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AUTOMORPHISMS OF SPATIAL CURVES

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ABSTRACT. Automorphisms of curves $y = y(x)$, $z = z(x)$ in $\mathbf{R}^3$ are investigated; i.e. invertible transformations, where the coordinates of the transformed curve $\bar{y} = \bar{y}(\bar{x})$, $\bar{z} = \bar{z}(\bar{x})$ depend on the derivatives of the original one up to some finite order m. While in the two-dimensional space the problem is completely resolved (the only possible transformations are the well-known contact transformations), the three-dimensional case proves to be much more complicated. Therefore, results (in the form of some systems of partial differential equations for the functions, determining the automorphisms) only for the special case $\bar{x} = x$ and order $m \le 2$ are obtained. Finally, the problem of infinitesimal transformations is briefly mentioned.

1. The problem of automorphisms

1.1. General formulation of the problem

Our aim is to investigate the group of automorphisms $\mathbf{h}$ of the family of smooth curves $y^i = y^i(x)$ of the underlying space $\mathbf{R}^{n+1}$ with coordinates $x, y^1, ..., y^n$ in a very broad sense which is as follows: The curve appearing after the automorphism is given by certain formulae $\bar{y}^i = \bar{y}^i(\bar{x})$ where

$$
\begin{aligned}
\bar{x}(x) &= h^0(x, y^1(x), ..., y^n(x), \tfrac{dy^1}{dx}(x), ..., \tfrac{dy^n}{dx}(x), ..., \tfrac{d^m y^1}{dx^m}(x), ..., \tfrac{d^m y^n}{dx^m}(x)), \\
\bar{y}^i(x) &= h^i(x, y^1(x), ..., y^n(x), \tfrac{dy^1}{dx}(x), ..., \tfrac{dy^n}{dx}(x), ..., \tfrac{d^m y^1}{dx^m}(x), ..., \tfrac{d^m y^n}{dx^m}(x))
\end{aligned}
\tag{1}
$$

and we suppose the existence of the inversion $x = x(\bar{x})$ of the first line of (1) to ensure the change of independent variable. If the functions h^i are arbitrarily chosen (in the sense of the note above), then then the formulae (1) make a good sense and determine a transformed curve; however, we are interested in the case

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of the automorphisms when the inversion

$$(2)\quad \begin{aligned} x(\bar x) &= g^0(\bar x, \bar y^1(\bar x), ..., \bar y^n(\bar x), \tfrac{d\bar y^1}{d\bar x}(\bar x), ..., \tfrac{d\bar y^n}{d\bar x}(\bar x), ..., \tfrac{d^m\bar y^1}{d\bar x^m}(\bar x), ..., \tfrac{d^m\bar y^n}{d\bar x^m}(\bar x)) , \\ y^i(\bar x) &= g^i(\bar x, \bar y^1(\bar x), ..., \bar y^n(\bar x), \tfrac{d\bar y^1}{d\bar x}(\bar x), ..., \tfrac{d\bar y^n}{d\bar x}(\bar x), ..., \tfrac{d^m\bar y^1}{d\bar x^m}(\bar x), ..., \tfrac{d^m\bar y^n}{d\bar x^m}(\bar x) \end{aligned}$$

of (1) in the family of all curves exists for appropriate functions g^i. Undoubtedly, this is a classical problem of fundamental importance. As yet, only partial results are known. For instance, if m=0 and we suppose $h^i = h^i(x, y^1, \dots, y^n)$, then the inversion exists in the case of the invertible point transformation. If n=1, it may be proved that the only case of automorphism is possible when m=1, which is the case of the familiar contact transformations (cf.[1]). If however $n > 1$, then there exist automorphisms with m arbitrarily large, for instance

$$\bar x(x) = x, \quad \bar y^1(x) = y^1(x), \quad \bar y^2(x) = y^2(x) + \frac{d^m y^1}{dx}(x)$$

(here n=2) with the obvious inversion

$$x(\bar x) = \bar x, \quad y^1(\bar x) = \bar y^1(\bar x), \quad y^2(\bar x) = \bar y^2(\bar x) - \frac{d^m \bar y^1}{d\bar x^m}(\bar x)$$

and many other analogous examples can be constructed by compositions.

An overwiev of such automorphisms is a highly non-trivial task. Therefore we shall restrict to the case of x-preserving automorphisms, i.e., the first line of (1) is chosen as $\bar x = x$ (hence $h^0 = x$) and even only some particular subcases with n=2 will be investigated. We shall also briefly mention the problem of infinitesimal automorphisms of curves later on. It is however desirable to delay the formulation to more convenient place since quite other principles are appearing.

Since the order of derivatives in (1) is not apriori limited, infinite-dimensional spaces involving the derivatives of all orders are employed. Our reasonings will be of local nature based on the category of C^∞ smooth real valued functions. Before proceeding to explicit calculations, we summarize some necessary concepts and tools.

1.2. Fundamental concepts

Our reasonings will be carried out in the space $\mathbf{R}^\infty$ of all real infinite sequences $P = (p^1, p^2, \dots)$. Denoting by $x^1, x^2, \dots,$ the coordinates in $\mathbf{R}^\infty$ defined by $x^i(P) = p^i$, we shall deal with the structural family $\mathcal{F}$ of all real valued smooth functions of the kind $f = f(x^1, \dots, x^m)$, $m = m(f)$, depending on a finite number of arguments. In order to keep brevity, the definition domains will not be explicitly mentioned. Then $\mathbf{R}^\infty$ may be equipped with the $\mathcal{F}$-module Φ of all *differential forms* $\varphi = \sum f^i dg^i$ (a finite sum, $f^i, g^i \in \mathcal{F}$) and the $\mathcal{F}$-module $\mathcal{T}$ of all *vector fields* $Z = \sum z^i \frac{\partial}{\partial x^i}$ (an infinite sum, $z^i \in \mathcal{F}$). If $f = f(x^1, \dots, x^m)$, then $df = \sum_{i=1}^m \frac{\partial f}{\partial x^i} dx^i$ denotes the common differential. The differentials of coordinates $dx^i (i = 1, 2, \dots)$ constitute a *basis* of Φ and the vector fields $\partial/\partial x^i$ constitute a basis of $\mathcal{T}$ in the weak sense (since infinite developments are allowed).

We shall deal with *exterior differentials* $d\varphi = \sum df^i \wedge dg^i$ with the well-known exterior multiplication, and *Lie derivatives* $\mathcal{L}_Z$ in the direction of vector fields. Remind the common duality relationships between differential forms and vector fields

$$Zf = df(Z) = \sum z^i \frac{\partial f}{\partial x^i}, \quad Z\rfloor\varphi = \varphi(Z) = \sum f^i dg^i(Z) = \sum f^i Zg^i \in \mathcal{F}$$

and the familiar rules for the Lie derivative

$$\mathcal{L}_Z\varphi = Z\rfloor d\varphi + d(Z\rfloor\varphi) = \sum Zf^i dg^i + f^i Zg^i\,,$$

$$\mathcal{L}_Z f = Zf\,, \quad \mathcal{L}_Z Y = [Z,Y] \quad (Y \in \mathcal{T})$$

where

$$Z\rfloor d\varphi = \sum Zf^i dg^i - Zg^i df^i$$

and $[Z,Y] = ZY - YZ$ is the *Lie bracket*.

We shall be interested in various *admissible mappings* $\mathbf{h} : U \to V$ where $U, V \subset \mathbf{R}^\infty$ are open subsets (which need not be explicitly specified); they are defined by the property $\mathbf{h}^*\mathcal{F} \subset \mathcal{F}$ where $\mathbf{h}^*f = f \circ \mathbf{h}$ is the *pull-back* of a function $f \in \mathcal{F}$. The pull-back of a differential form $\varphi = \sum f^i dg^i$ is given by $\mathbf{h}^*\varphi = \sum \mathbf{h}^*f^i\, d\mathbf{h}^*g^i$ (in particular $\mathbf{h}^*df = d\mathbf{h}^*f$). The vector fields cause some troubles but if the inversion $\mathbf{h}^{-1}$ exists then Z is transformed into the vector field $\mathbf{h}_*Z$ defined by

$$(\mathbf{h}_*Z)f = (\mathbf{h}^{-1})^* Z(\mathbf{h}^*f)\,.$$

Remind the familiar relationship

$$Z\rfloor(\mathbf{h}^*\omega) = (\mathbf{h}_*Z)\rfloor\omega$$

where $Z \in \mathcal{T}$, $\omega \in \Phi$. In terms of coordinates, denoting by $h^i = \mathbf{h}^*x^i \in \mathcal{F}$, we have

$$\bar{P} = (\bar{p}^1, \bar{p}^2, \dots) = \mathbf{h}P = (h^1(p), h^2(p), \dots).$$

Quite explicitly, if $h^i = h^i(x^1, \dots, x^{m_i})$, then $\bar{p}^i = h^i(p^1, \dots, p^{m_i})$. The invertibility of $\mathbf{h}$ means that there are some inverse substitutions $p^i = g^i(\bar{p}^1, \dots, \bar{p}^{m_{j(i)}})$ where $g^i = (\mathbf{h}^{-1})^*x^i$. If the functions h^i are given in advance, it is not quite easy to decide whether such functions g^i exist; even to find some non-trivial examples of invertible mappings of $\mathbf{R}^\infty$ is not a trivial task.

1.3. Automorphisms of curves

In this section, we shall express the problem of automorphisms of family of curves in $\mathbf{R}^{n+1}$ in explicit and geometrical terms. For this aim, alternatively denote by $x, y^i_s \quad (i = 1, \dots, n; s = 0, 1, \dots)$ the coordinates in $\mathbf{R}^\infty$, introduce the submodule $\Omega \subset \Phi$ of all forms $\omega = \sum a^i_s\omega^i_s$ (finite sum, $a^i_s \in \mathcal{F}$) where $\omega^i_s = dy^i_s - y^i_{s+1}dx$ are the familiar *contact forms*, and the submodule $\Omega^\perp \subset \mathcal{T}$ of all vector fields satisfying $\omega(Z)=0$ for all $\omega\in\Omega$. One can see that $\Omega^\perp$ consists of all multiples of the vector field

$$X=\frac{\partial}{\partial x}+\sum^{\infty} y^i_{s+1}\frac{\partial}{\partial y^i_s}\,,$$

which is the familiar *formal* (or *total*) *derivative operator*. The coordinates x, y^i_s and the contact forms are regarded for a mere technical tool; in reality only the submodule $\Omega\subset\Phi$ (equivalently, the submodule $\Omega^\perp\subset\mathcal{T}$) is an intrisical object. We shall deal with the family Δ of all curves $x=x(t)$, $y^i_s=y^i_s(t)$ (the domain of t will not be specified) satisfying the Pfaffian system $\omega=0\quad(\omega\in\Omega)$, that is, satisfying the recurrences

$$y^i_{s+1}(t)=\frac{dy^i_s(t)}{dt}\Big/\frac{dx(t)}{dt}\,.$$

We shall deal only with curves which can be parametrized by means of x; then the recurrences simplify as $y^i_{s+1}(x)=\frac{dy^i_s}{dx}(x)$ so that

$$y^i_s(x)=\frac{d^s y^i_0}{dx^s}(x)\,.$$

The family Δ is defined by means of Ω; on the other hand, Δ determines the submodule Ω:

Lemma 1. *If a Pfaffian equation $\varphi=0$ is satisfied for all curves from Δ, where $\varphi\in\Phi$ is fixed, then $\varphi\in\Omega$.*

We are interested in automorphisms of the family of all curves in $\mathbf{R}^{n+1}$, equipped with coordinates $x, y^1_0,\dots,y^n_0$, given by certain formulae (1), (2). This means, we search for invertible mappings $\mathbf{h}$: $U\to V$ $(U,V\subset\mathbf{R}^\infty)$ transforming the family Δ into itself.

