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TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER AND ADJOINED NONLINEAR EQUATIONS

L. M. Berkovich, N. H. Rozov

Dedicated to the memory of Professor Otakar Borůvka

Abstract. Transformations of linear differential equations were studied by Euler, Kummer, Liouville, Lyapunov, S. Lie, Darboux, Halphen, Imshenetskii, Bohl and others. Otakar Borůvka's name takes a deserved place even in this list of famous mathematicians. His work [1] is prominent contribution to classic theory development. In this paper a short review of some works (including not well-known ones) dedicated to stated subject is given. Moreover, Kummer–Liouville's transformations and Euler–Imshenetskii–Darboux's transformations of second order linear equations are considered. Algorithmic procedures of related equations construction are indicated. Also adjoined nonlinear equations are investigated, namely Ermakov's equation and Kummer–Schwarz's equation for which one or other principles of nonlinear superposition take place.

Introduction

Euler [25] (1780), Kummer [34] (1834) and Liouville [37] (1837) set up problems about reductions of linear ordinary differential equations of second order (LODE-2) with variable coefficients to LODE-2 of a preassigned form, in other words, problems of linear equations equivalence. These problems have not only theoretical, but also practical significance, because a constructive solution of many natural science and engineering problems depends on the following problem: could the present equation be transformed to a known form? However, Euler and Kummer have considered transformations of different types. Euler have applied linear differential transformation of dependent variable, Kummer have applied transformation of dependent and independent variables. These transformations have a great significance not only for integration of differential equations. They have a great significance in investigating such problems of qualitative theory of differential equations as oscillating properties, boundedness, stability and solutions

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asymptotic behavior. Already Kummer, with some limitations to equation's coefficients, has shown, that equivalence problem has always solution for local change of variables (Kummer–Liouville transformation (KL)). This result is applied in geometrical (qualitative) theory LODE (see, for example, Arnold, [6]). Kummer's problem for global transformations was solved by Borůvka [1], [2], [3]. This problem for local transformations was investigated methodically by Berkovich [8], [9], [10], who paid a special attention to effective finding the transformations. It should be noted that KL transformation is the most general transformation preserving linearity and the order of equation, since Stäckel–Lie's theorem (Stäckel [46]). Solution of Kummer's equivalence problem was the finding of all corresponding set of transformations KL. And one more: Kummer's problem inevitably leads to nonlinear equations (Ermakov's and Kummer–Schwarz's equations), for which one or other nonlinear superposition principle take place. Thus, it is proved that problems of linear equations transformations could not be solved without using of nonlinear equations.

According to what has been said there was a great interest in transformation problems which resulted in creation of an extensive bibliography. However, many works were forgotten. It is also true for Kummer's work [34], which Borůvka have opened for all mathematicians, and it is true for many other works. Between these forgotten works are works of Euler [25], Imshenetskii [33], in which differential transformation was applied, reopened by Darboux [21]. This transformation should be naturally called Euler–Imshenetskii–Darboux's transformation (EID). There was Ermakov's work [20] in which he introduced and integrated nonlinear equation for the first time, and it was also forgotten. Later this equation arose many times in the qualitative theory of differential equations (Lyapunov [32], Bohl [18], [19], Hamel [36], Yakubovich [48],[49] et al.). However it's Pinney's work [43] that is better known and in this work he had shown explicitly nonlinear superposition principle which took place in Ermakov's equation. In Berkovich and Rozov's work [7] it is stated for the first time that Ermakov has priority in the mentioned equation.

In particular this paper is about these works. The mentioned works were not well known (some of them were even forgotten). They have not an influence on Borůvka, so he has created algebraic theory of global transformations of LODE-2 independently. However the authors' opinion is that they must be considered in the following development of this theory.

The main purposes of this paper are consideration KL and EID transformations, including finding of their mutual relations, and finding of algorithmic procedures for equations "reproduction".

Contents.

In §1 a brief historical review of some works which have any relations with Kummer's problem is given. Apparently, their authors had not known about these relations. In this review, which is not complete, the works, preceding Borůvka's work [1], made by Russian and Soviet mathematicians are considered.

In §2 Kummer's problem, Kummer–Liouville's transformation and adjoined nonlinear equations are considered in detail.

In §3 one way of equations reproduction (related equations construction) is proposed, which is based on using of *basic* differential equation; this basic equation describes KL transformation.

In §4 related equations, that has relation through EID transformation are considered. Also relation between KL transformation and EID transformation is noted. Borůvka's accompanying equation is considered to be an example.

§1. A brief review of some works, that have relations with Kummer's problem

Kummer's [34] and Liouville's [37] works are fundamental in Kummer's problem. However these works are cited frequently at present. We shall start our brief (and incomplete) historical review from Ermakov's work [24].

Let there is an equation

$$y'' + a_0(x)y = 0,\ a_0(x) \in \mathbf{C}(I),\ I = \{x | a \le x \le b\}. \tag{1.1}$$

Ermakov's equation is the equation

$$v'' + a_0(x)v - b_0 v^{-3} = 0,\ b_0 \ne 0 \tag{1.2}$$

Theorem 1 (Ermakov). 1) *General solution of the equation (1.2) can be written in the form*

$$v^2(x) = c_1(y_1 \int \frac{dx}{y_1^2} + c_2 y_1)^2 + \frac{b_0}{c_1} y_1^2,\ c_1 \ne 0; \tag{1.3}$$

where $y_1(x)$ is any nontrivial solution of the equation (1.1), and $c_1 \ne 0$, c_2 are arbitrary constants;

2) *in case where $c_1 = 0$ there are one-parameter solutions families of*

$$v^2(x) = 2\sqrt{-b_0}\, y_1^2 \int \frac{dx}{y_1^2} + c_2 y_1^2; \tag{1.4}$$

3) *The solution of nonsingular Cauchy problem for (1.2) with initial conditions*

$$v(x_0) = v_0 \ne 0,\ v'(x_0) = v_0' \tag{1.5}$$

can be written in the form

$$v^2(x) = y_1^2 + b_0 w^{-2} y_2^2, \tag{1.6}$$

where y_1, y_2 is the equation's (1.1) base, satisfying to conditions

$$y_1(x_0) = y_{10} \ne 0,\ y_2(x_0) = y_{20},\ y_1'(x_0) = y_{10}',\ y_2'(x_0) = y_{20}' \ne 0, \tag{1.7}$$

and $w_0 = y_{10}y'_{20} - y_{20}y'_{10} = const \neq 0$ *is the equation's (1.1) Wronskian.*

Note. Ermakov's work [24] presentation can be found in Berkovich works [8], [10]. The formula (1.4) makes possible to find the solution of the singular Cauchy problem for (1.2), satisfying the conditions $v(x_0) = 0,\ v'(x_0) = \infty$. This formula is absent in [24]. General solution for the equation (1.2) can be written in the form (Pinney [43])

$$v(x) = \sqrt{Ay_2^2 + By_2y_1 + Cy_1^2},\ -4b_0 = B^2 - 4AC, \tag{1.8}$$

where y_1, y_2 is the equation's (1.1) base.

