\mathcal{F}(Y_1, Y_2) \otimes \bigotimes^2 T^*M$.

6. The absolute differentiation on $\mathcal{F}(Y_1, Y_2)$

We are going to study the second order absolute differentiation of a section s of $\mathcal{F}(Y_1, Y_2)$ with respect to a finite order 2-connection Δ on $\mathcal{F}(Y_1, Y_2)$. We start with the case of a k-th order 1-connection $\overline{\Gamma}$ with the associated map (48) and we summarize the results of [2] from our present viewpoint.

We define

$$J^1_{\text{fib}}\mathcal{F}(Y_1, Y_2) = \bigcup_{x \in M} J^1(M, C^\infty(Y_{1x}, Y_{2x}))\,. \tag{51}$$

This is a smooth space in the sense of Frölicher. By [3], an element $X \in J^1(M, C^\infty(Y_{1x}, Y_{2x}))$ is characterized by the associated map

$$\widetilde{X} : Y_{1x} \to J^1(M, Y_{2x})\,.$$

Hence we have

$$J^1_{\text{fib}}\mathcal{F}(Y_1, Y_2) = \mathcal{F}(Y_1, J^1_{\text{fib}}Y_2)\,. \tag{52}$$

Consider a section $s : M \to \mathcal{F}(Y_1, Y_2)$, whose associated base-preserving morphism $\widetilde{s} : Y_1 \to Y_2$ has the coordinate form

$$z^a = s^a(x, y)\,. \tag{53}$$

Write $S^a_{0i}(x, y) = \Psi^a_i(j^k s)$, where $j^k s : Y \to \mathcal{F}J^r(Y_1, Y_2)$ is the k-th jet prolongation of s constructed fiberwise. By [2], the coordinate form of $\nabla_{\overline{\Gamma}} s$ is (53) and

$$Z^a_{0i} = \frac{\partial s^a(x, y)}{\partial x^i} - S^a_{0i}(x, y)\,. \tag{54}$$

The connection Γ^1 on $J^1_{\text{fib}}Y$ of Section 3 was constructed by means of the exchange map. By [12], this idea works in the infinite dimension as well. Let Γ be an r-th order 1-connection on $\mathcal{F}(Y_1,Y_2)$ with the associated map (47). Let Γ^1 be the connection induced on $J^1_{\text{fib}}\mathcal{F}(Y_1,Y_2)$ by the procedure of Section 3. This construction is analogous to the case of the vertical prolongation $\mathcal{V}\Gamma : V\mathcal{F}(Y_1,Y_2) \to J^1V\mathcal{F}(Y_1,Y_2)$ from [3]. This implies that Γ^1 is an r-th order connection on $J^1_{\text{fib}}\mathcal{F}(Y_1,Y_2)$. Let $u^i_p,\dots,u^i_\alpha, Z^a_{0ip},\dots,Z^a_{0i\alpha}$, $|\alpha|=r$, be the induced coordinates on $\mathcal{F}J^r(Y_1,J^1_{\text{fib}}Y_2)$. By [3], the associated map of Γ^1 is given by (47), $u^i_j=0$ and

$$\frac{\partial\Psi^a_i}{\partial z^b}\,Z^b_{0j}+\frac{\partial\Phi^a_i}{\partial z^b_p}\,Z^b_{0jp}+\cdots+\frac{\partial\Phi^a_i}{\partial z^b_\alpha}Z^b_{0j\alpha}\,. \tag{55}$$

In particular, the coordinate form of the iterated absolute differential $\nabla_{\Gamma^1}(\nabla_{\overline{\Gamma}}s)$ of a section of $\mathcal{F}(Y_1,Y_2)$ is (53), (54),

$$Z^a_i=\frac{\partial s^a(x,y)}{\partial x^i}-\Phi^a_i(j^r s)\,, \tag{56}$$

$$\begin{aligned} Z^a_{ij}=\frac{\partial^2 s^a}{\partial x^i\partial x^j}-\frac{\partial S^a_i}{\partial x^j}-\frac{\partial\Phi^a_i}{\partial z^b}\,(j^r s)S^b_j-\\ -\frac{\partial\Phi^a_i}{\partial z^b_p}\,(j^r s)\partial_p S^b_j-\cdots-\frac{\partial\Phi^a_i}{\partial z^b_\alpha}\,(j^r s)\partial_\alpha S^b_j\,, \end{aligned} \tag{57}$$

where $\partial_p,\dots,\partial_\alpha$ denote the partial derivatives with respect to $y's$.

In the case of an arbitrary 2-connection Δ on $\mathcal{F}(Y_1,Y_2)$, we use the decomposition $\Delta=\Delta_1 * \Delta_2 \dotplus \Sigma$. Taking into account Propositions 4 and 6, we define

$$\nabla_\Delta s=\nabla_{\Delta^1_1}(\nabla_{\Delta_2}s)\dotplus\Sigma(s) \tag{58}$$

for every section s of $\mathcal{F}(Y_1,Y_2)$. This operation has several properties analogous to Section 3.

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