The requirement of invertibility will be expressed by the condition that $\mathbf{h}^*$ is an invertible mapping of Φ. In terms of coordinates, let $\mathbf{h}$ be given by certain formulae

$$\mathbf{h}^*x=\bar{x}\in\mathcal{F},\quad \mathbf{h}^*y^i_s=\bar{y}^i_s\in\mathcal{F}\,.$$

The invertibility of $\mathbf{h}^*:\ \Phi\to\Phi$ is ensured if a certain basis of Φ is again transformed into a basis of Φ ; it follows that the forms

$$\mathbf{h}^*dx=d\mathbf{h}^*x=d\bar{x}\,,\quad \mathbf{h}^*dy^i_s=d\mathbf{h}^*y^i_s=d\bar{y}^i_s$$

should constitute a basis of Φ.

The invariance of Δ will be explicitly expressed in following lemma; its proof is quite easy.

Lemma 2. *Let an invertible mapping* $\mathbf{h} : U \to V$ $(U, V \subset \mathbf{R}^\infty)$ *be given. Then the following conditions are equivalent:*

(i) $\mathbf{h}$ *transforms* Δ *into itself;*
(ii) $\mathbf{h}_* X = \lambda X$ *where* $\lambda \in \mathcal{F}$, $\lambda \neq 0$;
(iii) $\mathbf{h}^* \Omega \subset \Omega$;
(iv) $\bar{y}^i_{s+1} = X\bar{y}^i_s / X\bar{x}$, $\quad i = 1, 2, \ldots, n;\quad s = 0, 1, \ldots$

1.4. The aim of present paper

We shall deal only with automorphisms $\mathbf{h}$ of the family of all curves in $\mathbf{R}^{n+1}$ which preserve the coordinate x, i.e., $\bar{x} = \mathbf{h}^* x = x$ and $X\bar{x} = Xx = 1$. Then the recurrences (iv) of lemma 2 simplify into

$$\bar{y}^i_{s+1} = X\bar{y}^i_s = X^s \bar{y}^i_0$$

so that the functions $\bar{y}^i_0, \ldots, \bar{y}^n_0 \in \mathcal{F}$ can be chosen, the remaining $\bar{y}^i_s$ $(s \geq 1)$ are uniquely determined, and our task is only to ensure that the differentials

$$d\bar{x} = dx,\ d\bar{y}^i_s = dX^s \bar{y}^i_0 = \mathcal{L}^s_X d\bar{y}^i_0$$

constitute a basis of Φ.

Recall that the main aim of the present paper will be a modest one: to examine some very particular examples especially for the case $n = 2$. We shall abbreviate our notation by $y^1_s = u_s$, $y^2_s = v_s$ so that the fundamental recurrences for the functions $\bar{u}_s = \mathbf{h}^* u_s$, $\bar{v}_s = \mathbf{h}^* v_s$ a little simplify as

$$\bar{u}_{s+1} = X\bar{u}_s, \quad \bar{v}_{s+1} = X\bar{v}_s$$

where

$$X = \frac{\partial}{\partial x} + \sum^\infty u_{s+1} \frac{\partial}{\partial u_s} + \sum^\infty v_{s+1} \frac{\partial}{\partial v_s}$$

is the total derivative operator after the change of notation.

Our task is to find the initial functions u_0, v_0 in such a manner that

$$dx, d\bar{u}_0, d\bar{v}_0, d\bar{u}_1, d\bar{v}_1, \ldots$$

may be used for a basis of Φ. We shall examine two methods of solution. The first one directly investigates the sought functions $\bar{u}_0, \bar{v}_0$, which leads to quite explicit results. In the second one, firstly we shall try to determine only the submodule of Φ generated by $dx, d\bar{u}_0, d\bar{v}_0$; then from the form of this submodule we obtain certain conditions for $\bar{u}_0, \bar{v}_0$.

2. The direct method

2.1. The point transformation

Now let us proceed to explicit calculations. Assuming $\mathbf{h}^*x = x$, we choose functions $\mathbf{h}^*u_0 = \bar{u}_0$, $\mathbf{h}^*v_0 = \bar{v}_0$ in accordance with the notation above; then

$$\mathbf{h}^*u_s = \bar{u}_s = X^s\bar{u}_0, \quad \mathbf{h}^*v_s = \bar{v}_s = X^s\bar{v}_0\,,$$

and we ask if

$$dx = d\bar{x},\ d\bar{u}_s,\ d\bar{v}_s$$

may serve for a basis of Φ. For convenience of notation, we shall occasionaly abbreviate

$$\bar{u}_0 = f,\ \bar{v}_0 = g \quad \text{and} \quad \partial F/\partial u_s = F_s,\ \partial F/\partial v_s = F^s$$

for various composed functions F (i.e., excepting $F = u$, $F = v$ where the lower indices distinquish the coordinates).

Firstly let us deal with the zeroth order case $f = f(x, u_0, v_0)$, $g = g(x, u_0, v_0)$ which is quite easy. Then

$$\begin{aligned}
&d\bar{u}_0 \simeq f_0 du_0 + f^0 dv_0 \quad (\mathrm{mod}\ \ dx)\,,\\
&d\bar{v}_0 \simeq g_0 du_0 + g^0 dv_0 \quad (\mathrm{mod}\ \ dx)\,,\\
&\quad \dots\\
&d\bar{u}_s = dX^s\bar{u}_0 = \mathcal{L}_X^s d\bar{u}_0 \simeq f_0 du_s + f^0 dv_s \quad (\mathrm{mod}\ \ dx, du_0, dv_0, \dots, du_{s-1}, dv_{s-1})\,,\\
&d\bar{v}_s = dX^s\bar{v}_0 = \mathcal{L}_X^s d\bar{v}_0 \simeq g_0 du_s + g^0 dv_s \quad (\mathrm{mod}\ \ dx, du_0, dv_0, \dots, du_{s-1}, dv_{s-1})\,,\\
&\quad \dots
\end{aligned}$$

and $dx, d\bar{u}_s, d\bar{v}_s$ make up a basis of Φ if and only if

$$\det\begin{pmatrix} f_0 & f^0\\ g_0 & g^0\end{pmatrix} \neq 0\,.$$

This is the classical case of the prolonged point transformation already mentioned above.

2.2. The first order case

Now let us assume that

$$\mathbf{h}^*u_0 = \bar{u}_0 = f(x, u_0, v_0, u_1, v_1), \quad \mathbf{h}^*v_0 = \bar{v}_0 = g(x, u_0, v_0, u_1, v_1).$$

Excluding the case $f_1 = f^1 = g_1 = g^1 = 0$, we may assume $f_1 \neq 0$. Analogously as above, we have

$$\begin{aligned}
d\bar{u}_s &\simeq f_1 du_{s+1} + f^1 dv_{s+1} \quad (\mathrm{mod}\ \ dx, du_0, dv_0, \dots, du_s, dv_s)\,,\\
d\bar{v}_s &\simeq g_1 du_{s+1} + g^1 dv_{s+1} \quad (\mathrm{mod}\ \ dx, du_0, dv_0, \dots, du_s, dv_s)\,.
\end{aligned}$$

If

$$\det\begin{pmatrix} f_1 & f^1\\ g_1 & g^1\end{pmatrix} \neq 0\,,$$

then du_0, dv_0 could not be expressed by means of the primed differentials, hence necessarily

$$\det \begin{pmatrix} f_1 & f^1 \\ g_1 & g^1 \end{pmatrix} = 0\,.$$

This condition means

$$g = G(x, u_0, v_0, f) \tag{3}$$

and then

$$d\bar{v}_0 \simeq G_0 du_0 + G^0 dv_0 \pmod{dx, d\bar{u}_0}\,,$$

$$d\bar{v}_1 = \mathcal{L}_X d\bar{v}_0 \simeq XG_0 du_0 + XG^0 dv_0 + G_0 du_1 + G^0 dv_1 \pmod{dx, d\bar{u}_0, d\bar{u}_1}\,,$$

and we can write

$$\begin{aligned}
&d\bar{u}_0 \simeq f_0 du_0 \ + f^0 dv_0 + f_1 du_1 + f^1 dv_1 \pmod{dx}\,,\\
&d\bar{v}_0 \simeq G_0 du_0 + G^0 dv_0 \pmod{dx, d\bar{u}_0}\,,\\
&d\bar{u}_1 \simeq f_1 du_2 + f^1 dv_2 \pmod{dx, du_0, dv_0, du_1, dv_1}\,,\\
&d\bar{v}_1 \simeq XG_0 du_0 + XG^0 dv_0 + G_0 du_1 + G^0 dv_1 \pmod{dx, d\bar{u}_0, d\bar{u}_1}\,,\\
&\dots\\
&d\bar{u}_s \simeq f_1 du_{s+1} + f^1 dv_{s+1} \pmod{dx, du_0, dv_0, \dots, du_s, dv_s}\,,\\
&d\bar{v}_s \simeq G_0 du_s + G^0 dv_s \pmod{dx, du_0, dv_0, \dots, du_{s-1}, dv_{s-1}, d\bar{u}_0, \dots, d\bar{u}_s},\\
&\dots
\end{aligned}$$

Analogously as above, necessarily

$$\det \begin{pmatrix} f_1 & f^1 \\ G_0 & G^0 \end{pmatrix} = 0$$

must hold. If $G_0 = 0$ then $G^0 = 0$ and $\bar{v}_0 = G(x, f)$, $d\bar{v}_0 = G_x dx + G_f d\bar{u}_0$ and the forms $dx, d\bar{u}_s, d\bar{v}_s$ could not constitute a basis of Φ ; so we may conclude $G_0 \neq 0$ and the last condition can be rewritten in the form

$$\frac{G^0}{G_0} = \frac{f^1}{f_1}\,. \tag{4}$$

Using (4), we obtain

$$\begin{aligned}
&d\bar{u}_0 \simeq f_0 du_0 + f^0 dv_0 + f_1 du_1 + f^1 dv_1 \pmod{dx},\\
&d\bar{v}_0 \simeq G_0 du_0 + G^0 dv_0 \pmod{dx, d\bar{u}_0},\\
&d\bar{u}_1 \simeq f_1 du_2 + f^1 dv_2 \pmod{dx, du_0, dv_0, du_1, dv_1},\\
&f_1 d\bar{v}_1 \simeq (f_1 X G_0 - G_0 f_0) du_0 + (f_1 X G^0 - G_0 f^0) dv_0 \pmod{dx, d\bar{u}_0, d\bar{u}_1},\\
&\dots\\
&d\bar{u}_s \simeq f_1 du_{s+1} + f^1 dv_{s+1} \pmod{dx, du_0, dv_0, \dots, du_s, dv_s},\\
&f_1 d\bar{v}_s \simeq (f_1 X G_0 - G_0 f_0) du_{s-1} + (f_1 X G^0 - G_0 f^0) dv_{s-1}\\
&\qquad\qquad\qquad \pmod{dx, du_0, dv_0, \dots, du_{s-1}, dv_{s-1}, d\bar{u}_0, \dots, d\bar{u}_s},\\
&\dots
\end{aligned}$$