It should be noted, that there are some useful transformations of the Hill's equation

$$\frac{d^2y}{dt^2} + p(t)y = 0, \tag{1.9}$$

in Lyapunov's work [38], where $p(t)$ is periodic function of period T. Also it should be noted, that [38] was particularly inspired by Joukovskii's work [50]. Lyapunov transformation, which leaves an equation in Hill's equations class, has the form

$$y(t) = \omega(t)y_1(\tau),\ \tau = \int\limits_0^t \frac{dt_1}{\omega^2(t_1)}. \tag{1.10}$$

Here $\omega(t)$ is positive T–periodic function, with continuous second derivative (or, in more general case, absolutely continuous first derivative). This indicated change transforms Hill's (1.9) equation to Hill's equation

$$\frac{d^2y_1}{d\tau^2} + p_1(\tau)y_1 = 0, \tag{1.11}$$

where

$$p_1(\tau) = \omega^3[\ddot{\omega} + p\omega]_{t=t(\tau)}. \tag{1.12}$$

Here $p_1(\tau)$ is periodic function of τ of period

$$T_1 = \int\limits_0^T \frac{dt}{\omega^2(t)}. \tag{1.13}$$

Lyapunov had stability criterion of Hill's equation solution for $p(t) \geq 0$ and he had used the transformation (1.10) to find stability criterion for those cases when function $p(t)$ changes its sign in period boundary.

Further, we shall mention Bohl's works [18], [19], [20]. In Bohl's work [18] (see also [19]) the equation

$$\frac{d^2z}{dt^2} + (\alpha + \varphi(t))z = \psi(t), \ \alpha = \text{const}, \tag{1.14}$$

was considered, where φ and ψ were continuous periodic functions. Also corresponding homogeneous equation

$$\frac{d^2z}{dt^2} + Xz = 0, \ X = \alpha + \varphi(t) \tag{1.15}$$

was considered. Then the equation (1.14) general solution can be founded easily if any particular solution of the equation (1.14) and linearly independent solutions of the equation (1.15) are known.

"However we also could investigate the equation (1.15) with the help of the following statement.

If X is a function, defined for all t, and function v, defined for all t and satisfying

$$v^3(v'' + Xv) = c, \ (c = \text{const} \neq 0), \tag{1.16}$$

then functions

$$v \exp(\sqrt{-c}\int\limits_a^t \frac{dt}{v^2}), \ v \exp(-\sqrt{-c}\int\limits_a^t \frac{dt}{u^2}) \tag{1.17}$$

in case of $c < 0$, and functions

$$v \cos(\sqrt{-c}\int\limits_a^t \frac{dt}{v^2}), \ v \sin(\sqrt{-c}\int\limits_a^t \frac{dt}{v^2}) \tag{1.18}$$

in case of $c > 0$, are linearly independent solutions of the equation (1.15)".

Also Bohl had studied solutions representations for linear equation of second order in another his work [20].

"Subsequently we shall use the linear equation (1.15) property, which states that the linear equation (1.15) can be "integrated", if the function $F(t)$ is known and this function can be written in the form

$$F(t) = c_1u^2 + c_2uv + c_3v^2, \tag{1.19}$$

where u, v are linearly independent solutions; and c_1, c_2, c_3 are constants not disappearing simultaneously".

It should be noted, that the function (1.19) is a solution of the selfadjoined linear equation

$$F''' + 4XF' + 2X'F = 0, \qquad (') = \frac{d}{dt}, \tag{1.20}$$

though the equation (1.20) was not written explicitly in [20]. In this work nonhomogeneous equation (1.14) was also considered.

Further, we shall consider Elshin's works. Started in [22], he wrote a whole sequence of works in which he had studied the solutions of linear differential equation of the second order

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{1.21}$$

with continuous in interval (finite or infinite) $a < t < b$ coefficients. A brief review of his investigations of qualitative problems for the equation (1.21) he has given in the work [23].

Solution of these problems by phase method is realized by means of the studying of those functions which are determined by characteristic operator

$$J[\Theta;(p,q)] = (\Theta - \frac{p}{2})' + \Theta^2 + q - \frac{p^2}{4}. \tag{1.22}$$

This operator on all admissible values of Θ is satisfying conditions:

1) Θ is continuous on (a,b); 2) Θ is the function, that $\Theta - p/2$ has continuous derivative.

The equation (1.21) transformation on all admissible values of Θ

$$x = y \exp \int_{t_0}^{t} (\Theta - \frac{p}{2}) d\xi \tag{1.23}$$

results in equation from which y can be found:

$$y'' + 2\Theta y' + J[\Theta;(p,q)]y = 0, \tag{1.24}$$

This equation can be reduced to Lagrangian selfadjoined form

$$(Ky')' + Gy = 0, \tag{1.25}$$

where

$$K[\Theta] = exp(2 \int_{t_0}^{t} \Theta d\xi), \; G[\Theta;(p,q)] = JK \tag{1.26}$$

For all continuous on (a,b) coefficients p and q, and for all admissible values of Θ those Θ for which $KG = 1$ are always exist.

In these conditions the equation (1.21) general integral is

$$x = \frac{C_1 \exp(-\frac{1}{2}\int_{t_0}^{t} p d\xi)}{\sqrt{|\,\omega\,|}} \cos(\int_{t_0}^{t} \omega d\xi + C_2), \tag{1.27}$$

where C_1 and C_2 are arbitrary constants, and variable frequency $\omega(t) = \pm K^{-1}(\Theta)$ can be found from the differential equation

$$\frac{1}{2}\frac{\omega''}{\omega} - \frac{3}{4}\left(\frac{\omega'}{\omega}\right)^2 + \omega^2 = q - \frac{1}{4}p^2 - \frac{1}{2}p' \tag{1.28}$$

If u and v are the equation (1.21) solutions fundamental system which has initial conditions: $u_0 = 1,\ u_0' = 0,\ v_0 = 0,\ v_0' = 1$, when $t = t_0$ then the equation (1.28) general integral is

$$\omega(t) = \frac{A\exp(-\int\limits_{t_0}^{t} p d\xi)}{(Au + Bv)^2 + v^2}, \tag{1.29}$$

A and B are arbitrary constants.

