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ANTI-HOLONOMIC JETS AND THE LIE BRACKET

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Abstract. Second order anti-holonomic jets as anti-symmetric parts of second order semi-holonomic jets are introduced. The anti-holonomic nature of the Lie bracket is shown. A general result on universality of the Lie bracket is proved.

1. Introduction

The concepts of *non-holonomic* (or *iterated*) and *semi-holonomic* jets, first introduced by Ehresmann in [1], are commonly used in differential geometry. In this paper, we use the concept of semi-holonomic jet to construct *second order anti-holonomic jets* as the anti-symmetric part of second order semi-holonomic jets. Further we introduce three differential operators between some holonomic, semi-holonomic, and anti-holonomic jets, namely the *prolongation*, *torsion*, and *curvature* operators. Finally, using these operators, we show a close relation between the Lie bracket and anti-holonomic jets and prove some universal property of the Lie bracket.

The definition of anti-holonomic jets has many similarities with the definition of *difference tensor* from [3], and of *dissymétrie* from [9] (in fact, the manifold anti$J^2(X_1, X_2)$ of anti-holonomic jets with source in X_1 and target X_2 can be identified with the space $J^1(X_1, X_2) \times_{X_1 \times X_2} (TX_2 \otimes \bigwedge^2 T^* X_1)$, where the difference tensor lie in the second factor of the product). In our approach, we want to emphasize the jet character of this object, namely the fact that anti-holonomic jets can be *composed* in a similar way as ordinary jets. This property of anti-holonomic jets is important, for example, when investigating natural differential operators of vector distributions [6].

Major part of presented results appeared first in the author's work [7]. More general results, as well as some further examples and applications, are contained in [6]. The result on the Lie bracket appeared first in [8]. We note that some results

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of [8] are not correct; however, this have no influence on the assertions on the Lie bracket in [8] (see also [5] for similar results).

In this paper, we use standard notions of the jet theory and basic results of the theory of natural bundles and operators (namely, the relation between natural differential operators and some invariant mappings). The reader can find them in [10], [4], [2], [7].

2. Second order semi-holonomic jets

In this paragraph, we briefly recall the standard notion of second order semi-holonomic jets. Let X_1, X_2 be two manifolds. By a (second order) *semi-holonomic jet* with source in X_1 and target in X_2 we understand a 1-jet

$$A = J^1_x g \in J^1(X_1, J^1(X_1, X_2)) \tag{1}$$

such that g is a local section of the fibration $J^1(X_1, X_2) \to X_1$ and

$$g(x) = J^1_x(\beta \circ g) \tag{2}$$

(β is the target projection, $\beta : J^1(X_1, X_2) \to X_2$). The point $x \in X_1$ is called the *source* of A, the point $\beta(g(x)) \in X_2$ the *target* of A. The second order semi-holonomic jets with source in X_1 and target in X_2 form a closed submanifold of $J^1(X_1, J^1(X_1, X_2))$ which is denoted by semi $J^2(X_1, X_2)$ (the standard symbol for this manifold, introduced in [1], is $\overline{J}^2(X_1, X_2)$; we prefer, however, using a different one since it is more understandable and admits a generalization to anti–holonomic jets, see Par. 3).

There is the canonical inclusion $J^2(X_1, X_2) \to \mathrm{semi}J^2(X_1, X_2)$, given by

$$J^2_x f \longrightarrow J^1_x(J^1 f). \tag{3}$$

Thus, for $f : X_1 \to X_2$, we can write $J^2_x f \in \mathrm{semi}J^2(X_1, X_2)$.

Semi-holonomic jets $A_1 \in \mathrm{semi}\, J^2(X_1, X_2)$, and $A_2 \in \mathrm{semi}\, J^2(X_2, X_3)$, $A_1 = J^1_{x_1} g_1$, $A_2 = J^1_{x_2} g_2$, are called *composable*, if the target of A_1 is equal to x_2. The *composition* $A_2 \circ A_1 \in \mathrm{semi}\, J^2(X_1, X_3)$ of these jets is defined by $A_2 \circ A_1 = J^1_{x_1} g_3$, where $g_3(x) = g_2(\beta(g_1(x))) \circ g_1(x)$. A semi-holonomic jet $J^1_x g \in \mathrm{semi}\, J^2(X_1, X_2)$ is called *regular*, if the 1-jet $g(x)$ is regular (i.e., if it is of maximal rank). The subset of $\mathrm{semi}J^2(X_1, X_2)$, consisting of regular jets is dense, and open, and is denoted by $\mathrm{regsemi}J^2(X_1, X_2)$.