If

$$\det \begin{pmatrix} G_0 & G^0 \\ f_1 X G_0 - G_0 f_0 & f_1 X G^0 - G_0 f^0 \end{pmatrix} \neq 0\,, \tag{5}$$

then the forms du_0, dv_0 can be calculated in terms of $dx, d\bar{u}_0, d\bar{v}_0, d\bar{u}_1, d\bar{v}_1$ and after applying $\mathcal{L}_X^s$, every

$$du_s = dX^s u_0 = \mathcal{L}_X^s du_0\,, \quad dv_s = dX^s v_0 = \mathcal{L}_X^s dv_0$$

can be calculated in terms of $dx, d\bar{u}_0, d\bar{v}_0, \dots, d\bar{u}_{s+1}, d\bar{v}_{s+1}$. Since (4) is valid,

$$\det \begin{pmatrix} f_0 & f^0 \\ f_1 X G_0 - G_0 f_0 & f_1 X G^0 - G_0 f^0 \end{pmatrix} \neq 0$$

is ensured and the differentials $dx, d\bar{u}_0, d\bar{v}_0, d\bar{u}_1, d\bar{v}_1, \dots$ are linearly independent.

So we may conclude:

Theorem 1. *Let $f = f(x, u_0, v_0, u_1, v_1)$, $\quad g = g(x, u_0, v_0, u_1, v_1)$, $\quad f_1 \neq 0$. These functions determine an x-preserving automorphism of curves in $\mathbf{R}^3$ by*

$$\mathbf{h}^* x = x, \quad \mathbf{h}^* u_s = X^s f, \quad \mathbf{h}^* v_s = X^s g \quad (s = 0, 1, \dots)$$

if and only if $g = G(x, u_0, v_0, f)$ where

$$G_0 \neq 0, \quad \frac{f^1}{f_1} = \frac{G^0}{G_0}, \quad \det \begin{pmatrix} G_0 & G^0 \\ f_1 X G_0 - G_0 f_0 & f_1 X G^0 - G_0 f^0 \end{pmatrix} \neq 0.$$

Except for a more explicit example, we shall not continue this elementary method at this place.

2.3. Example

Let us examine the linear choice

$$\begin{aligned} &\bar{u}_0 = f(x, u_0, v_0, u_1, v_1) = A + ax + bu_0 + cv_0 + du_1 + ev_1 \,, \\ &\bar{v}_0 = g(x, u_0, v_0, u_1, v_1) = P + px + qu_0 + rv_0 + su_1 + tv_1 \,, \\ &A, a, b, c, d, P, p, q, r, s, t \in \mathbf{R}, \quad d \neq 0 \end{aligned}$$

Then due to (3) and (4) we have

$$\bar{v}_0 = G(x, u_0, v_0, f) = H + hx + ku_0 + \frac{e}{d} kv_0 + l\bar{u}_0, \quad h, k, l \in \mathbf{R}, \quad k \neq 0 \,.$$

The condition (5)

$$\det \begin{pmatrix} k & \frac{e}{d}k \\ -kb & -kc \end{pmatrix} \neq 0$$

gives

$$cd - eb \neq 0 \,.$$

In this case it is quite easy to compute explicitly the inverse transformation; by direct calculation we obtain

$$\begin{aligned} v_0 &= \tilde{f}(\bar{x}, \bar{u}_0, \bar{v}_0, \bar{u}_1, \bar{v}_1) = \\ &= \frac{d}{k(cd - eb)} (dh - kA + bH + (-ka + bh)\bar{x} + (k + bl)\bar{u}_0 - b\bar{v}_0 + dl\bar{u}_1 - d\bar{v}_1) \,, \\ u_0 &= \tilde{G}(\bar{x}, \bar{u}_0, \bar{v}_0, \tilde{f}) = \\ &= ebH + ecdH + (ebh - cdh)\bar{x} + (ebl - cdl)\bar{u}_0 + (cd - eb)\bar{v}_0 - \frac{e}{d} v_0 \,, \\ u_s &= X^s u_0, \ v_s = X^s v_0 \,. \end{aligned}$$

Notice that after the transformation the variables u_s, v_s changed their roles and

$$\begin{aligned} &\tfrac{\partial \tilde{f}}{\partial \bar{v}_1} = -d \neq 0, \quad \tfrac{\partial \tilde{G}}{\partial \bar{v}_0} = cd - eb \neq 0 \,, \\ &\det \begin{pmatrix} \frac{\partial \tilde{G}}{\partial \bar{v}_0} & \frac{\partial \tilde{G}}{\partial \bar{u}_0} \\ X \frac{\partial \tilde{G}}{\partial \bar{v}_0} - \frac{\partial \tilde{G}}{\partial \bar{v}_0} \frac{\partial \tilde{f}}{\partial \bar{v}_0} & X \frac{\partial \tilde{G}}{\partial \bar{u}_0} - \frac{\partial \tilde{G}}{\partial \bar{v}_0} \frac{\partial \tilde{f}}{\partial \bar{u}_0} \end{pmatrix} = \\ &= \tfrac{d}{k(cd - eb)} \det \begin{pmatrix} cd - eb & ebl - cdl \\ (cd - eb)b & -(cd - eb)(k + bl) \end{pmatrix} = \\ &= d(eb - cd) \neq 0 \end{aligned}$$

is ensured.

3. The method of submodules

3.1. Introduction into the method

In this chapter, we shall not investigate directly the differentials $d\bar{u}_s, d\bar{v}_s, dx$, but only the submodules $\{dx, d\bar{u}_0, d\bar{v}_0, \dots, d\bar{u}_s, d\bar{v}_s\}$ and $\{dx, du_0, dv_0, \dots, du_s, dv_s\}$. Especially denoting by $\Theta = \{dx, du_0, dv_0\} \subset \Phi$, the following lemma gives us the method of calculations:

Lemma 3. *If a mapping* $\boldsymbol{h}$ *performs an* x*-preserving automorphism of curves in* $\boldsymbol{R}^3$*, then the submodule* $\bar{\Theta} = \mathbf{h}^*\Theta$ *has these properties:*

(i) $dx \in \bar{\Theta}$;
(ii) $\bar{\Theta}$ *is completely integrable;*
(iii) $\cup_0^\infty \mathcal{L}_X^s \bar{\Theta} = \Phi$.

Conversely, if $\bar{\Theta} \subset \Phi$ *is any submodule meeting the conditions* (i), (ii), (iii), *then there exists an* x*-preserving automorphism* $\boldsymbol{h}$ *of curves in* $\boldsymbol{R}^3$ *such that* $\mathbf{h}^*\Theta = \bar{\Theta}$.

Proof. If $\mathbf{h}$ is an x-preserving automorphism of curves in $\mathbf{R}^3$, then

$$\bar{\Theta} = \mathbf{h}^*\Theta = \{dx, d\bar{u}_0, d\bar{v}_0\}$$

is completely integrable and

$$\mathcal{L}_X^s \bar{\Theta} = \{dx, d\bar{u}_0, d\bar{v}_0, \dots, d\bar{u}_s, d\bar{v}_s\}$$

hence $\cup_0^\infty \mathcal{L}_X^s \bar{\Theta} = \Phi$.

On the other hand, let $\bar{\Theta} \subset \Phi$ be a submodule satisfying (i), (ii), (iii). The conditions (i), (ii) mean that $\bar{\Theta} = \{dx, df, dg\}$ for appropriate f, g. Denoting by

$$\bar{u}_0 = f, \quad \bar{v}_0 = g, \quad \bar{u}_s = X^s u_0, \quad \bar{v}_s = X^s v_0\,,$$

consider the mapping $\mathbf{h}$ defined by

$$\mathbf{h}^* x = x, \quad \mathbf{h}^* u_s = \bar{u}_s, \quad \mathbf{h}^* v_s = \bar{v}_s\,.$$

According to the definition $\mathbf{h}$ preserves x, transformes the family Δ into itself and since

$$\{dx, d\bar{u}_0, d\bar{v}_0, \dots\} = \cup_0^\infty \mathcal{L}_X^s \bar{\Theta} = \Phi\,,$$

the differentials $dx, d\bar{u}_s, d\bar{v}_s$ $(s = 0, 1, \dots)$ generate Φ.

At last, we have to prove that the differentials $dx, d\bar{u}_s, d\bar{v}_s$ are linearly independent. Let us assume that there is an identity

$$d\bar{u}^s = \sum_{r=0}^{s-1} A_r d\bar{u}^r + \sum_{r=0}^{R} B_r d\bar{v}_r + C dx\,.$$

Then

$$\mathcal{L}_s^K d\bar{u}^s = d\bar{u}^{s+K} = \sum_{r=0}^{s-1+K} A_r^K d\bar{u}^r + \sum_{r=0}^{R+K} B_r^K + C^K dx$$

hence
$$l(\mathcal{L}_X^K\bar\Theta) \le \text{const} + K$$
for K large enough (where $l(\mathcal{L}_X^K\bar\Theta)$ is the dimension of $\mathcal{L}_X^K\bar\Theta$). On the other hand, since (iii) is valid, $\Theta \subset \mathcal{L}_X^p\bar\Theta$ for appropriate p so that
$$\mathcal{L}_X^K\Theta \subset \mathcal{L}_X^{p+K}\bar\Theta$$
for all $K \ge 0$. We obtain
$$3 + 2K = l(\mathcal{L}_X^K\Theta) \le l(\mathcal{L}_X^{p+K}\bar\Theta) \le \text{const} + K$$
for K large enough, which is a contradiction concluding the proof.

The method of submodules will consist of two steps. In the first one, we shall search for submodules
$$\bar\Theta = \{dx, \alpha, \beta\} \subset \Phi \text{ such that } \cup\mathcal{L}_X^s\bar\Theta = \Phi\,,$$
and, according to the Frobenius theorem,
$$d\alpha \simeq 0 \;(\mathrm{mod}\;\bar\Theta)\,,\;\; d\beta \simeq 0 \;(\mathrm{mod}\;\bar\Theta)\,.$$
We obtain certain as a rule overdetermined systems of partial differential equations for the coefficients of the forms α and β. After this step, the existence of the automorphisms already is ensured.