The following expressions

$$\rho(t) = \frac{C_1 \exp(-\frac{1}{2}\int\limits_{t_0}^{t} p d\xi)}{\sqrt{|\,\omega\,|}},\ \varphi(t) = \int\limits_{t_0}^{t} \omega d\xi \tag{1.30}$$

are called **an amplitude** and **a phase** respectively, the function $\omega(t)$ is called **a variable frequency**, and Elshin has noted that they were deduced by Bohl for the first time.

Some important results in qualitative theory were obtained due to the phase method. Elshin's particular approach to this problem was the equation (1.21) reduction not to canonical form (Jacobi's form), but to the equation (1.24). The second specialty of the phase method is transition from the estimate of the equation roots (zeroes) distances to the phases relations.

We finish our brief review of the works preceding Borůvka's work [1] (see also [2] and the papers [3], [4]), by the indication of Yakubovich's work [48] (see also Yakubovich and Starzhinskii's work [49]). Investigating the equation (1.9) solutions stability he used the following expression for the y_1, y_2 base:

$$y_1 = r\cos(n\pi\int\limits_0^t \frac{dt}{r^2}),\ y_2 = r\sin(n\pi\int\limits_0^t \frac{dt}{r^2}). \tag{1.31}$$

Yakubovich again came to Ermakov's equation written in the form

$$p(t) = \frac{n^2\pi^2}{r^4(t)} - \frac{\ddot{r}(t)}{r(t)}, n = 1, 2, ... \tag{1.32}$$

Borůvka [1] has developed original and fruitful theory of LODE–2 transformations apparently from the mentioned works. However the authors suppose that these works must be considered for the further development of the DE qualitative theory.

§2. Kummer's problem

Setting of the problem. Let there are the equations

$$(2.1)\quad y'' + a_1(x)y' + a_0(x)y = 0,\ a_1(x) \in \mathbf{C}^1(I),\ a_0(x) \in \mathbf{C}(I),\ \mathbf{I} = \{x|a < x < b\}$$

$$(2.2)\quad \ddot{z} + b_1(t)\dot{z} + b_0(t)z = 0,\ b_1(t) \in \mathbf{C}^1(J),\ b_0(t) \in \mathbf{C}(\mathbf{J}),\ \mathbf{J} = \{t|\alpha < t < \beta\}$$

where I and J are opened (finite or infinite) intervals and there is Kummer–Liouville's transformation (KL)

$$(2.3)\quad y = v(x)z,\ dt = u(x)dx,\ v, u \in \mathbf{C^2(I)},\ uv \neq 0,\ \forall x \in \mathbf{I} = \{x|a < x < b\}$$

This transformation is the most general point transformation which preserves the order and linearity of equation (Stäckel [46]).

Kummer's problem is to find all set of Kummer–Liouville transformations (2.3) which transform (2.1) to (2.2). LODE–2 global transformations were considered in Borůvka's works [1], [2] (see also [3]), as mentioned above. In Berkovich's works [8], [9], [10] a special attention paid to the effective finding of KL transformation.

Remark 1. Considering Kummer's problem we usually restrict ourselves to the equations (2.1) and (2.2) canonical forms ($a_1 = 0,\ b_1 = 0$). However, not to lose some admissible transformations, we must take into account the possibilities of $b_1 \neq 0$ (even if $a_1 = 0$) and $b_1 = 0$.

Lemma 1. *To reduce (2.2) from (2.1) by means of transformation (2.3) it is necessary and sufficient that factorization through differential operators (noncommutative ones in general case) should take place*

$$(2.4)\qquad Ly \equiv (D - \frac{v'}{v} - \frac{u'}{u} - r_2(t)u)(D - \frac{v'}{v} - r_1(t)u)y = 0,\ D = d/dx,$$

where $r_1(t)$ and $r_2(t)$ are satisfying Riccati equations respectively

$$(2.5)\qquad \dot{r}_1 + r_1^2 + b_1(t)r_1 + b_0(t) = 0,\ \dot{r}_2 - r_2^2 - b_1(t)r_2 + \dot{b}_1 - b_0(t) = 0.$$

Lemma 1 is proved by means of Mammana's theorem about factorization existence (Mammana [39]).

Lemma 2. *To reduce (2.2) from (2.1) by means of transformation (2.3) it is necessary and sufficient that the following formulas should take place*

$$(2.6)\qquad -2v'v^{-1} - u'u^{-1} + b_1(t)u = a_1(x),$$

$$(2.7)\qquad v'' + a_1v' + a_0v - b_0(t)u^2v = 0.$$

This lemma is proved through straightforward calculations or by means of lemma 1.

Theorem 2. *The equation (2.1) can be reduced to (2.2) by means of KL transformation (2.3) if and only if the following conditions take place:*

$$v(x) = |u(x)|^{-1/2} \exp(-\frac{1}{2}\int a_1(x)dx + \frac{1}{2}\int b_1(t)dt); \tag{2.8}$$

$$\{t, x\} + B_0(t)t'^2 = A_0(x), \tag{2.9}$$

where $\{t, x\} = \frac{1}{2}\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2$ *is Schwarz's derivative;*

$$A_0(x) = a_0 - \frac{1}{4}a_1^2 - \frac{1}{2}a_1', \; B_0(t) = b_0 - \frac{1}{4}b_1^2 - \frac{1}{2}\dot{b}_1 \tag{2.10}$$

are the equations (2.1), (2.2) semiinvariants, respectively, over the dependent variables $y = \lambda(x)z, \; z = \mu(t)\xi$ *transformations, where* $\lambda(x), \; \mu(t)$ *are arbitrary functions;*

$$v'' + a_1 v' + a_0 v - b_0 v^{-3} \exp(-2\int_{x_0}^{x} a_1 dx) = 0, \qquad b_1 = 0, \tag{2.11}$$

$$\begin{aligned} &v'' + a_1 v' + a_0 v - \\ &- b_0 v^{-3} \exp(-2\int_{x_0}^{x} a_1 dx)\left(\int_{x_0}^{x} b_1(t(x)) v^{-2} \exp(-\int_{x_0}^{x} a_1 dx) dx\right)^{-2} = 0; \\ &b_1 \neq 0. \end{aligned} \tag{2.12}$$

Proof. Having solved (2.6) over v we get (2.8). Then substituting (2.7) by (2.8) and using the relation $u = t'$, we get (2.9). At least, solving (2.6) over u and substituting (2.7) by obtained expression, we get (2.11), when $b_1 = 0$, or we get (2.12), when $b_1 \neq 0$. □

Lemma 3 (Cayley). *1)* $\{x, t\} = -\{t, x\}\dot{x}^2$, *2)* $\{t(\tau), x\} = \{t, \tau\}\tau_x'^2 + \{\tau, x\}$.