A semi-holonomic jet $A \in \mathrm{semi}J^2_{x_1,x_2}(X_1, X_2)$ is called *invertible* if there is a jet $A^{-1} \in \mathrm{semi}J^2_{x_2,x_1}(X_2, X_1)$ such that $A^{-1} \circ A = J^2_{x_1} \mathrm{id}_{X_1}$, and $A \circ A^{-1} = J^2_{x_2} \mathrm{id}_{X_2}$. The jet A is invertible if and only if it is regular and $\dim X_1 = \dim X_2$.

One can consider semi-holonomic analogy to all standard jet spaces, such as the space

$$\mathrm{semi}T^2_m X = \mathrm{semi}J^2_0(R^m, X) \tag{4}$$

of *semi-holonomic* $(2, m)$*-velocities of manifold* X, its subspace

$$\mathrm{regsemi}T^2_m X, \tag{5}$$

consisting of regular semi-holonomic jets, and the *second semi-holonomic differential group*

$$\mathrm{semi}L^2_n = \mathrm{regsemi}J^2_{0,0}(R^n, R^n). \tag{6}$$

For a mapping $f : X_1 \to X_2$, we define a mapping

$$\mathrm{semi}T^2_m f : \mathrm{semi}T^2_m X_1 \to \mathrm{semi}T^2_m X_2 \tag{7}$$

by

$$\mathrm{semi}T^2_m f(A) = J^2_{\beta(A)} f \circ A \tag{8}$$

and obtain a second order bundle functor $\mathrm{semi}T^2_m$ (over the category $\mathcal{M}f$ of smooth manifolds and smooth mappings). Analogously it can be defined the functor $\mathrm{regsemi}T^2_m$ (over the category $\mathcal{D}_n$ of n-dimensional manifolds and their embeddings). The type fiber of the functor $\mathrm{semi}T^2_m$ is the manifold $\mathrm{semi}T^2_{n,m} = \mathrm{semi}J^2_{0,0}(R^m, R^n)$ with the following left action of the group L^2_n:

$$(g, A) \to g \circ A. \tag{9}$$

For some product chart $(U \times V, \chi)$ on a manifold $X_1 \times X_2$, and for some semi-holonomic jet $A = J^1_{x_1} g \in \mathrm{semi}J^2_{x_1,x_2}(X_1, X_2)$, we set

$$\begin{aligned}
\chi^i(A) &= \chi^i(x_1), && (10)\\
\chi^s(A) &= \chi^s(x_2), && (11)\\
\chi^s_k(A) &= \chi^s_k(g(x_1)), && (12)\\
\chi^s_{kl}(A) &= D_l(\chi^s_k \circ g)(x_1), && (13)
\end{aligned}$$

where $i, k, l = 1, \ldots, n_1$, $s = n_1 + 1, \ldots, n_1 + n_2$, $n_1 = \dim X_1$, and $n_2 = \dim X_2$, and on the first three lines we have the standard induced coordinates on $J^1(X_1, X_2)$. The system $(\chi^i, \chi^s, \chi^s_k, \chi^s_{kl})$ form a coordinate system on $\mathrm{semi}J^2(X_1, X_2)$ over $U \times V$, called *induced* by $(U \times V, \chi)$.

Coordinate expression of the composition of semi-holonomic jets is the following. Let $(U \times V \times W, \chi)$ be a product chart on a manifold $X_1 \times X_2 \times X_3$, $A_1 \in \mathrm{semi}J^2_{x_1,x_2}(X_1, X_2)$, $A_2 \in \mathrm{semi}J^2_{x_2,x_3}(X_2, X_3)$. Then

$$\begin{aligned}
\chi^u_k(A_2 \circ A_1) &= \chi^u_s(A_2)\chi^s_k(A_1), && (14)\\
\chi^u_{k_1k_2}(A_2 \circ A_1) &= \chi^u_{s_1s_2}(A_2)\chi^{s_1}_{k_1}(A_1)\chi^{s_2}_{k_2}(A_1) + \chi^u_s(A_2)\chi^s_{k_1k_2}(A_1) && (15)
\end{aligned}$$