In the second step, we search for explicit equations of the automorphisms. The identity
$$\{dx, \alpha, \beta\} = \{dx, df, dg\}$$
provide us necessary and sufficient conditions (again in the form of systems of partial differential equations) for the functions f, g which can be (in principle) obtained by solving ordinary differential equations.

The following reasoning will facilitate us to describe systematicaly the class of all automorphisms under consideration. If $\bar\Theta = \{\alpha, \beta, dx\} \subset \mathcal{L}_X^m\Theta$, then the forms α, β cannot be linearly independent (mod $\mathcal{L}_X^{m-1}\Theta$) or else $\mathcal{L}_X\alpha,\ \mathcal{L}_X\beta$ would be linearly independent (mod $\mathcal{L}_X^m\Theta$), $\mathcal{L}_X^2\alpha,\ \mathcal{L}_X^2\beta$ would be linearly independent (mod $\mathcal{L}_X^{m+1}\Theta$),... and $\cup_0^\infty\mathcal{L}_X^s\bar\Theta = \Phi$ could not hold. Thus we can introduce following notation:

An x-preserving automorphism of curves in $\mathbf{R}^3$ is of the type $[p, q]$, if $\alpha \in \mathcal{L}_X^p\Theta,\ \alpha \notin \mathcal{L}_X^{p-1}\Theta,\ \beta \in \mathcal{L}_X^q\Theta,\ \beta \notin \mathcal{L}_X^{q-1}\Theta$ and $p < q$.

From now on, we shall abbreviate our terminology and speak of an "automorphism" instead of an "x-preserving automorphism of curves in $\mathbf{R}^3$".

3.2. The point transformation

Let us examine the most simple case $\bar{\Theta} \subset \Theta$ just corresponding to the zeroth order case in 2.1. Then necessarily $\bar{\Theta} = \Theta = \{dx, du_0, dv_0\}$ and we obtained the identity. However, the choice of the first integrals is not unique and $\{dx, du_0, dv_0\} = \{dx, df, dg\}$ if and only if $f = f(x, u_0, v_0), g = g(x, u_0, v_0)$ are such functions that

$$\det \begin{pmatrix} 1 & 0 & 0 \\ f_x & f_0 & f^0 \\ g_x & g_0 & g^0 \end{pmatrix} = \det \begin{pmatrix} f_0 & f^0 \\ g_0 & g^0 \end{pmatrix} \neq 0\,.$$

Therefore automorphisms $\mathbf{h}$ with the property $\mathbf{h}^*\Theta = \Theta$ are the invertible prolonged point transformations. One can observe that the reasonings were shorter than in section 2.1.

If a submodule $\bar{\Theta} \subset \Phi$ satisfying (i), (ii) and (iii) is given and $dx, d\bar{u}_0, d\bar{v}_0$ are its first integrals then

$$\bar{\Theta} = \{dx, d\bar{u}_0, d\bar{v}_0\} = \{dx, df, dg\}$$

if and only if the functions f, g can be expressed in the form

$$f = f(x, \bar{u}_0, \bar{v}_0),\;\; g = g(x, \bar{u}_0, \bar{v}_0)\,,\;\; \det \begin{pmatrix} \frac{\partial f}{\partial \bar{u}_0} & \frac{\partial f}{\partial \bar{v}_0} \\ \frac{\partial g}{\partial \bar{u}_0} & \frac{\partial g}{\partial \bar{v}_0} \end{pmatrix} \neq 0$$

It means that the automorphism $\mathbf{h}$ given by $\bar{\Theta}$ is given up to a transposition with an invertible prolonged point transformation.

3.3. Automorphisms of the type [0,1]

Let

$$\alpha \in \Theta,\;\; \beta \in \mathcal{L}_X\Theta,\;\; \beta \notin \Theta\,;$$

then we can take

$$\alpha = Adu_0 + Bdv_0 + Cdx\,,$$
$$\beta = \bar{A}du_1 + \bar{B}dv_1 + \bar{C}du_0 + \bar{D}dv_0 + \bar{E}dx$$

where $A, B, C, \bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E} \in \mathcal{F}$. We can suppose $A \neq 0, \bar{A} \neq 0$ and omit the terms with dx, hence we may choose

$$\alpha = du_0 + adv_0\,,$$
$$\beta = du_1 + \bar{a}dv_1 + \bar{b}dv_0$$

for appropriate $a, \bar{a}, \bar{b}, \in \mathcal{F}$. Since the forms $\mathcal{L}_X\alpha$ and β cannot be linearly independent (mod Θ) (according to the note in 3.1) and

$$\mathcal{L}_X\alpha = du_1 + adv_1 + Xadv_0\,,$$

we can take

$$\alpha = du_0 + adv_0\,, \qquad (6)$$
$$\beta = du_1 + adv_1 + bdv_0$$

for appropriate $a, b \in \mathcal{F}$ and $\cup_0^\infty \mathcal{L}_X^s \Theta = \Phi$ is ensured if

$$Xa - b \neq 0\,. \tag{7}$$

The condition of complete integrability gives

$$\begin{aligned}
d\alpha = da \wedge dv_0 &= (a_x dx + \sum_{s=0} a_s du_s + \sum_{s=0} a^s dv_s) \wedge dv_0 = \\
&= (a_x dx + a_0(\alpha - a dv_0) + a_1(\beta - a dv_1 - b dv_0) \\
&\quad + \sum_{s=2} a_s du_s + \sum_{s=0} a^s dv_s) \wedge dv_0 \simeq \\
&\simeq (-a_1 a + a^1) dv_1 \wedge dv_0 + \sum_{s=2} a_s du_s \wedge dv_0 + \sum_{s=2} a^s dv_s \wedge dv_0 = 0\,,
\end{aligned}$$

where $\simeq$ means (mod Θ). Then necessarily

$$-a_1 a + a^1 = 0, \quad a = a(x, u_0, v_0, u_1, v_1)\,. \tag{8}$$

Furthermore,

$$\begin{aligned}
d\beta &= da \wedge dv_1 + db \wedge dv_0 = \\
&= (a_x dx + a_0(\alpha - a dv_0) + a_1(\beta - a dv_1 - b dv_0) + \\
&\quad + \sum_{s=2} a_s du_s + \sum_{s=0} a^s dv_s) \wedge dv_1 + \\
&\quad + (b_x dx + b_0(\alpha - a dv_0) + b_1(\beta - a dv_1 - b dv_0) + \\
&\quad + \sum_{s=2} b_s du_s + \sum_{s=0} b^s dv_s) \wedge dv_0 \simeq \\
&\simeq (-a_0 a - a_1 b + a^0 + b_1 a - b^1) dv_0 \wedge dv_1 + \\
&\quad + \sum_{s=2} a_s du_s \wedge dv_1 + \sum_{s=2} a^s dv_s \wedge dv_1 + \\
&\quad + \sum_{s=2} b_s du_s \wedge dv_0 + \sum_{s=2} b^s dv_s \wedge dv_0 = 0
\end{aligned}$$

hence

$$-a_0 a - a_1 b + a^0 + b_1 a - b^1 = 0, \quad b = b(x, u_0, v_0, u_1, v_1). \tag{9}$$

Notice that the equation (8) for the unknown function a is solvable by

$$\Phi(a u_1 + v_1, a, x, u_0, v_0) = 0$$

where Φ is arbitrary (smooth) function and then b can be determined from (9).

If we take any couple of functions a, b, satisfying (7), (8), (9), then some relationships

$$
\begin{aligned}
\alpha &= du_0 + a dv_0 = P df + Q dg = \\
&= P(f_0 du_0 + f^0 dv_0 + f_1 du_1 + f^1 dv_1 + \dots) + \\
&\quad + Q(g_0 du_0 + g^0 dv_0 + g_1 du_1 + g^1 dv_1 + \dots)\,, \\
\beta &= du_1 + a dv_1 + b dv_0 = R df + S dg = \\
&= R(f_0 du_0 + f^0 dv_0 + f_1 du_1 + f^1 dv_1 + \dots) + \\
&\quad + S(g_0 du_0 + g^0 dv_0 + g_1 du_1 + g^1 dv_1 + \dots)
\end{aligned} \tag{10}
$$

are valid for appropriate $P, Q, R, S, f, g \in \mathcal{F}$. Then

$$
\begin{aligned}
1 &= P f_0 + Q g_0\,, & 0 &= R f_0 + S g_0\,, \\
a &= P f^0 + Q g^0\,, & b &= R f^0 + S g^0\,, \\
0 &= P f_1 + Q g_1\,, & 1 &= R f_1 + S g_1\,, \\
0 &= P f^1 + Q g^1\,, & a &= R f^1 + S g^1
\end{aligned} \tag{11}
$$

and necessarily

$$\det \begin{pmatrix} f_1 & g_1 \\ f^1 & g^1 \end{pmatrix} = 0 \quad (\text{or else } P = Q = 0)$$

and

$$f^1 = a f_1\,.$$

We may assume that $f = f(x, u_0, v_0, u_1, v_1)$, $g = g(x, u_0, v_0, u_1, v_1)$: If, e.g., $f_2 \neq 0$, then from (10) we have

$$
\begin{aligned}
0 &= P f_2 + Q g_2, \\
0 &= R f_2 + S g_2
\end{aligned}
$$

hence

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = 0$$

and the first line of (11) implies $R = S = 0$. Moreover, from (11) it follows that either $f_1 \neq 0$ or $g_1 \neq 0$; we may suppose $f_1 \neq 0$ and then the condition above can be expressed in the form

$$g = G(x, u_0, v_0, f)\,. \tag{12}$$

Substituting this into (10), we have

$$
\begin{aligned}
\alpha &= du_0 + a dv_0 = (P + G_f Q) df + Q G_0 du_0 + Q G^0 dv_0\,, \\
\beta &= du_1 + a dv_1 + b dv_0 = (R + G_f S) df + S G_0 du_0 + S G^0 dv_0\,,
\end{aligned} \tag{13}
$$

and combining

$$\begin{aligned}&(R+G_fS)(du_0+adv_0)-(P+G_fQ)(du_1+adv_1+bdv_0)=\\&=((R+G_fS)QG_0-(P+G_fQ)SG_0)du_0+\\&+((R+G_fS)QG^0-(P+G_fQ)SG^0)dv_0\end{aligned}$$

we obtain

$$\begin{aligned}P+G_fQ&=0\,,\\R+G_fS&=(R+G_fS)QG_0\,,\\a(R+G_fS)&=(R+G_fS)QG_0\end{aligned}$$

hence

$$\frac{f^1}{f_1}=a=\frac{G^0}{G_0}\,. \tag{14}$$

Furthermore, (13) gives

$$\begin{aligned}1&=(R+G_fS)f_1\,,\\0&=(R+G_fS)f_0+SG_0\,,\\b&=(R+G_fS)f^0+SG^0\,,\end{aligned}$$

which implies (using (14))

$$b=\frac{f^0f_1-f_0f^1}{f_1f_1}\,. \tag{15}$$

The condition $Xa-b\neq 0$ can be expressed in the form

$$X\left(\frac{G^0}{G_0}\right)-\frac{f^0-f_0a}{f_1}=\frac{XG^0G_0-G^0-XG_0}{G_0G_0}-\frac{f^0-f_0a}{f_1}\neq 0\,,$$

which is equivalent to

$$\det\begin{pmatrix}G_0 & G^0\\ f_1XG_0-G_0f_0 & f_1XG^0-g_0f^0\end{pmatrix}\neq 0\,. \tag{16}$$

If we take any functions a,b, satisfying (7), (8), (9) and search for functions f,g, determining an automorphism such that $\{dx,df,dg\}=\{dx,\alpha,\beta\}$, where α,β are given by (6), then the functions f,g must be of the form (12) and satisfy (14), (15) and (16). The validity of (7), (8), (9) ensure us the existence of such functions.