Lemma 4. *The equation (2.9) general solution can be written in the form of the composition* $t = t \circ \tau \circ \xi \circ x$, *where* $t(\tau)$ *is the inversion of some solution* $\tau = w_0(t)$ *of the equation* $\{\tau, t\} = B_0(t)$,

$$\tau(\xi) = \frac{c_1 + c_2\xi}{c_3 + c_4\xi}$$

is the equation $\{\tau, \xi\} = 0$ *general solution, and* $\xi(x) = \omega_0(x)$ *is some particular solution of the equation* $\{\xi, x\} = A_0(x)$, *i. e. in the form*

$$w_0(t) = \frac{c_1 + c_2\omega_0(x)}{c_3 + c_4\omega_0(x)}, \; c_1c_4 - c_2c_3 \neq 0.$$

The proof is based on lemma 3.

Theorem 3. *The set of all transformations (2.3), that give us the Kummer problem solution, is described by the formulas (2.8) and*

$$\int \exp\left(-\int b_1(t)dt\right) z_1^{-2}dt = \frac{c_1 + c_2 \int \exp(-\int a_1 dx) y_1^{-2} dx}{c_3 + c_4 \int \exp(-\int a_1 dx) y_1^{-2} dx} \tag{2.13}$$

where y_1 and z_1 are some particular solutions of the equations (2.1) and (2.2) respectively.

Proof. The equation (2.1) transformation to (2.2) process can be carried out through intermediate equations $\eta''(\xi) = 0$ and $\zeta''(\tau) = 0$. During this process independent variables are making a chain $x \to \xi \to \tau \to t$. Using the theorem 2 and lemma 4, we get the theorem statement. □

An category approach. It is known that the category theory approach can be applied to LODE–2. Borůvka [5] is one of the founders of this theory (see also Neuman [41]). We shall recall some of its notions.

Category – is notion which selects some algebraic properties of collection of morphisms (transformations) of mathematical objects of the same name to each other, with the condition that morphisms collections contain identity mappings and they are closed over consecutive fulfilment of mappings.

A category A consists of $\mathrm{Ob}\, A$ class, which elements are called category A objects, and $\mathrm{Mor}\, A$ class, which elements are called category A morphisms. To every ordered couple of objects $P, Q \in \mathrm{Ob}\, A$ the set $\mathrm{Hom}(P, Q)$ from $\mathrm{Mor}\, A$ is associated. If $\alpha \in \mathrm{Hom}(P, Q)$, then we say, that P is an origin, or domain of definition, of the α morphism, and Q is an endpoint, or range of values, of the α morphism. A morphism can be also designated by means of arrows: $P \to Q$, or $P \xrightarrow{\alpha} Q$.

The following axioms are true in these conditions:

1. Every morphism α belongs to one and only one set $\mathrm{Hom}(P, Q)$.

2. For any two morphisms $\alpha \in \mathrm{Hom}(P, Q)$ and $\beta \in \mathrm{Hom}(Q, S)$ the composition law $\alpha \circ \beta \in \mathrm{Hom}(P, S)$ is defined so, that

a) associative law is fulfilled

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

for every $\gamma \in \mathrm{Hom}(S, U)$.

b) There exist such identical morphisms $I_p \in \mathrm{Hom}(P, P)$, $I_q \in \mathrm{Hom}(Q, Q)$, (which are also called *unitary* morphisms, or *unity elements*), that $I_p\alpha = \alpha I_q = \alpha$.

Ehresmann's groupoid. A category A is called *Ehresmann's* groupoid, if the following axiom takes place equally with the above mentioned axioms:

3. For any $\alpha \in \mathrm{Hom}(P, Q)$ there exists $\alpha^{-1} \in \mathrm{Hom}(Q, P)$ so, that

$$\alpha \circ \alpha^{-1} = I_p, \ \alpha^{-1} \circ \alpha = I_q\,.$$

Brandt's groupoid. Ehresmann's groupoid is called *Brandt's* groupoid if the following axiom take place equally with the three above mentioned axioms:

4. $\mathrm{Hom}(P, Q) \neq \oslash$ for $\forall P, Q$.

Theorem 4. *A LODE–2 category (with KL transformations as the morphisms) is Brandt's groupoid, moreover KL transformations (2.3) are defined by the formulas (2.8) and (2.13).*

The transition from the A equation to B equation can be written in the form

$$AfG(S)h^{-1} = B$$

where f is some transformation which transfers the (2.1) equation to $\ddot{z} = 0$ equation; $G(S)$ is the set of all transformations which transfer the $\ddot{z}(t) = 0$ equation to the $\zeta''(\tau) = 0$ equation; h is some transformation which transfers the (2.2) equation to the $\zeta''(\tau) = 0$ equation, and h^{-1} is inverse transformation to h.

Reduction to the equations with the constant coefficients. We shall consider as the preassigned equation the equation with the constant coefficients

$$\ddot{z} + b_1\dot{z} + b_0 z = 0, \tag{2.14}$$

where b_0 is real constant, and b_1 can be either real or pure imaginary constant.

Lemma 5. *The (2.1) equation, reducible to (2.14) by means of KL transformation, can be factorized:*

a) through the noncommutative operators of the 1st order:

$$Ly \equiv (D - \frac{v'}{v} - \frac{u'}{u} - r_2 u)(D - \frac{v'}{v} - r_1 u)y = 0, \tag{2.15}$$

b) through the commutative operators of the 1st order:

$$\frac{1}{u^2}Ly \equiv (\frac{1}{u}D - \frac{v'}{uv} - r_2)(\frac{1}{u}D - \frac{v'}{uv} - r_1)y = 0, \tag{2.16}$$

where r_1, r_2 are the roots of the characteristic equation

$$r^2 + b_1 r + b_0 = 0. \tag{2.17}$$

The proof is following from lemma 1.