($k, k_1, k_2 = 1, \ldots, n_1$, $s, s_1, s_2 = n_1 + 1, \ldots, n_1 + n_2$, $u = n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3$, $n_1 = \dim X_1$, $n_2 = \dim X_2$, and $n_3 = \dim X_3$). Especially, for $g \in L^2_n$, and $A \in \mathrm{semi}T^2_{n,m}$ from (9) it follows

$$\begin{aligned}
\chi^s_k(g \cdot A) &= a^s_t(g)\chi^t_k(A), && (16)\\
\chi^s_{k_1k_2}(g \cdot A) &= a^s_{t_1t_2}(g)\chi^{t_1}_{k_1}(A)\chi^{t_2}_{k_2}(A) + a^s_t(g)\chi^t_{k_1k_2}(A). && (17)
\end{aligned}$$

($(a^s_t, a^s_{t_1t_2})$ are the first canonical coordinates of L^2_n).

In the induced coordinates, the inclusion (3) is given by the canonical inclusion.

3. Second order anti-holonomic jets

In this paragraph, we introduce the notion of second order anti-holonomic jet.

Let X_1, and X_2 be two manifolds. We introduce an equivalence $\sim$ on the manifold $\mathrm{semi}J^2(X_1, X_2)$ by means of coordinates as follows: For elements $A, \bar{A} \in \mathrm{semi}J^2(X_1, X_2)$ we set $A \sim \bar{A}$ if A and $\bar{A}$ have the same source x_1 and target x_2 and if for some product chart $(U \times V, \chi)$ at (x_1, x_2) the anti-symmetric parts of $\chi^s_{kl}(A)$ and $\chi^s_{kl}(\bar{A})$ are equal:

$$\chi^s_{kl}(A) - \chi^s_{lk}(A) = \chi^s_{kl}(\bar{A}) - \chi^s_{lk}(\bar{A}). \tag{18}$$

Evidently, this condition does not depend on the choice of the chart χ.

There evidently exists the quotient space $\mathrm{semi}J^2(X_1, X_2)/\sim$ with the canonical smooth structure (such that the quotient projection is a submersion). It is called the manifold of *second order anti-holonomic jets with source in* X_1 *and target in* X_2 and denoted by the symbol $\mathrm{anti}J^2(X_1, X_2)$. The canonical projection

$$\mathrm{semi}J^2(X_1, X_2) \to \mathrm{anti}J^2(X_1, X_2) \tag{19}$$

will be denoted by Tors and called the *torsion mapping.*

The notions of source, and target of anti-holonomic jet, composition of anti-holonomic jets, as well as of the regular and inverse anti-holonomic jet are defined by means of representatives.

The inclusion (3) leads to the mapping $J^2(X_1, X_2) \to \mathrm{anti}J^2(X_1, X_2)$. One can easily obtain that this mapping is factored through the canonical projection

$$J^2(X_1, X_2) \to J^1(X_1, X_2) \tag{20}$$

and get an inclusion

$$J^1(X_1, X_2) \to \mathrm{anti}J^2(X_1, X_2). \tag{21}$$

Thus, for $f : X_1 \to X_2$ we can write $J^1_x f \in \mathrm{anti}J^2(X_1, X_2)$.

We shall use anti-holonomic counter-parts of many standard jet spaces. We set, for example

$$\mathrm{anti}T^2_m X = \mathrm{anti}J^2_0(R^m, X). \tag{22}$$

This space is called the space of *anti-holonomic* $(2, m)$*-velocities of manifold* X. Further we have its subspace $\mathrm{reganti}T^2_m X$, consisting of regular anti-holonomic jets, and the *second anti-holonomic differential group*

$$\mathrm{anti}L^2_n = \mathrm{reganti}J^2_{0,0}(R^n, R^n). \tag{23}$$

Analogously as in the previous paragraph we can define bundle functors $\mathrm{anti}\,T^2_m$, and $\mathrm{reg\,anti}\,T^2_m$. The type fiber of the functor $\mathrm{anti}T^2_m$ is the manifold $\mathrm{anti}\,T^2_{n,m} = \mathrm{anti}\,J^2_{0,0}(R^m, R^n)$ with the following canonical left action of the group L^1_n:

$$(g, A) \to g \circ A. \tag{24}$$

For some product chart $(U \times V, \chi)$ on $X_1 \times X_2$, the *induced coordinates* on $\operatorname{anti} J^2(X_1, X_2)$ form the system $(\chi^i, \chi^s, \chi^s_k, \chi^s_{kl})$, where $\chi^s_{kl} = -\chi^s_{lk}$, and for $A \in \operatorname{semi} J^2(X_1, X_2)$ it holds