On the other hand, let us take any functions f,g, satisfying (12), $f^1/f_1=G^0/G_0$ and (16) and define a and b by (14), (15). Then

$$\{dx,df,dg\}=\{du_0+adv_0,du_1+adv_1+bdv_0\}\,,$$

the relationships (7), (8), (9) are valid and f,g determine an automorphism of the type [0,1].

Not surprisingly, this concludes Theorem 1 from section 2.2. For this case, the method of submodules could seem a little artifical. However, we obtained some additional information about the structure of the automorphisms, given by expressions $f^1/f_1 = G^0/G_0 = a, (f^0 f_1 - f_0 f^1)/f_1 f_1 = b$ determining the submodule $\{dx, \alpha, \beta\}$, which satisfy (8) and (9). Moreover, while the reasonings would become very complicated for the higher order cases if using the direct method, the former calculations will a little prolong but remain rather straightforward.

3.4. Automorphisms of the type [1,2]

Let

$$\bar{\Theta} = \{\alpha, \beta, dx\} \subset \mathcal{L}_X^2\Theta\,, \quad \alpha \in \mathcal{L}_X\Theta\,, \quad \alpha \notin \Theta\,, \quad \beta \in \mathcal{L}_X^2\Theta\,, \quad \beta \notin \mathcal{L}_X\Theta\,.$$

Then, analogously as above, we can take α and β in the form

$$\begin{aligned} \alpha &= du_1 + adv_1 + bdu_0 + cdv_0\,, \\ \beta &= du_2 + adv_2 + edv_1 + hdu_0 + kdv_0 \end{aligned} \tag{17}$$

where $a, b, c, e, h, k \in \mathcal{F}$. The forms

$$\begin{aligned} &\mathcal{L}_X\alpha - \beta = bdu_1 + (Xa + c - e)dv_1 + (Xb - h)du_0 + (xc - k)dv_0\,, \\ &\alpha = du_1 + dv_1 + bdu_0 + cdv_0 \end{aligned}$$

must be linearly dependent (mod Θ) and so

$$e = Xa + c - ab\,. \tag{18}$$

Then the forms

$$\begin{aligned} \mathcal{L}_X(\mathcal{L}_X\alpha - \beta - b\alpha) &= \mathcal{L}_X((Xb - h - bb)du_0 + (Xc - k - bc)dv_0) = \\ &= (Xb - h - bb)du_1 + (Xc - k - bc)dv_1 + \\ &+ X(Xb - h - bb)du_0 + X(Xc - k - bc)dv_0\,, \\ \alpha &= du_1 + adv_1 + bdu_0 + cdv_0 \end{aligned}$$

must be linearly dependent (mod Θ), which gives

$$a(Xb - h - bb) = Xc - k - bc\,. \tag{19}$$

Since

$$\mathcal{L}_X\alpha - \beta - b\alpha = (Xb - h - bb)(du_0 + adv_0)\,,$$

we have

$$Xb - h - bb \neq 0\,. \tag{20}$$

To ensure the equality $\cup\mathcal{L}_X^s\bar{\Theta} = \Phi$, the forms

$$\begin{aligned} du_0 + adv_0 \;&(= \frac{1}{Xb - h - bb}((Xb - h - bb)du_0 + (Xc - k - bc)dv_0))\,, \\ \mathcal{L}_X(du_0 + adv_0) - \alpha &= -bdu_0 + (Xa - c)dv_0 \end{aligned}$$

must be linearly independent, i.e. ,

$$Xa + ab - c \neq 0\,. \tag{21}$$

Applying the condition of complete integrability,

$$\begin{aligned}
d\alpha = {} & da \wedge dv_1 + db \wedge du_0 + dc \wedge dv_0 \simeq \\
& \simeq (a_0 du_0 + a_1(-bdu_0 - cdv_0) + a_2(-adv_2 - hdu_0 - kdv_0) + \sum_{s=3} a_s du_s + \\
& + \sum_{s=0} a^s dv_s) \wedge dv_1 + (b_1(-adv_1 - cdv_0) + b_2(-adv_2 - edv_1 - kdv_0) + \\
& + \sum_{s=3} b_s du_s + \sum_{s=0} b^s dv_s) \wedge du_0 + \\
& + (c_0 du_0 + c_1(-adv_1 - bdu_0) + c_2(-adv_2 - edv_1 - hdu_0) + \\
& + \sum_{s=3} c_s du_s + \sum_{s=1} c^s dv_s) \wedge dv_0 = 0\,,
\end{aligned}$$

$$\begin{aligned}
d\beta = {} & da \wedge dv_2 + de \wedge dv_1 + dh \wedge du_0 + dk \wedge dv_0 \simeq \\
& \simeq (a_0 du_0 + a_1(-adv_1 - bdu_0 - cdv_0) + a_2(-edv_1 - hdu_0 - kdv_0) + \\
& + \sum_{s=3} a_s du_s + \sum_{s=0} a^s dv_s) \wedge dv_2 + \\
& + (e_0 du_0 + e_1(-bdu_0 - cdv_0) + e_2(-adv_2 - hdu_0 - kdv_0) + \\
& + \sum_{s=3} e_s du_s + \sum_{s=0} e^s dv_s) \wedge dv_1 + \\
& + (h_1(-adv_1 - cdv_0) + h_2(-adv_2 - edv_1 - kdv_0) + \\
& + \sum_{s=3} h_s du_s + \sum_{s=0} h^s dv_s) \wedge du_0 + \\
& + (k_0 du_0 + k_1(-adv_1 - bdu_0) + k_2(-adv_2 - edv_1 - hdu_0) + \\
& + \sum_{s=3} k_s du_s + \sum_{s=1} k^s dv_s) \wedge dv_0 = 0\,;
\end{aligned}$$

then from (18) it follows that $a = a(x, u_0, v_0, u_1, v_1)$ and we obtain conditions for the coefficients $a = a(x, u_0, v_0, u_1, v_1)$ and b, c, e, h, k of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$

$$
\begin{aligned}
&b_2 a - b^2 = 0\\
&c_2 a - c^2 = 0\\
&b_1 c + b_2 k - b^0 + c_0 - c_1 b - c_2 h = 0\\
&a_0 - a_1 b + b_1 a + b_2 e - b^1 = 0\\
&a^0 - a_1 c + c_1 a + c_2 e - c^1 = 0\\
&h_1 c + h_2 k - h^0 + k_0 - k_1 b - k_2 h = 0\\
&e_0 - e_1 b - e_2 h + h_1 a + h_2 e - h^1 = 0\\
&a_0 - a_1 b + h_2 a - h^2 = 0\\
&e^0 - e_1 c - e_2 k + k_1 a + k_2 e - k^1 = 0\\
&a^0 - a_1 c + k_2 a - k^2 = 0\\
&a^1 - a_1 a + e_2 a - e^2 = 0\,,
\end{aligned}
\tag{22}
$$

(18) and (19). Moreover, the inequalities (20) and (21) must hold.

If a, b, c, d, e, h, k are such functions, then

$$
\begin{aligned}
\alpha &= du_1 + a dv_1 + b du_0 + c dv_0 = P df + Q dg\,,\\
\beta &= du_2 + a dv_2 + e dv_1 + h du_0 + k dv_0 = R df + S dg
\end{aligned}
\tag{23}
$$

where f, g are functions of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$ (analogously as in the previous section) . Comparing the coefficients at du_2, dv_2 we obtain

$$
\begin{aligned}
0 &= P f_2 + Q g_2\,, & 1 &= R f_2 + S g_2\,,\\
0 &= P f^2 + Q g^2\,, & a &= R f^2 + S g^2
\end{aligned}
$$

hence

$$
\det \begin{pmatrix} f_2 & g_2 \\ f^2 & g^2 \end{pmatrix} = 0 \quad (\text{or else } P = Q = 0)
$$

and

$$
f^2 = a f_2\,.
$$

We may assume $f_2 \neq 0$ so that

$$
g = G(x, u_0, v_0, u_1, v_1, f)
\tag{24}
$$

Substituting this into (23) we have

$$
\begin{aligned}
\alpha &= du_1 + a dv_1 + b du_0 + c dv_0 =\\
&= (P + Q G_f) df + Q G_0 du_0 + Q G^0 dv_0 + Q G_1 dv_1 + Q G^1 dv_1\,,\\
\beta &= du_2 + a dv_2 + e dv_1 + h du_0 + k dv_0 =\\
&= (R + S G_f) df + S G_0 du_0 + S G^0 dv_0 + S G_1 du_1 + S G^1 dv_1\,,
\end{aligned}
$$

and after combining

$$\begin{aligned}
&(R+SG_f)(du_1+adv_1+bdu_0+cdv_0)-\\
&-(P+QG_f)(du_2+adv_2+edv_1+hdu_0+kdv_0)=\\
&=((R+SG_f)QG_0-(P+QG_f)SG_0)du_0+\\
&+((R+SG_f)QG^0-(P+QG_f)SG^0)dv_0+\\
&+((R+SG_f)QG_1-(P+QG_f)SG_1)du_1+\\
&+((R+SG_f)QG^1-(P+QG_f)SG^1)dv_1\,,
\end{aligned}$$

necessarily

$$\begin{aligned}
P+QG_f&=0\,,\\
R+SG_f&=(R+SG_f)QG_1\,,\\
a(R+SG_f)&=(R+SG_f)QG^1\,,\\
b(R+SG_f)&=(R+SG_f)QG_0\,,\\
c(R+SG_f)&=(R+SG_f)QG^0
\end{aligned}$$

must hold, which gives

$$\frac{G^1}{G_1}=a=\frac{f^2}{f_2},\quad b=\frac{G_0}{G_1},\quad c=\frac{G^0}{G_1}\,, \tag{25}$$

where $G_1\neq 0$ is ensured. Substituting this again into (23), then the first line of (23) is fulfilled identicaly and

$$\begin{aligned}
\beta&=du_2+adv_2+edv_1+hdu_0+kdv_0=\\
&=(R+SG_f)(f_0du_0+f^0dv_0+f_1du_1+f^1dv^1+f_2du_2+f^2dv_2)+\\
&\quad+SG_0du_0+SG^0dv_0+SG_1du_1+SG^1dv_1
\end{aligned}$$

implies

$$\begin{aligned}
1&=(R+SG_f)f_2\quad\text{hence } R+SG_f=1/f_2\,,\\
0&=f_1/f_2+SG_1\quad\text{hence } S=-f_1/(G_1f_2)\,,
\end{aligned}$$

and we obtained

$$\begin{aligned}
e&=f^1/f_2-G^1f_1/G_1f_2=(G_1f^1-G^1f_1)/G_1f_2\,,\\
h&=f_0/f_2-G_0f_1/G_1f_2=(G_1f_0-G_0f_1)/G_1f_2\,,\\
k&=f^0/f_2-G^0f_1/G_1f_2=(G_1f^0-G^0f_1)/G_1f_2\,.
\end{aligned} \tag{26}$$