Theorem 5. *The (2.1) equation can be reduced to (2.14) by means of KL transformation and the following conditions take place at the same time:*

a) The (2.1) equation allows a one-parameter Lie group with generator

$$X = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv}y\frac{\partial}{\partial y}; \tag{2.18}$$

b) $u(x)$ is satisfying to the equations

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\delta u^2 = A_0(x),\ \ \delta = b_1^2 - 4b_0, \tag{2.19}$$

$$u''' + 6\frac{u'u''}{u} + 6\frac{u'^3}{u^2} + 4A_0u' - 2A_0'u = 0; \tag{2.20}$$

c) the multiplier v and the kernel u of the KL transformation are related through the formulas

$$v(x) = |u(x)|^{-1/2}\exp(-\frac{1}{2}\int a_1(x)dx + \frac{1}{2}b_1\int udx); \tag{2.21}$$

$$v'' + a_1v' + a_0v - b_0u^2v = 0; \tag{2.22}$$

d) v is satisfying one of the following nonlinear equations

$$v'' + a_1v' + a_0v - b_0v^{-3}\exp(-2\int_{x_0}^{x} a_1dx) = 0,\ b_1 = 0, \tag{2.23}$$

$$\begin{gathered} v'' + a_1v' + a_0v- \\ - b_0v^{-3}\exp(-2\int_{x_0}^{x} a_1dx)\left(b_1\int_{x_0}^{x} v^{-2}\exp(-\int_{x_0}^{x} a_1dx)dx\right)^{-2} = 0; \\ b_1 \neq 0; \end{gathered} \tag{2.24}$$

e) the function

$$R(x) = \exp(-\int a_1dx)u^{-1} \tag{2.25}$$

is the equation (2.1) resolvent and satisfies the equation

$$R''' + 3a_1R'' + (4a_0 + a_1' + 2a_1^2)R' + (2a_0' + 4a_0a_1)R = 0. \tag{2.26}$$

Proof. The one-parametric group G_1 with the generator (2.18) existence follows directly from the (2.1) reducibility to autonomous form (2.14) (see Berkovich [9], [10]). It can be checked by mere calculation that (2.20) can be reduced to (2.26) by means of (2.25). The latter fact can be easily checked up by the following method. It is known that the equation (2.26) general solution can be presented in the form $y = c_1y_1^2 + c_2y_1y_2 + c_3y_2^2$, where $y_1,\ y_2$ are linear independent solutions of the (2.1) equation, and c_1, c_2, c_3 are arbitrary constants. Because of (2.1) reducibility to (2.14) we have

$$y_{1,2}(x) = |\ u\ |^{-1/2}\exp(-\frac{1}{2}\int a_1dx \pm \frac{\sqrt{\delta}}{2}\int udx), \tag{2.27}$$

whence $y_1y_2 = u^{-1}\exp(-\int a_1dx)$, and its coincidence with (2.25).

The symmetries of the linear equation of the 2nd order

Definition 1. We shall say that the (2.1) equation allows Lie one-parameter group of symmetries:

$$x_1 = f(x, y; a), \; y_1 = \varphi(x, y; a), \; a \text{ is a parameter} \tag{2.28}$$

if the (2.28) transformations are the group and the (2.1) equation is invariant to the (2.28).

The Lie algebra for the (2.1) equation has dimension equals to eight, corresponding to Lie group $SL(3, \mathbf{R})$, and has the generators

$$X_1 = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv} y \frac{\partial}{\partial y}; \; X_2 = v\frac{\partial}{\partial y} = y_1 \frac{\partial}{\partial y}, \; X_3 = X_1 \int u dx;$$

$$X_4 = X_2 \int u dx = y_2 \frac{\partial}{\partial y}, \; X_5 = \frac{y}{v} X_1, \; X_6 = \frac{y}{v} X_2 = y \frac{\partial}{\partial y},$$

$$X_7 = (\int u dx)^2 X_1 + (\frac{y}{v} \int u dx) X_2, \; X_8 = \frac{y}{v} \int u dx) X_1 + (\frac{y}{v})^2 X_2,$$

as a base, where $u(x)$ is satisfying to Kummer-Schwarz's equation of 2nd order

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = A_0(x),$$

and $v(x)$ – is satisfying to the formula

$$v(x) = |u(x)|^{-1/2} \exp(-\frac{1}{2} \int a_1(x) dx)$$

In the work (Samokhin [44]) Lie algebra generators are represented in other form.

Kummer's problem turned out to be connected with adjoined nonlinear equations, for which some or other principle of nonlinear superposition takes place.

Definition 2 (Schneider, Winternitz [45])**.** We shall say that for ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

nonlinear superposition principle is true if its general solution can be represented in the form of nonlinear function
a) of particular solutions of nonlinear equation;
b) of arbitrary constants;
c) of particular solutions of adjoined linear equation.

Lemma 7. *The Kummer–Schwarz's equation (2.19) has the following general solution which depends from δ:*

$$u^{(1)}(x) = F(\alpha_1 y_2 + \beta_1 y_1)^{-1}(\alpha_2 y_2 + \beta_2 y_1)^{-1},\ \delta_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0;$$
$$u^{(2)}(x) = F(Ay_2^2 + By_1y_2 + Cy_1^2)^{-1},\ \delta_2 = B^2 - 4AC < 0;$$
$$u^{(3)}(x) = F(\alpha y_2 + \beta y_1)^{-2},\ \delta_3 = 0.$$

Also there are special cases:

$$u^{(4)}(x) = F(\alpha y_2 + \beta y_1)^{-1}y_i^{-1}, i = 1,2;\ \delta_4 = \alpha^2;$$
$$u^{(5)}(x) = Fy_i^{-2},\ \delta_5 = 0.$$

Here $y_2 = y_1\int Fy_1^{-2}dx$, $F = e^{-\int a_1 dx}$, and y_1, y_2 form the base for the (2.1) equation.

The (2.22) equation, which has the basic role in KL transformation theory, will be called **B-equation**.