$$\chi^s_{kl}(\mathrm{Tors}(A)) = \chi^s_{kl}(A) - \chi^s_{lk}(A). \tag{25}$$

For $A_1 \in \mathrm{anti}J^2_{x_1,x_2}(X_1, X_2)$, $A_2 \in \mathrm{anti}J^2_{x_2,x_3}(X_2, X_3)$ we have

$$\chi^u_k(A_2 \circ A_1) = \chi^u_s(A_2)\chi^s_k(A_1), \tag{26}$$
$$\chi^u_{k_1k_2}(A_2 \circ A_1) = \chi^u_{s_1s_2}(A_2)\chi^{s_1}_{k_1}(A_1)\chi^{s_2}_{k_2}(A_1) + \chi^u_s(A_2)\chi^s_{k_1k_2}(A_1) \tag{27}$$

(ranges of the indices as above).

In the induced coordinates the inclusion (21) is the canonical inclusion

$$(\chi^i, \chi^s, \chi^s_k) \to (\chi^i, \chi^s, \chi^s_k, 0). \tag{28}$$

4. Lie bracket as an anti-holonomic jet

Consider an n-dimensional manifold X. The bundle $T^1_m X$ is in a canonical diffeomorphism with the bundle $\bigoplus^m TX$. Thus, sections of $T^1_m X$ can be considered as m-tuples of vector fields. We shall construct the Lie bracket as a part of a composition of some operators between semi-holonomic and anti-holonomic velocities.

In the first step we define an operator $\mathrm{Prol} : T^1_m \to \mathrm{semi}T^2_m$. For a section $\gamma : X \to T^1_m X$ we define a section $\mathrm{Prol}\gamma : X \to \mathrm{semi}T^2_m X$ by

$$(\mathrm{Prol}\gamma)(x_0) = J^1_0 h, \tag{29}$$

where $h(x) = \gamma(\bar{h}(x)) \circ J^1_x \mathrm{tr}_{-x}$ (tr_{-x} is the translation $y \to y - x$), and $J^1_0 \bar{h} = \gamma(x_0)$. We obtain a first order differential operator $\mathrm{Prol} : T^1_m \to \mathrm{semi}T^2_m$, called the *(semi-holonomic) prolongation of fields of velocities.*

The associated mapping of type fibers $p : T^1_n T^1_{n,m} \to \mathrm{semi}T^2_{n,m}$ (where

$$T^1_{n,m} = J^1_{0,0}(R^m, R^n) \tag{30}$$

is the type fiber of T^1_m and

$$\mathrm{semi}T^2_{n,m} = \mathrm{semi}J^2_{0,0}(R^m, R^n) \tag{31}$$

is the type fiber of $\mathrm{semi}T^2_m$) has in the canonical coordinates (i.e., in the coordinates, induced by the canonical coordinates on R^m, and R^n) the form

$$\chi^s_k(p(A)) = \chi^s_k(A), \tag{32}$$
$$\chi^s_{kl}(p(A)) = \chi^s_{k,t}(A)\chi^t_l(A) \tag{33}$$

(ranges of the indices as above).

In the second step we apply the torsion mapping Tors to the elements of $\mathrm{semi}T^2_m X$. Thus, we get a zero order differential operator $\mathrm{Tors} : \mathrm{semi}T^2_m \to \mathrm{anti}T^2_m$, called *torsion operator*. The associated mapping of type fibers $t : \mathrm{semi}\,T^2_{n,m} \to \mathrm{anti}T^2_{n,m}$ has, according to (25) in the canonical coordinates the form

$$\chi^s_k(t(A)) = \chi^s_k(A), \tag{34}$$
$$\chi^s_{kl}(t(A)) = \chi^s_{kl}(A) - \chi^s_{lk}(A). \tag{35}$$

Finally, we define a first order operator $\mathrm{Curv} : T^1_m \to \mathrm{anti}T^2_m$ as the composition $\mathrm{Curv} = \mathrm{Tors} \circ \mathrm{Prol}$. This operator is called the *curvature of fields of velocities*, or, using the identification of $T^1_m X$ and $\bigoplus^m TX$, the *curvature of m-tuples of vector fields*.