Comparing (18), (19), (25) and (26), we obtain

$$(G_1f^1-G^1f_1)G_1=(G_1XG^1-G^1XG_1+G^0G_1-G^1G_0)f_2\,; \tag{27}$$

$$
\begin{aligned}
(28)\qquad & G^1 f_2(G_1XG_0 - G_0XG_1) - G^1G_1(G_1f_0 - G_0f_1) - f_2G^1G_0G_0 = \\
& = G_1f_2(G_1XG^0 - G^0XG_1) - G_1G_1(G_1f^0 - G^0f_1) - f_2G_1G_0G^0 \,.
\end{aligned}
$$

The condition (20) can be rewritten in the form

$$(29)\qquad f_2(G_1XG_0 - G_0XG_1 - G_0G^0) + G_1(G^0f_1 - G_1f^0) \neq 0\,,$$

and similarly (21) gives

$$(30)\qquad G_1XG^1 - G^1XG_1 + G^1G_0 - G^0G_1 \neq 0\,.$$

So we may conclude: If functions $a(x, u_0, v_0, u_1, v_1)$ and b, c, e, h, k of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$ satisfy the conditions (18)–(22), then there exist functions f, g of the form (24) such that $\{dx, \alpha, \beta\} = \{dx, df, dg\}$ where the forms α, β are given by (17), and the relationships (25)–(30) are valid. The functions f, g represent an automorphism of the type [1,2].

On the other hand, if f, g are functions of the form (24) such that the relationships $G^1/G_1 = f^2/f_2$, (27)–(30) are valid and we define a, b, c, e, h, k by (25) and (26), then $\{dx, \alpha, \beta\} = \{dx, df, dg\}$ where α, β are defined by (17), the relationships (18)–(22) hold and f, g determine an automorphism of the type [1,2].

3.5. Example

As in section 2.3, let us examine the linear case

$$
\begin{aligned}
\alpha &= du_1 + adv_1 + bdu_0 + cdv_0\,,\\
\beta &= du_2 + adv_2 + edv_1 + hdu_0 + kdv_0\,,
\end{aligned}
$$

where

$$a, b, c, e, h, k \in \mathbf{R},\ \ e = c - ab,\ \ k = a(h + bb) - bc,\ \ ab - c \neq 0,\ \ h + bb \neq 0$$

due to (18), (19), (20) and (21), so that

$$
\begin{aligned}
\alpha &= d(u_1 + av_1 + bu_0 + cv_0)\,,\\
\beta &= d(u_2 + av_2 + ev_1 + hu_0 + kv_0)\,.
\end{aligned}
$$

The linear automorphisms of the type [1,2] are of the form

$$
\begin{aligned}
\bar u_0 &= \mathbf{h}^* u_0 = \mathbf{h}_p^*\mathbf{H}^* u_0 = \mathbf{h}_p^*\bar U_0,\ \ \bar u_s = X^s\bar u_0\,,\\
\bar v_0 &= \mathbf{h}^* v_0 = \mathbf{h}_p^*\mathbf{H}^* v_0 = \mathbf{h}_p^*\bar V_0,\ \ \bar v_s = X^s\bar v_0\,,
\end{aligned}
$$

where $\mathbf{H}$ is given by

$$
\begin{aligned}
&\bar U_0 = u_1 + av_1 + bu_0 + cv_0\,,\\
&\bar V_0 = u_2 + av_2 + (c - ab)v_1 + hu_0 + (a(h + bb) - bc)v_0\,,\\
&\bar U_s = X^s\bar U_0,\ \ \bar V_s = X^s\bar V_0\,,\\
&ab - c \neq 0,\ \ h + bb \neq 0\,,
\end{aligned}
$$

and $\mathbf{h}_p$ is the prolonged point transformation

$$\begin{aligned}
&\bar{u}_0 = A + Bx + C\bar{U}_0 + D\bar{V}_0\,,\\
&\bar{v}_0 = E + Fx + G\bar{U}_0 + H\bar{V}_0\,,\\
&\bar{u}_s = X^s\bar{u}_0,\ \bar{v}_s = X^s\bar{v}_0\,,\\
&A, B, C, D, E, F, G, H \in \mathbf{R}\,,\\
&\det\begin{pmatrix} C & D\\ G & H\end{pmatrix} = CH - GD \neq 0\,.
\end{aligned}$$

Then the inversion is given by

$$u_0 = (\mathbf{h}^*)^{-1}\bar{u}_0 = (\mathbf{H}^*)^{-1}(\mathbf{h}_p^*)^{-1}\bar{u}_0 = (\mathbf{H}^*)^{-1}\bar{U}_0,\ u_s = X^s u_0\,,$$

$$v_0 = (\mathbf{h}^*)^{-1}\bar{v}_0 = (\mathbf{H}^*)^{-1}(\mathbf{h}_p^*)^{-1}\bar{v}_0 = (\mathbf{H}^*)^{-1}\bar{V}_0,\ v_s = X^s v_0\,,$$

and after some calculations we obtain the inverse relationships for $\mathbf{h}_p^*$ and $\mathbf{H}^*$: The transformation $\mathbf{h}_p^{-1}$ is expressed by

$$\begin{aligned}
\bar{U}_0 &= \frac{1}{CH - GD}(-AH + ED + (-HB + DF)x + H\bar{u}_0 - D\bar{v}_0)\,,\\
\bar{V}_0 &= \frac{1}{CH - GD}(-CE + GA + (-CF + GB)x - G\bar{u}_0 + C\bar{v}_0)\,,\\
\bar{U}_s &= X^s\bar{U}_0,\ \bar{V}_s = X^s\bar{V}_0\,,
\end{aligned}$$

and $\mathbf{H}^{-1}$ is given by

$$u_0 = \frac{1}{(h+bb)(c-ab)}((cb - a(h+bb))\bar{U}_0 + c\bar{V}_0 + (-c+ab)\bar{U}_1 + a\bar{V}_1 - a\bar{U}_2)\,,$$

$$v_0 = \frac{1}{(h+bb)(c-ab)}(h\bar{U}_0 - b\bar{V}_0 + 2b\bar{U}_1 + \bar{V}_1 - \bar{U}_2)\,,$$

$$u_s = X^s u_0,\ v_s = X^s v_0\,.$$

3.6. Automorphisms of the type [0,2]

Let

$$\alpha \in \Theta\,,\ \beta \in \mathcal{L}_X^2\Theta\,,\ \beta \notin \mathcal{L}_X\Theta\,.$$

Then we may assume that

$$\begin{aligned}
\alpha &= du_0 + adv_0\,,\\
\beta &= du_2 + Adv_2 + Bdu_1 + Cdv_1 + Ddv_0\,.
\end{aligned}$$

The forms

$$\mathcal{L}_X^2\alpha = du_2 + adv_2 + 2Xadv_1 + X^2adv_0$$

and β must be linearly dependent (mod $\mathcal{L}_X\Theta$) hence

$$\begin{aligned} \alpha &= du_0 + adv_0\,, \\ \beta &= du_2 + adv_2 + bdu_1 + cdv_1 + edv_0\,. \end{aligned} \tag{31}$$

Analogously the forms

$$\begin{aligned} \mathcal{L}_X^2\alpha - \beta &= -bdu_1 + (2Xa - c)dv_1 + (X^2a - e)dv_0\,, \\ \mathcal{L}_X\alpha &= du_1 + adv_1 + Xadv_0 \end{aligned}$$

must be linearly dependent (mod Θ) but linearly independent so that

$$2Xa + ab - c = 0\,, \tag{32}$$

$$X^2a + bXa - e \neq 0\,. \tag{33}$$

Analogously as in the previous sections, from the complete integrability conditions we obtain the system of partial differential equations

$$\begin{aligned} b_2a - b^2 &= 0 \\ c_2a - c^2 &= 0 \\ e_2a - e^2 &= a^0 - a_0a \\ e_1 + b_2e - e_2b \ &= b^0 - b_0a \\ e^1 + c_2e - e_2c \ &= c^0 - c_0a \end{aligned} \tag{34}$$

where $a = a(x, u_0, v_0)$ (which follows from (32) and $a_1 - a_2b = 0,\ a^1 - a_2c = 0$) and b, c, e are functions of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$.

If we take any functions, satisfying (32)–(34), then

$$\begin{aligned} \alpha &= du_0 + adv_0 = Pdf + Qdg, \\ \beta &= du_2 + adv_2 + bdu_1 + cdv_1 + edv_0 = Rdf + Sdg \end{aligned}$$

where f, g are functions of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$, which implies

$$\begin{aligned} 1 &= Pf_0 + Qg_0\,, & 0 &= Rf_0 + Sg_0\,, \\ a &= Pf^0 + Qg^0\,, & e &= Rf^0 + Sg^0\,, \\ 0 &= Pf_1 + Qg_1\,, & b &= Rf_1 + Sg_1\,, \\ 0 &= Pf^1 + Qg^1\,, & c &= Rf^1 + Sg^1\,, \\ 0 &= Pf_2 + Qg_2\,, & 1 &= Rf_2 + Sg_2\,, \\ 0 &= Pf^2 + Qg^2\,, & a &= Rf^2 + Sg^2\,. \end{aligned}$$

Thus

$$h\begin{pmatrix} f_1 & f^1 & f_2 & f^2 \\ g_1 & g^1 & g_2 & g^2 \end{pmatrix} = 1$$

hence (assuming $f_2 \neq 0$)

$$f = f(x, u_0, v_0, u_1, v_1, u_2, v_2)\,, \qquad g = G(x, u_0, v_0, f) \tag{35}$$

and

$$a = \frac{f^2}{f_2}\,, \quad b = \frac{f_1}{f_2}\,, \quad c = \frac{f^1}{f_2}\,. \tag{36}$$

As in sections 3.3 and 3.4, after some calculations we derive

$$\frac{G^0}{G_0} = a = \frac{f^2}{f_2}\,, \tag{37}$$

$$e = \frac{f^0}{f_2} - \frac{f^0 G^0}{G_0 f_2} = \frac{f^0 G_0 - f^0 G^0}{G_0 f_2}\,, \tag{38}$$

and notice that the second equation of (37) is solvable by

$$f = \Phi(u_2 + a v_2, x, u_0, v_0, u_1, v_1)\,.$$

The condition (32) can be expressed in the form

$$\frac{f^1}{f_2} = 2X(\frac{f^2}{f_2}) + \frac{f^2}{f_2}\frac{f_1}{f_2} \tag{39}$$

and the condition (33) means

$$X^2(\frac{f^2}{f_2}) + \frac{f_1}{f_2}X(\frac{f^2}{f_2}) - \frac{f^0}{f_2} + \frac{f_0 G^0}{G_0 f_2} \neq 0\,. \tag{40}$$

If we take any functions $a(x, u_0, v_0)$ and b, c, e of variables $x, u_0, v_0, u_1, v_1, u_2, v_2$, satisfying (32)–(34), then there exist functions f, g of the form (35) such that $\{dx, \alpha, \beta\} = \{dx, df, dg\}$ where α, β are defined by (31), and the conditions (36), (37)–(40) hold. The functions f, g determine an automorphism of the type [0,2].