Lemma 8. *B - equation (2.22) has general solution*

$$v^{(1)}_{(1,2)} = (\alpha_1 y_2 + \beta_1 y_1)^{\frac{1}{2}\pm\frac{b_1}{2\sqrt{\delta_1}}}(\alpha_2 y_2 + \beta_2 y_1)^{\frac{1}{2}\mp\frac{b_1}{2\sqrt{\delta_1}}},\ \delta_1 > 0;$$
$$v^{(2)}_{(1,2)} = \sqrt{Ay_2^2 + By_2y_1 + Cy_1^2}\exp\left(\pm\frac{b_1}{\sqrt{-\delta_2}}\arctan\frac{2Ay_2 + By_1}{\sqrt{-\delta_2}y_1}\right),\ \delta_2 < 0$$
$$v^{(3)}_{(1,2)} = (\alpha y_2 + \beta y_1)\exp\left(\mp\frac{b_1 y_1}{2\alpha(\alpha y_2 + \beta y_1)}\right),\ \delta_3 = 0;$$
$$v^{(4)}_{(1,2)} = (\alpha y_2 + \beta y_1)^{\frac{1}{2}\pm\frac{b_1}{2\alpha}}y_i^{\frac{1}{2}\mp\frac{b_1}{2\alpha}},\ \delta_4 = \alpha^2 > 0,\ i = 1,2;$$
$$v^{(5)}_{(1,2)} = y_i\exp\left(\pm\frac{b_1 y_2}{2y_1}\right),\ \delta_5 = 0,$$

where y_1, y_2 form the base for the (2.1) equation.

If $a_1 = 0$, then B-equation has the form

$$v'' + (a_0(x) - b_0u^2(x))v = 0.$$

If, in addition, $b_1 = 0$, then B-equation corresponds to Ermakov's equation

$$v'' + a_0(x)v - b_0v^{-3} = 0, \tag{2.29}$$

and the (2.19) equation has the form

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + b_0u^2 = a_0(x). \tag{2.30}$$

Corollary. *General solutions for the (2.29) and (2.30) equations have the forms*

$$v^2(x) = c_1y_1^2 + c_2y_1y_2 + c_3y_2^2,\quad -4b_0 = c_2^2 - 4c_1c_3,$$

$$u(x) = (c_1y_1^2 + c_2y_1y_2 + c_3y_2^2)^{-1},\quad -4b_0 = c_2^2 - 4c_1c_3, \tag{2.31}$$

respectively, where y_1, y_2 is the base of the equation $y'' + a_0(x)y = 0$.

§3. Second order related linear differential equations connected by Kummer-Liouville's transformation

It was mentioned in §2 that constructive investigation of LODE-2 can be realized very effectively by means of the KL transformations and in many important cases the equation can be integrated in quadrature or special functions by means of this transformation. How the bilateral sequences of linear equations (related equations families) can be constructed from one generative equation is shown in this paragraph.

The following two equations:

$$y''+a_1(x)y'+a_0(x)y=0,\ a_1(x)\in \mathbf{C^1(I)},\ a_0(x)\in \mathbf{C(I)},\ \mathbf{I}=\{x|a<x<b\}, \tag{3.1}$$

$$y_1''+a_1y_1'+a_0y_1-b_0u^2y_1=0 \tag{3.2}$$

will be called the *related* equations. (They correspond to the equations (2.1) and (2.22)). They are indirectly connected with each other by means of KL transformation.

The procedure for related equations construction will be called *basic procedure* (**B-procedure**).

3.1 Basic procedure and related linear differential equations of 2nd order.

Let us consider an equation:

$$y''+a_0(x)y=0. \tag{a_0}$$

Theorem 6 (Berkovich [10], [11]). *The equation (a_0) induces the following equations sequence*

$$y_k''+a_ky_k=0, \tag{a_k}$$

$$a_k=a_0-\sum_{s=1}^{k}b_{0s}u_s^2,$$

$$b_{0s}=const\ \neq 0,\ a_k=a_{k-1}-b_{0k}u_k^2,$$

where $u_s(x)$ is satisfying to the following sequence of the KS-2 equations:

$$\frac{1}{2}\frac{u_s''}{u_s}-\frac{3}{4}\left(\frac{u_s'}{u_s}\right)^2-\frac{1}{4}\delta_su_s^2=a_{s-1},$$

$$\delta_s=b_{1s}^2-4b_{0s}$$

is discriminant of the characteristic equation:

$$r_s^2\pm b_{1s}r_s+b_{0s}=0.$$

Then linear independent solutions $y_{k(1,2)}$ have the form:

$$y_{k(1,2)} = |u_k|^{-1/2} \exp(\pm(1/2b_{1k} \int u_k dx), \ b_{1k} \neq 0,$$

$$y_{k1} = |u_k|^{-1/2}, \ y_{k2} = |u_k|^{-1/2} \int u_k dx, \ b_{1k} = 0$$

This equations sequence (a_k) can be written in the form

$$y_k'' + [a_0 - \sum_{s=1}^{k} b_{0s}(\alpha_{s1}y_{2s-1} + \beta_{s1}y_{1s-1})^{-2}(\alpha_{s2}y_{2s-1} + \beta_{s2}y_{1s-1})^{-2}]y_k = 0,$$

where y_{1s}, y_{2s} are the base of the equation $y_s'' + a_s y_s = 0$, $(\alpha_{s1}\beta_{s2} - \alpha_{s2}\beta_{s1})^2 = \delta_s$.

Theorem 7. *The equation (a_k) induces the following equations sequence (a_s), $s = \overline{k-1, 0}$, where*

$$a_s = a_k - \sum_{m=s+1}^{k} b_{-m}^0 u_{-m}^2, \ a_{s-1} = a_s - b_s^0 u_{-s}^2,$$

$$u_{-m} = (\alpha_{-m}^1 y_{1m-1} + \beta_{-m}^1 y_{2m-1})^{-1}(\alpha_{-m}^2 y_{1m-1} + \beta_{-m}^2 y_{2m-1})^{-1}$$

$$b_m^0 = -b_m^0, \ b_{-m}^1 = \pm\sqrt{\delta_m}, \ \delta_{-m} = (\alpha_{-m}^1 \beta_{-m}^2 - \alpha_{-m}^2 \beta_{-m}^1)^2 = (b_{-m}^1)^2 - 4b_{-m}^0.$$

and y_{1m-1}, y_{2m-1} form the base of the equation (a_{m-1}).

3.2 Examples.

We shall consider the sequence (we shall call it 0 - *sequence*), which is induced by the equation $y'' = 0$ (with support $a_0 \equiv 0$) by means of B - procedure.