From the above expressions it follows, that the associated mapping of type fibers $c = t \circ p : T^1_n T^1_{n,m} \to \mathrm{anti}T^2_{n,m}$ has in the canonical coordinates the form

$$\chi^s_k(c(A)) = \chi^s_k(A), \tag{36}$$
$$\chi^s_{kl}(c(A)) = \chi^s_{k,t}(A)\chi^t_l(A) - \chi^s_{l,t}(A)\chi^t_k(A). \tag{37}$$

As we shall see, the similarity of the second expression with the coordinate expression of the Lie bracket is not accident.

We show, that sections of the bundle $\mathrm{anti}T^2_m X \to X$ can be considered as collections of vector fields over X.

Theorem 4.1. *Let $M = \frac{1}{2}m(m-1)$. The bundle functors $T^1_m \oplus T^1_M$ and $\mathrm{anti}T^2_m$ are isomorphic.*

Proof. Let us define a chart on the manifold $T^1_{n,M}$ in the following way. Consider an index $a \in \{1, \dots, M\}$ as a multi-index, consisting of two indices $k_1, k_2 \in \{1, \dots, m\}$, $k_1 < k_2$, sorted in some fixed order, and write the canonical coordinate functions (χ^s_a) on $T^1_{n,M}$ as follows:

$$(\chi^s_a) = (\chi^s_{k_1 k_2}). \tag{38}$$

Now the action of L^1_n on the type fiber $T^1_{n,m} \times T^1_{n,M}$ of the functor $T^1_m \oplus T^1_M$ has in the canonical coordinates the form

$$\chi^s_k(g \cdot A) = a^s_t(g)\chi^t_k(A), \tag{39}$$
$$\chi^s_{k_1 k_2}(g \cdot A) = a^s_t(g)\chi^t_{k_1 k_2}(A). \tag{40}$$

This is exactly the coordinate expression of the action (24). Thus we have proved that the type fibers $T^1_{n,m} \times T^1_{n,M}$ and $\mathrm{anti}T^2_{n,m}$ are isomorphic, which proves the theorem. □

From the above theorem and from (36,37) it follows that the curvature of an m-tuple of vector fields can be divided to two parts: the vector fields themselves (36) and all their Lie brackets (37). Note, however, that in this approach we lose the anti-holonomic nature of the curvature.

5. Universality of the Lie bracket

In this paragraph, we shall show universality (in a sense) of the operator Curv (and hence of the Lie bracket), using similar properties of the operators Prol, and Tors. In all this paragraph, we shall concentrate only to regular velocities (and consider our operators restricted to sections of appropriate bundles) and suppose $m \leq n$.

We shall often use actions of the kernel $K_n^{2,1}$ of the projection $L_n^2 \to L_n^1$ on some type fibers. As it is discussed in [2], or [7], these actions are related to the problem of universality of our operators. $\mathcal{D}_n$ will denote the category of n-dimensional manifolds and their embeddings, $\mathcal{FM}$ the category of fibered manifolds and their smooth, projectable mappings.

Theorem 5.1. *Any first order natural differential operator*

$$D : \mathrm{reg}T_m^1 \to F, \tag{41}$$

where $F : \mathcal{D}_n \to \mathcal{FM}$ is a first order bundle functor, is a composition of the semi-holonomic prolongation operator Prol, and some zero order operator

$$D_0 : \mathrm{regsemi}T_m^2 \to F. \tag{42}$$

The operator D_0 is unique.

Proof. In this proof, we shall use indices with the following ranges: $s, t = \{1, \dots, n\}, i, j, k, l = \{1, \dots, m\}, \sigma, \nu = \{n-m+1, \dots, n\}$.

It is sufficient to show that the type fiber representation $p : T_n^1 T_{n,m}^1 \to \mathrm{semi}T_{n,m}^2$ of the operator Prol is a surjective submersion whose fibers are subsets of the orbits of the group $K_n^{2,1}$.

Surjectivity of p follows easily from (32,33). In the second component (33), p is a surjective linear mapping, i.e., a submersion.

Now, let us prove the second statement. If $g \in K_n^{2,1}$, then we have

$$\chi_k^s(g \cdot A) = \chi_k^s(A), \tag{43}$$

$$\chi_{k,u}^s(g \cdot A) = a_{tu}^s(g)\chi_k^t(A) + \chi_{k,u}^s(A) \tag{44}$$

(these equations follow from the equations of prolonged actions of differential groups. See, for example [4], [2], or [7]).