On the other hand, if the functions f, g satisfy the conditions (35), $G^0/G_0 = f^2/f_2$, (39) and (40) and we define a, b, c, e by (36) and (38), then $\{dx, \alpha, \beta\} = \{dx, df, dg\}$ where α, β are defined by (31), the relationships (32)–(34) are valid and f, g determine an automorphism of the type [0,2].

Using the achievements of sections 3.4 and 3.6, we can state the following theorem:

Theorem 2. *Let* $f = f(x, u_0, v_0, u_1, v_1, u_2, v_2)$, $\quad g = g(x, u_0, v_0, u_1, v_1, u_2, v_2)$, $f_2 \neq 0$. *These functions determine an x-preserving automorphism of curves in* $\mathbf{R}^3$, *if and only if one (and only one) of the following conditions holds:*

(i) $\quad g = G(x, u_0, v_0, u_1, v_1, f), \quad G_1 \neq 0\,, \quad \dfrac{G^1}{G_1} = \dfrac{f^2}{f_2}, \quad (27){-}(30).$

$$\text{(ii)} \quad g = G(x, u_0, v_0, f)\,, \quad G_0 \neq 0\,, \quad \frac{G^0}{G_0} = \frac{f^2}{f_2}\,, \quad (39), (40)\,.$$

4. The infinitesimal symmetries

4.1. The infinitesimal symmetries in $\mathbf{R}^\infty$

For the convenience of exposition, let us briefly mention the finite-dimensional space $\mathbf{R}^n$ with points $x = (x^1, \dots, x^n)$, differential forms $\varphi = \sum_{i=1}^m f_i dg^i$ and vector fields $Z = \sum_{i=1}^n z^i \frac{\partial}{\partial x^i}$, where $f_i = f_i(x), g^i = g^i(x), z^i = z^i(x)$ are smooth functions. Given a submodule Ω of differential forms, a vector field Z is called an *infinitesimal symmetry of* Ω, if

$$\mathcal{L}_Z \Omega \subset \Omega\,. \tag{41}$$

Recall that such a vector field generates a *one-parameter group of transformations* in the sense

$$Zf(x) = \frac{\partial}{\partial\lambda} f(\mathbf{h}(x,\lambda))|_{\lambda=0}, \quad \mathbf{h}(x,0) = id, \quad \mathbf{h}(x,\lambda+\mu) = \mathbf{h}(\mathbf{h}(x,\lambda),\mu)\,, \tag{42}$$

where λ and μ are real parameters near enough to zero, and, for fixed λ the transformation $\mathbf{h}(x,\lambda)$ preserve the module Ω in the sense

$$\mathbf{h}^*(x,\lambda)\Omega \subset \Omega\,.$$

Now let us proceed to the infinite-dimensional space $\mathbf{R}^\infty$ with the coordinates $x^1, x^2, \dots$, the structural family $\mathcal{F}$ and the $\mathcal{F}$-modules Φ, $\mathcal{T}$ of differential forms and vector fields according to the notation of section 1.2. In $\mathbf{R}^\infty$, given a submodule $\Omega \subset \Phi$, a vector field $Z \in \mathcal{T}$ is called a *generalized infinitesimal symmetry of* Ω, if (41) holds and an *infinitesimal symmetry of* Ω, if (41) holds and Z generates a one-parameter group of transformations in the sense (42). The generalized infinitesimal symmetries can be in principle calculated by finite algorithms; however, as yet no method exists to determine the infinitesimal symmetries or even to prove or disprove their existence for a given module Ω in the infinite-dimensional space. Our approach will be based on the following lemma. We state only the proof of the necessity; the sufficiency is more delicate and lengthy and cannot be discussed here (see, e.g., [6]).

Lemma 4. *A vector field Z generates a one-parameter group of transformations if and only if for any function $f \in \mathcal{F}$ all the functions of the sequence $f, Zf, Z^2 f, \dots$ can be expressed in a finite number of coordinates.*

Proof. If $f \in \mathcal{F}$ and $\mathbf{h}(x,\lambda)$ generates a one- parameter group of transformations, then $f(\mathbf{h}(x,\lambda))$ depends on a finite number of coordinates, hence $\frac{\partial^n}{\partial\lambda^n} f(\mathbf{h}(x,\lambda))|_{\lambda=0}$

depends on a finite number of coordinates; therefore it is sufficient to prove that

$$Z^n f(x) = \frac{\partial^n}{\partial \lambda^n} f(\mathbf{h}(x,\lambda))|_{\lambda=0} \tag{43}$$

for all $n \in \mathbf{N}$.

If $n = 1$, then $Z(f(x)) = \frac{\partial}{\partial \lambda} f(\mathbf{h}(x,\lambda))|_{\lambda=0}$ by definition. Let us assume that the formula (43) is valid for some $n \in \mathbf{N}$, $n \geq 1$; then

$$\begin{aligned} Z^{n+1} f(x) &= Z(Z^n f(x)) = Z(\frac{\partial^n}{\partial \lambda^n} f(\mathbf{h}(x,\lambda))|_{\lambda=0}) = \\ &= \frac{\partial}{\partial \mu}(\frac{\partial^n}{\partial \lambda^n} f(\mathbf{h}(\mathbf{h}(x,\mu),\lambda))|_{\lambda=0})|_{\mu=0} = \\ &= \frac{\partial}{\partial \mu}(\frac{\partial^n}{\partial \lambda^n} f(\mathbf{h}(x,\lambda+\mu))|_{\lambda=0})|_{\mu=0} = \frac{\partial^{n+1}}{\partial \lambda^{n+1}} f(\mathbf{h}(x,\lambda))|_{\lambda=0} \end{aligned}$$

and (43) is valid for all $n \in \mathbf{N}$, which concludes the proof.

If $x^1, x^2, \ldots$ are coordinates in $\mathbf{R}^\infty$, it is sufficient to verify the requirement only for the coordinate functions x^i $(i = 1, 2, \ldots)$: If the finiteness condition of lemma 4 holds for the coordinate functions x^i $(i = 1, 2, \ldots)$, then for any other function $f \in \mathcal{F}$, $f = f(x^1, \ldots, x^k)$ clearly is

$$Zf = \sum f_{x^i} Z x^i, \; Z^2 f = \sum (\sum f_{x^i x^j} Z x^j Z x^i + f_{x^i} Z^2 x^i), \ldots$$

and the functions $f, Zf, Z^2 f, \ldots$ depend on a finite number of coordinates.

4.2. The infinitesimal symmetries in our case

We shall be interested in infinitesimal symmetries of the very special submodule $\Omega \subset \Phi$ defined in the section 1.4. Returning to the coordinates x, u_s, v_s and to the notation from section 1.4, let us remind the contact forms

$$du_s - u_{s+1}dx, \quad dv_s - v_{s+1}dx \; (s = 0, 1, \ldots),$$

generating the submodule $\Omega \subset \Phi$, and the formal derivative operator

$$X = \frac{\partial}{\partial x} + \sum^\infty u_{s+1} \frac{\partial}{\partial u_s} + \sum^\infty v_{s+1} \frac{\partial}{\partial v_s}.$$

Before proceeding to another concept of our theory, we shall establish a useful assertion; its proof is quite easy.

Lemma 5. *Let $f, g \in \mathcal{F}$. Then $df - g dx \in \Omega$ if and only if $g = Xf$.*

In accordance with our task, we shall be interested in vector fields

$$Z = z\frac{\partial}{\partial x} + \sum u^s \frac{\partial}{\partial u_s} + \sum v^s \frac{\partial}{\partial v_s}$$

preserving the coordinate x, i.e., satisfying

$$0 = Zx = z\,.$$

Then the condition (41) is fulfilled if and only if

$$\mathcal{L}_z(du_s - u_{s+1}dx) \in \Omega\,, \quad \mathcal{L}_z(dv_s - v_{s+1}dx) \in \Omega\,.$$

Since

$$\mathcal{L}_z du_s = dZu_s = du^s, \quad \mathcal{L}_z dv_s = dZv_s = dv^s\,,$$

we obtain the requirements

$$du^s - u^{s+1}dx \in \Omega, \quad dv^s - v^{s+1}dx \in \Omega\,,$$

which give, according to lemma 5,

$$u^{s+1} = Xu^s, \quad v^{s+1} = Xv^s\,.$$

Thus it is quite easy to find the x-preserving generalized infinitesimal symmetries in $\mathcal{T}$: The initial values

$$u^0 = a \in \mathcal{F}, \quad v^0 = b \in \mathcal{F}$$

may be arbitrarily chosen and then

$$Z = \sum_0^\infty X^s a \frac{\partial}{\partial u_s} + \sum_0^\infty X^s b \frac{\partial}{\partial v_s}$$

are the sought generalized infinitesimal symmetries in $\mathcal{T}$.

Such vector fields satisfy the following important condiditon, which can be easily derived by direct calculation.

Lemma 6. *If Z is an x-preserving generalized infinitesimal symmetry and X is the formal derivative operator, then $[X, Z] = 0$.*

We have seen that generalized infinitesimal symmetries can be easily found. Passing to infinitesimal symmetries, the problem becomes much more difficult. Recall that according to lemma 4 and to the note below, such a generalized infinitesimal symmetry generates a one-parameter group of transformations if and only if for any fixed $s \in \mathbf{N}$ the functions

$$u_s,\ Zu_s,\ Z^2u_s,\ v_s,\ Zv_s,\ Z^2v_s$$

can be expressed by means of a finite number of coordinates (we need not consider the coordinate x since $Zx = 0$). It turns out that that the condition of lemma 4 has to be verified even only for the coordinates u_0, v_0 : If the functions

$$Zu_0, \quad Z^2u_0, \ldots, \quad Zv_0, \quad Z^2v_0, \ldots$$

are depending on a finite number of coordinates and $s \in \mathbf{N}$ is fixed, then (using lemma 5) the functions

$$Zu_s = ZX^su_0 = X^sZu_0, \quad Z^2u_s = Z^2X^su_0 = X^sZ^2u_0, \ldots,$$

$$Zv_s = ZX^s v_0 = X^s Z v_0, \quad Z^2 v_s = Z^2 X^s v_0 = X^s Z^2 v_0, \dots$$

are depending on a finite number of coordinates, too. Consequently, since $Zu_0 = a,\ Zv_0 = b$, we search for all couples $a, b \in \mathcal{F}$ such that all functions

$$a,\ Za,\ Z^2 a,\ \dots,\ b,\ Zb,\ Z^2 b,\ \dots$$

can be expressed by a finite number of coordinates. We are not able to solve this problem in full generality. For this reason, we shall discuss only the particular cases when

$$a = a(x, u_0, v_0),\ b = b(x, u_0, v_0)$$

are of the zeroth order, or,

$$a = a(x, u_0, v_0, u_1, v_1),\ b = b(x, u_0, v_0, u_1, v_1)$$

are of the first order. The zeroth order case is quite trivial: If $a = a(x, u_0, v_0)$, then

$$Za = aa_0 + ba^0 = \bar{a}(x, u_0, v_0),\ Z^2 a = Z\bar{a} = \tilde{a}(x, u_0, v_0), \dots$$

are depending only on zeroth order variables and the same reasoning can be carried out with b. The vector field Z generates a one-parameter group of point transformations.