Example 1. Liouville equation:

$$y'' + d(ax^2 + bx + c)^{-2}y = 0.$$

It corresponds to the basic equation $v'' - b_0 u^2 v = 0$, where $d = -b_0$, $u = (ax^2 + bx + c)^{-1}$.

Example 2.

$$y'' - \left[\frac{m(m+1)}{x^2} + \frac{1}{T^4}\right] y = 0, \ T = -\frac{\alpha}{2m+1} x^{-m} + \beta x^{m+1}, m \neq -\frac{1}{2}.$$

This equation has the general solution

$$y = T[M \cosh \frac{x^m}{\alpha T} + N \sinh \frac{x^m}{\alpha T}].$$

Example 3.

$$y'' + \left(\frac{1}{4x^2} + \frac{1}{x^2S^4}\right) y = 0,\ S = \alpha \log x + \beta.$$

This equation has the general solution

$$y = \sqrt{x}S(M \cos \frac{1}{\alpha S} + N \sin \frac{1}{\alpha S}).$$

The following two examples which belong to Ince's equations class also belong to Schrödinger's equations class:

Example 4.

$$y'' + m^2 y + d(a \sin^2 mx + b \sin mx \cos mx + c \cos^2 mx)^{-2} y = 0.$$

Example 5.

$$y'' - m^2 y + d(a \sinh^2 mx + b \sinh mx \cosh mx + c \cosh^2 mx)^{-2} y = 0.$$

Examples 4 and 5 belong to bilateral (0) - sequence

$$(-m^2 + d(a \sinh 2mx + b \cosh 2mx + c)^{-2}) \longleftarrow (-m^2) \longleftarrow (0)$$
$$(0) \longrightarrow (m^2) \longrightarrow (m^2 + d(a \sin 2mx + b \cos 2mx + c)^{-2}).$$

Example 6.

The following equation

$$y'' - [f^2(x) + f'(x) + b_{01} F^{-4} (\alpha_1 \Phi + \beta_1)^{-2} (\alpha_2 \Phi + \beta_2)^{-2}] y = 0,$$

where $f(x)$ a rather arbitrary function, $F = \exp(-\int f dx)$, $\Phi = \int F^2 dx$, is induced by the equation $(-f^2(x) - f'(x))$.

Solutions of all mentioned examples are given in details in Berkovich [10], [11].

§4. Second order related linear differential equations connected with Euler-Imshenetskii-Darboux's transformation

We shall consider known method of integrable equations reproduction. This method is based on differential transformation which is often called Darboux transformation but correctly it should be called Euler–Imshenetskii–Darboux's transformation [25], [33], [21]. Also the connection between the KL and EID transformation will be indicated.

4.1 Euler problem and Euler–Imshenetskii–Darboux's transformation for canonical linear equation of second order.

Let we have the equation

$$y'' + a_0(x) y = 0,\ a_0(x) \in C(I) \tag{4.1}$$

By means of EID transformation

$$z = \beta(x) y' + \alpha(x) y,\ \beta(x),\ \alpha(x) \in C^2(I) \tag{4.2}$$

reduce (4.1) to the preassigned form

$$z'' + b_0(x) z = 0,\ b_0(x) \in C(I). \tag{4.3}$$

Theorem 8 (Euler, Imshenetskii, Darboux).
The equation (4.1) induces the following equations sequence

$$y_k'' + a_k y_k = 0, \tag{4.4}$$

where

$$a_k = a_0 + 2\sum_{s=1}^{k} \alpha'_{s-1}, \tag{4.5}$$

and a_{s-1} is satisfying Riccati equation

$$\alpha'_{s-1} + \alpha^2_{s-1} + a_{s-1} = \lambda_{s-1}, \tag{4.6}$$

or, denoting

$$\alpha_{s-1} = (\ln \tilde{y}_{s-1})' = \frac{\tilde{y}'_{s-1}}{\tilde{y}_{s-1}} \tag{4.7}$$

we shall get

$$a_k = a_0 + 2\sum_{s=1}^{k} (\ln \tilde{y}_{s-1})'' = a_0 + 2\sum_{s=1}^{k} \left(\frac{\tilde{y}'_{s-1}}{\tilde{y}_{s-1}}\right)', \tag{4.8}$$

where $\tilde{y}_{s-1}$ is the equation's

$$y''_{s-1} + (a_{s-1} - \lambda) y_{s-1} = 0, \tag{4.9}$$

eigenfunction corresponding to eigenvalue $\lambda = \lambda_{s-1}$.

4.2 Euler problem and Euler-Imshenetskii-Darboux's transformation for complete linear equations of second order.

Let we have the equation

$$y'' + a_1(x)y' + a_0(x)y = 0,\ a_1 \in C^1(I),\ a_0(x) \in C(I) \tag{4.11}$$

To reduce it to a preassigned form

$$z'' + b_1(x)z' + b_0(x)z = 0,\ b_1(x) \in \mathbf{C^1(I)},\ b_0(x) \in \mathbf{C(I)} \tag{4.13}$$

by means of EID transformation, which is written in the form

$$z = \beta(x)y' - \alpha(x)y,\ \beta(x),\ \alpha(x) \in C^2(I) \tag{4.12}$$

In other words

a) to find (4.12) by means of assigned (4.11) and (4.13).

The mentioned problem has others formulations:

b) to find (4.13) by means of assigned (4.11), (4.12);

c) to find (4.11) by means of assigned (4.12), (4.13).

Theorem 9 (Heading [32], Whiting [47]). *For transformation (4.11) to (4.13) by means of transformation (4.12) it is necessary and sufficient that the formula*

$$\alpha'\beta - \alpha\beta' + \alpha^2 + a_0\beta^2 + a_1\alpha\beta = K\exp(\int(a_1 - b_1)dx) \tag{4.14}$$

takes place, where K is an integration constant. The expression (4.14) is the first integral of the equations system

$$\alpha'' + b_1\alpha' + (b_0 - a_0)\alpha + \beta(a_0' + a_0b_1 - a_0a_1) + 2a_1\beta' = 0, \tag{4.15}$$

$$\beta'' + (b_1 - 2a_1)\beta' + (b_0 - a_0 + a_1^2 - a_1b_1 - a_1')\beta + (a_1 - b_1)\alpha - 2\alpha' = 0 \tag{4.16}$$

4.3 About relation between KL and EID transformations.

Theorem 10. *For the equation (4.11) transformation to itself*

$$z'' + a_1(x)z' + a_0(x)z = 0,\ a_1(x) \in \mathbf{C^1(I)},\ a_0(x) \in \mathbf{C(I)} \tag{4.17}$$

by means of EID transformation (4.12) it is necessary and sufficient that

$$\beta(x) = u^{-1}(x),\ \alpha(x) = v'v^{-1}u^{-1}, \tag{4.18}$$

where $u(x)$ and $v(x)$ are the kernel and multiplier of the KL transformation, respectively, which transforms (4.11) to an equation with constant coefficients.