Let $A, \bar{A} \in T_n^1 T_{n,m}^1$ be elements such that $p(A) = p(\bar{A})$. Then, according to (32,33),

$$\chi_k^s(\bar{A}) - \chi_k^s(A) = 0, \tag{45}$$

$$(\chi_{k,t}^s(\bar{A}) - \chi_{k,t}^s(A))\chi_l^t(A) = 0. \tag{46}$$

Without loss of generality, we shall suppose that the matrix $(\chi_k^l(A)) = (\chi_k^l(\bar{A}))$ is regular. Denote by (κ_l^k) its inverse. Then from (46) it easily follows that the expression

$$H_{lj}^s = ((\chi_{k,j}^s(\bar{A}) - \chi_{k,j}^s(A)) - (\chi_{l,\sigma}^s(\bar{A}) - \chi_{l,\sigma}^s(A))\kappa_j^l \chi_k^\sigma(A))\kappa_l^k \tag{47}$$

satisfies $H^s_{lj} = H^s_{jl}$. Thus, there exists an element $g \in K^{2,1}_n$ such that

$$\begin{aligned} a^s_{\sigma\nu}(g) &= 0, && (48)\\ a^s_{l\nu}(g) &= a^s_{\nu l}(g) = (\chi^s_{k,\nu}(\bar{A}) - \chi^s_{k,\nu}(A))\kappa^k_l, && (49)\\ a^s_{lj}(g) &= H^s_{lj}. && (50) \end{aligned}$$

Direct computation gives $\bar{A} = g \cdot A$. □

Theorem 5.2. *Any zero order natural differential operator*

$$D : \mathrm{regsemi}T^2_m \to F, \tag{51}$$

where $F : \mathcal{D}_n \to \mathcal{FM}$ is a first order bundle functor, is a composition of the torsion operator Tors, *and some zero order operator*

$$D_0 : \mathrm{reganti}T^2_m \to F. \tag{52}$$

The operator D_0 is unique.

Proof. It is sufficient to show that the equivalence $\approx$, induced by the left action of the group $K^{2,1}_n$ on $\mathrm{regsemi}T^2_{n,m}$, and the equivalence $\sim$ from Par. 3 coincide. Since $\mathrm{reganti}T^2_m$ is a first order bundle functor, then we have $\approx \subset \sim$.

Let $A, \bar{A} \in \mathrm{regsemi}T^2_{n,m}$ be elements such that $t(\bar{A}) = t(A)$. We shall find $g \in K^{2,1}_n$, satisfying $\bar{A} = g \cdot A$, which, according to (16,17), means

$$\begin{aligned} \chi^s_k(\bar{A}) &= \chi^s_k(A), && (53)\\ \chi^s_{k_1k_2}(\bar{A}) &= a^s_{t_1t_2}(g)\chi^{t_1}_{k_1}(A)\chi^{t_2}_{k_2}(A) + \chi^s_{k_1k_2}(A). && (54) \end{aligned}$$

Denote by (κ^k_s) a left inverse to the matrix $(\chi^s_k(\bar{A})) = (\chi^s_k(A))$.

From (34,35) we have

$$\begin{aligned} \chi^s_k(\bar{A}) &= \chi^s_k(A), && (55)\\ \chi^s_{k_1k_2}(\bar{A}) - \chi^s_{k_2k_1}(\bar{A}) &= \chi^s_{k_1k_2}(A) - \chi^s_{k_2k_1}(A). && (56) \end{aligned}$$

From (56) it follows that we can take $g \in K^{2,1}_n$ such that

$$a^s_{t_1t_2}(g) = (\chi^s_{k_1k_2}(\bar{A}) - \chi^s_{k_1k_2}(A))\kappa^{k_1}_{t_1}\kappa^{k_2}_{t_2}. \tag{57}$$

Now we have $\bar{A} = g \cdot A$. □

Theorem 5.3. *Any first order natural differential operator*

$$D : \mathrm{reg}T^1_m \to F, \tag{58}$$

where $F : \mathcal{D}_n \to \mathcal{FM}$ is a first order bundle functor, is a composition of the curvature operator Curv, *and some zero order operator*

$$D_0 : \mathrm{reganti}T^2_m \to F. \tag{59}$$

The operator D_0 is unique.

Proof. Follows from Theorem 5.1 and Theorem 5.2. □

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