4.3. The first order case

Let

$$a = a(x, u_0, v_0, u_1, v_1),\ b = b(x, u_0, v_0, u_1, v_1),\ a_1 \neq 0.$$

The conditions for a and b are are equivalent to the condition that

$$Z\begin{pmatrix} a \\ b \end{pmatrix},\ Z^2\begin{pmatrix} a \\ b \end{pmatrix},\ \dots$$

are vectors depending only on a finite number of coordinates. In all the following calculations, we shall consider only the top terms, i.e., the terms depending on the variables of the highest order. We have

$$Z\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dots + Xa \cdot a_1 + Xb \cdot a^1 \\ \dots + Xa \cdot b_1 + Xb \cdot b^1 \end{pmatrix} = \begin{pmatrix} \dots + (u_2 a_1 + v_2 a^1) a_1 + (u_2 b_1 + v_2 b^1) a^1 \\ \dots + (u_2 a_1 + v_2 a^1) b_1 + (u_2 b_1 + v_2 b^1) b^1 \end{pmatrix} =$$

$$= \dots + \begin{pmatrix} a_1 a_1 + b_1 a^1 & a^1 a_1 + b^1 a^1 \\ a_1 b_1 + b_1 b^1 & a^1 b_1 + b^1 b^1 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} =$$

$$= \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

$$Z^2\begin{pmatrix} a \\ b \end{pmatrix} = \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^2 Z\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^2 \begin{pmatrix} X^2 a \\ X^2 b \end{pmatrix} =$$

$$= \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^2 \begin{pmatrix} u_3 a_1 + v_3 a^1 \\ u_3 b_1 + v_3 b^1 \end{pmatrix} = \dots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^3 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix},$$

and in general

$$Z^s\begin{pmatrix} a \\ b \end{pmatrix} = \cdots + \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^{s+1} \begin{pmatrix} u_{s+1} \\ v_{s+1} \end{pmatrix},$$

hence necessarily

$$\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^{s} = 0$$

for s large. However, then even

$$\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^{2} = 0\,, \tag{44}$$

so that

$$\det \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} = 0\,,$$

which can be expressed by

$$b = B(x, u_0, v_0, a)\,. \tag{45}$$

Furthermore, the condition (44) is expressed by

$$a_1a_1 + a^1b_1 = a_1a^1 + a^1b^1 = 0\,,$$

$$b_1a_1 + b^1b_1 = b_1a^1 + b^1b^1 = 0\,,$$

which is equivalent to

$$a_1 + b^1 = 0$$

(since $a^1 \neq 0$ or else $a_1 = 0$), and we may express this condition in the form

$$a_1 + B_a a^1 = 0\,.$$

Let us assume that the condition (44) holds; then the functions $a,\ \bar a = Za,\ b,\ \bar b = Zb$ are of order ≤ 1 and we can continue our calculations with

$$Z^2\begin{pmatrix} a \\ b \end{pmatrix} = Z\begin{pmatrix} \bar a \\ \bar b \end{pmatrix} = \cdots + \begin{pmatrix} Xa\cdot\bar a_1 + Xb\cdot\bar a^1 \\ Xa\cdot\bar b_1 + Xb\cdot\bar b^1 \end{pmatrix} =$$

$$= \cdots + \begin{pmatrix} (u_2a_1 + v_2a^1)\bar a_1 + (u_2b_1 + v_2b^1)\bar a^1 \\ (u_2a_1 + v_2a^1)\bar b_1 + (u_2b_1 + v_2b^1)\bar b^1 \end{pmatrix} =$$

$$= \cdots + \begin{pmatrix} a_1\bar a_1 + b_1\bar a^1 & a^1\bar a_1 + b^1\bar a^1 \\ a_1\bar b_1 + b_1\bar b^1 & a^1\bar b_1 + b^1\bar b^1 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \cdots + \begin{pmatrix} \bar a_1 & \bar a^1 \\ \bar b_1 & \bar b^1 \end{pmatrix}\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

$$Z^3\begin{pmatrix} a \\ b \end{pmatrix} = \cdots + \begin{pmatrix} \bar a_1 & \bar a^1 \\ \bar b_1 & \bar b^1 \end{pmatrix}\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix},$$

so that the functions $Z^3a,\ Z^3b$ are of order ≤ 2. Furthermore, denoting by $\tilde a = Z^3a,\ \tilde b = Z^3b$, we have

$$Z^4\begin{pmatrix} a \\ b \end{pmatrix} = Z\begin{pmatrix} \tilde a \\ \tilde b \end{pmatrix} = \cdots + \begin{pmatrix} (u_3a_1 + v_3a^1)\tilde a_2 & (u_3b_1 + v_3b^1)\tilde a^2 \\ (u_3a_1 + v_3a^1)\tilde b_2 & (u_3b_1 + v_3b^1)\tilde b^2 \end{pmatrix} =$$

$$= \cdots + \begin{pmatrix} \tilde{a}_2 & \tilde{a}^2 \\ \tilde{b}_2 & \tilde{b}^2 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix},$$

$$Z^5 \begin{pmatrix} a \\ b \end{pmatrix} = \cdots + \begin{pmatrix} \tilde{a}_2 & \tilde{a}^2 \\ \tilde{b}_2 & \tilde{b}^2 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} Z \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} =$$

$$= \cdots + \begin{pmatrix} \tilde{a}_2 & \tilde{a}^2 \\ \tilde{b}_2 & \tilde{b}^2 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

and the functions $Z^5a,\ Z^5b$ are of order ≤ 3.

In general, the functions $Z^{2k}a,\ Z^{2k}b,\ Z^{2k+1}a,\ Z^{2k+1}b$ are of order $\le k+1$ $(k = 1, 2, \ldots)$. If Z generates a one-parameter group of transformations, then necessarily there exist $k \in \mathbf{N}$ such that the functions $Z^{2k}a,\ Z^{2k}b$ are of order $\le k$.

At last, to obtain an explicit result, let us discuss the case when Z^2a is of order ≤ 1. Then

$$Z^2b = Z^2B(x, u_0, v_0, a) = Z(B_0a + B^0b + B_aZa) =$$
$$= ZB_0a + B_0Za + ZB^0B + B^0ZB + ZB_aZa + B_aZ^2a$$

is of order ≤ 1, too. According to the calculations above, we have

$$\begin{pmatrix} \bar{a}_1 & \bar{a}^1 \\ \bar{b}_1 & \bar{b}^1 \end{pmatrix} \begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix} = 0,$$

which implies

$$\det \begin{pmatrix} \bar{a}_1 & \bar{a}^1 \\ \bar{b}_1 & \bar{b}^1 \end{pmatrix} = 0,$$

so that

$$\bar{b} = \bar{B}(x, u_0, v_0, \bar{a}), \tag{46}$$

excluding the case $\bar{a}_1 = \bar{a}^1 = 0$, i.e. ,

$$(aa_0 + ba^0)_1 = (aa_0 + ba^0)^1 = 0.$$

Now let us assume that

$$\det \begin{pmatrix} a_1 & a^1 \\ \bar{a}_1 & \bar{a}^1 \end{pmatrix} \ne 0.$$

Then the functions u_1, v_1 can be expressed in terms of $x, u_0, v_0, a, \bar{a}$ from the relationships

$$a = a(x, u_0, v_0, u_1, v_1),$$
$$\bar{a} = \bar{a}(x, u_0, v_0, u_1, v_1),$$

hence the functions Zu_1, Zv_1 can be expressed in terms of $x, u_0, v_0, a, \bar{a}, Z\bar{a}$. Since $Z\bar{a} = Z^2a$ is of order ≤ 1, we obtained that the functions Zu_1, Zv_1 are depending only on variables x, u_0, v_0, u_1, v_1, which is a contradiction since $Zu_1 = Xa = = u_2a_1 + v_2a^1$ is depending on u_2. Consequently

$$\det \begin{pmatrix} a_1 & a^1 \\ \bar{a}_1 & \bar{a}^1 \end{pmatrix} = 0,$$

which means

$$\bar{a} = \bar{A}(x, u_0, v_0, a) \tag{47}$$

and now the condition of lemma 4 (equivalently, the condition of the note below) is fulfilled : Since (47),(46) and (45) are valid and $Zu_0 = a, Zv_0 = b, Za = \bar{a}$,the functions

$$a,\ Za = \bar{a},\ Z\bar{a},\ Z^2\bar{a}, \ldots,\quad b,\ Za = \bar{b},\ Z\bar{b},\ Z^2\bar{b}, \ldots$$

are depending only on the variables x, u_0, v_0, a, i.e., only on the variables x, u_0, v_0, u_1, v_1 .

Let us summarize the achievements of this section:

Theorem 3. *Let* $a = a(x, u_0, v_0, u_1, v_{1)}),\ b = b(x, u_0, v_0, u_1, v_1),\ a_1 \neq 0$.

(i) *If a vector field*

$$Z = \sum_0^\infty X^s a \frac{\partial}{\partial u_s} + \sum_0^\infty X^s b \frac{\partial}{\partial v_s}$$

performs an infinitesimal symmetry of Ω*, then necessarily*

$$\begin{pmatrix} a_1 & a^1 \\ b_1 & b^1 \end{pmatrix}^2 = 0\,,$$

which is equivalent to

$$b = B(x, u_0, v_0, a) \quad \text{and} \quad a_1 + a^1 \partial B/\partial a = 0\,.$$

(ii) *If a vector field*

$$Z = \sum_0^\infty X^s a \frac{\partial}{\partial u_s} + \sum_0^\infty X^s b \frac{\partial}{\partial v_s}$$

is given such that $Z^2 a$ *is of order* ≤ 1, $Z^2 b$ *is of order* ≤ 2*, then even* $Z^2 b$ *is of order* ≤ 1 *and if*

$$(aa_0 + ba^0)_1 \neq 0 \quad \text{or} \quad (aa_0 + ba^0)^1 \neq 0\,,$$

then Z *is an infinitesimal symmetry of* Ω.

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