4.4 Amplitudes. Let y_1, y_2 is the equation (a_0) base, w is its Wronskian. The functions

$$v = \sqrt{y_1^2 + y_2^2},\ v_1 = \sqrt{y_1'^2 + y_2'^2}, \tag{4.19}$$

are called *the first* and *the second amplitudes* of the y_1, y_2 base. The functions $v(x),\ v_1(x)$ are satisfying to nonlinear equations of the second order

$$v'' + a_0(x)v - \frac{w^2}{v^3} = 0, \tag{4.20}$$

$$v_1'' - \frac{a_0'}{a_0}v_1' + a_0(x)v_1 - \frac{w^2a_0^2}{v_1^3} = 0. \tag{4.21}$$

Let apply EID transformation to the equation (4.1)

$$z = \beta(x)y'. \tag{4.12'}$$

Then the equation (4.14) (the first integral) has the form

$$a_0\beta^2 = K\exp\Big(\int \frac{a_0'}{a_0}dx\Big), \tag{4.14'}$$

whence β can be assumed equal to unit. Integration constant K can also be assumed equal to unit. The equation (4.1) is transformed into the equation

$$z'' - \frac{a_0'}{a_0}z' + a_0 z = 0. \tag{4.22}$$

The equation (4.22) can be transformed to

$$\ddot{\zeta} + w^2\zeta = 0,\ \ w = \text{const}\,.$$

by means of KL transformation

$$z = v_1(x)\zeta,\ \ dt = u_1(x)dx,$$

where

$$v_1 =\mid u_1 \mid^{-1/2} \exp\left(-\frac{1}{2}\int -\frac{a_0'}{a_0}dx\right) =\mid u_1 \mid^{-1/2} \sqrt{\mid a_0 \mid}.$$

Corresponding to (2.15) the equation (4.22) factorization has the form

$$\left(D - \frac{v_1'}{v_1} - \frac{a_0'}{a_0} - r_2 a_0 v_1^{-2}\right)\left(D - \frac{v_1'}{v_1} - r_1 a_0 v_1^{-2}\right) z = 0,\ \ r^2 - w^2 = 0.$$

Uncovering factorization, due to differential analogy of Vieta's theorem, we get

$$\left(\frac{v_1'}{v_1} + \frac{a_0'}{a_0} + r_2 a_0 v_1^{-2}\right)\left(\frac{v_1'}{v_1} + r_1 a_0 v_1^{-2}\right) - \left(\frac{v_1'}{v_1} + r_1 a_0 v_1^{-2}\right)' = a_0.$$

The equation (4.21) for v_1 follows from the latest formula.

Using the substitution

$$z = \exp\left(-\frac{1}{2}\int -\frac{a_0'}{a_0}dx\right) Y = \sqrt{|a_0|}Y \tag{4.23}$$

in the (4.22) equation we get the equation

$$Y'' + \Big(a_0 + \frac{1}{2}\frac{a_0''}{a_0} - \frac{3}{4}\frac{a_0'^2}{a_0^2}\Big)Y = 0 \tag{4.24}$$

which, according to Borůvka [1], we call *the accompanying* equation.

The accompanying equation (4.24) solution has the form

$$Y = \frac{y'}{\sqrt{|a_0|}}$$

Let the equation (4.1) be assigned in the form (a_0) by means of the "support" a_0. Then the accompanying equation can be assigned in the form $(\hat{a}_1)$, where the support $\hat{a}_1$ has the form

$$\hat{a}_1(x) = a_0(x) + \frac{1}{2}\frac{a_0''}{a_0} - \frac{3}{4}\left(\frac{a_0'}{a_0}\right)^2,$$

or more concisely

$$\hat{a}_1(x) = a_0(x) + \{\int\limits_{x_0}^{x} a_0(\sigma)d\sigma, x\},\ x_0 \in I$$

where the symbol $\{,\}$ designates Schwarz's derivative of $\int\limits_{t_0}^{x} a_0(\sigma)d\sigma$, which is calculated at the point x.

The following theorem displays the relation between (a_0) and $(\hat{a}_1)$.

Theorem 11 (Borůvka [1]). *For any integral of the equation (a_0), the function $Y(x) = y'(x) : \sqrt{|a_0(x)|}$ is an integral of the equation $(\hat{a}_1)$; conversely, for any integral Y of the equation $(\hat{a}_1)$ the function $Y\sqrt{|a_0(x)|}$ is the derivative y' of exactly one integral y of the equation (a_0).*

Conclusion

Kummer–Liouville's transformation are also applied in investigations of linear differential equations of the order $n > 2$ (see Laguerre [35], Halphen [29], Forsyth [26], Berkovich [10], [12]), including global transformations (see Birkhoff [17], Greguš [28] and especially Neuman [42], who has developed Borůvka's approach). KL transformation is also applied in investigation of nonlinear equations (see, for example, Berkovich [13], Berkovich and Rozov [14], [15], [16], Hanon [31]). It is a very extensive and poorly investigated theme. But consideration of mentioned theme is not the subject of this paper. Kummer's problem has been applied a new in investigations of Sturm–Liouville's and Hill's equations (Lazutkin and Pankratova [36]), and also in investigations of nonlinear equations (Gelfand, Dikii [27]), including Korteveg–de Vries's periodical problem investigation (see., for example, Marchenko [40]).

Application of linear differential transformation to higher order equations was initiated by Imshenetskii [33] and has been waiting for its further development.

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Berkovich Lev M.
Samara State University
Department of Algebra and Geometry
Acad. Pavlov street, 1
443 011 Samara, RUSSIA
E-mail: berk@info.ssu.samara.ru

Rozov Nikolai H.
Moscow State University
Department of Differential Equations
Vorobievy gory
119 899 Moscow, RUSSIA