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SOME OSCILLATORY PROPERTIES OF THE PERTURBED LINEAR DIFFERENTIAL EQUATIONS OF ORDER n

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Abstract. In this paper there are generalized some results on oscillatory properties of the binomial linear differential equations of order $n(\geq 3)$ for perturbed iterative linear differential equations of the same order.

T.A. Čanturija ([1]) and R. Oláh ([8]) have published some remarkable results on oscillatory properties of the solutions of the binomial linear differential equation

$$z^{(n)}(t) + p(t)z(t) = 0, \tag{p}$$

where $p : J = [c, \infty) \to \mathbb{R}, c \geq 1$; is continuous function. F. Neuman has summarized results of the theory of the global equivalence of linear differential equations of the n-th order in the monography [7]. These results enable in certain sense some of the above mentioned results to be generalized for perturbed linear differential equations of the n-th order $(n \geq 3)$

$$I_n(y; a_2) + q(x)y = 0, \tag{q}$$

where $a_2 : [a, \infty) \to \mathbb{R}, q : [a, \infty) \to \mathbb{R}, I_n(y; a_2) = 0$ is the iterative differential equation of the n-th order ([2]) on the interval $I = [a, \infty)$.

The differential equation of the second order

$$u'' + \frac{3}{n+1} a_2(x) u = 0 \tag{a}$$

is called the accompanying equation of the equation (q). A solution y or u of the equation (q) or (a), respectively, is called oscillatory on the interval I if it has arbitrarily large zeros, and it is called, non-oscillatory on I otherwise. Equation (q), resp. (a) is called oscillatory on I if it has at least one oscillatory solution on this interval. Equation (q) is called strictly oscillatory on the interval I if every non-trivial solution of this equation is oscillatory on I.

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Definition 1. We say that the equation (p) is globally equivalent on J to the differential equation (q) on I if there exists an ordered pair $\{t, A\}$ of functions such that

1) $t : I \to \mathbb{R}, A : I \to \mathbb{R}, t \in C^{n+1}(I), A \in C^n(I), t'(x)A(x) \neq 0$ for all $x \in I, t(I) = J$;

2) the function $y : I \to \mathbb{R}$,

$$y(x) = A(x)z[t(x)] \tag{T}$$

is a solution of the equation (q) on I whenever z is a solution of the equation (p) on J.

We shall call the ordered pair $\{t, A\}$ of the functions a carrier of the equivalence of the equation (p) on J to the equation (q) on I and we shall denote this equivalence by

$$(p)J \sim (q)I\{t(x)\} \tag{e}$$

because the following relation holds

$$A(x) = C|t'(x)|^{(1-n)/2}, \qquad x \in I,$$

where $C \neq 0$ is a real constant (see e.g. [3], [6] or [7]).

Let the equation (a) have a solution without zeros on I. There exists subsequently (see [4], p.503) a fundamental system of solutions u_1, u_2 of the equation (a) on interval I with following properties:

(i) $u_1(x) > 0$ on I; $\int_a^\infty [u_1(t)]^{-2} dt = \infty$;
(ii) $u_2(x) > 0$ on I; $\int_a^\infty [u_2(t)]^{-2} dt < \infty$;
(iii) $w[u_1(x), u_2(x)] \equiv 1, x \in I$;

where $w[u_1, u_2] = u_1 u_2' - u_1' u_2$ is the wronskian of the functions u_1, u_2.

Definition 2. The solution u_1 with the property (i) is called the principal solution of the equation (a). The base u_1, u_2 of the linear space of all solutions of the equation (a) with the properties (i)–(iii) is called a normed principal base of solutions of the equation (a) on I.

Lemma (see [5], th.1). *If the accompanying equation (a) of the equation (q) has a solution without zeros on I and*

$$p(t) = q[x(t)][u_1(x(t))]^{2n},$$

where $x : J \to \mathbb{R}$ is the inverse function to the function $t : I \to \mathbb{R}$, $t(x) = u_2(x) : u_1(x), t(I) = J$ and u_1, u_2 is the normed principal base of the solutions of the equation (a) on I, then the relation (e) is true.

Definition 3. The equation (q) is said to have the property A_1 if it holds:

a) if n is even, then the equation (q) is strictly oscillatory on I;

b) if n is odd, then either the equation (q) is strictly oscillatory on I or every non-oscillatory solution y of this equation tends to zero for $x \to \infty$, [along with the first derivative $y\prime$], if the principal solution u_1 of the equation (a) is bounded [and non-increasing] on I.

We introduce the notation (according to [8]):
M_n is a maximum of the function $P_n(x) = x(1-x)\dots(n-1-x)$ on (0,1). Let α_1, α_2 be fixed points of the function $f_n : (0,1) \to \mathbb{R}$,

$$f_n(x) = \frac{\alpha}{(1-x)\dots(n-1-x)}, \qquad 0 < \alpha \le M_n.$$

Theorem 1. *Let the equation (a) have on I a solution without zeros and u_1, u_2 be the normed principal base of solutions of this equation on I. Let $q(x) > 0$ for all $x \in I$ and the following conditions be fulfilled:*

$$\liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \int_x^\infty q(s)[u_1(s)]^{2(n-1)} ds = \frac{\alpha}{n-1}, 0 < \alpha \le M_n; \tag{2}$$

$$\begin{gathered}\liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1-\alpha_1+\epsilon} \int_x^\infty q(s)[u_2(s)]^{\alpha_1-\epsilon}[u_1(s)]^{2(n-1)-\alpha_1+\epsilon} ds > \\ > \alpha_2(n-2)!, \qquad 0 < \epsilon < \alpha_1.\end{gathered} \tag{3}$$

Then the equation (q) has the property A_1.

Proof. By the lemma, equations (p) and (q) are equivalent and the coefficient p of the equation (p) is determined by (1) on I. From this follows that $p(t) > 0$ on J, because is $q(x) > 0$ on I. Further we obtain

$$\begin{aligned}\liminf_{t\to\infty} t^{n-1} \int_t^\infty p(s) ds &= \liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \int_{t(x)}^\infty q[x(s)][u_1(x(s))]^{2n} ds = \\ &= \liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \int_x^\infty q(x)[u_1(x)]^{2n}[u_1(x)]^{-2} dx = \\ &= \liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \int_x^\infty q(x)[u_1(x)]^{2(n-1)} dx = \frac{\alpha}{n-1}, 0 < \alpha \le M_n;\end{aligned}$$

and similarly

$$\begin{aligned}&\liminf_{t\to\infty} t^{n-1-\alpha_1+\epsilon} \int_t^\infty s^{\alpha_1-\epsilon} p(s) ds = \\ &= \liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1-\alpha_1+\epsilon} \int_{t(x)}^\infty \left[\frac{u_2(x(s))}{u_1(x(s))}\right]^{\alpha_1-\epsilon} q(x(s))[u_1(x(s))]^{2n} ds = \\ &= \liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1-\alpha_1+\epsilon} \int_x^\infty q(x)[u_2(x)]^{\alpha_1-\epsilon}[u_1(x)]^{2(n-1)-\alpha_1+\epsilon} dx > \\ &> \alpha_2(n-2)!, \qquad 0 < \epsilon < \alpha_1.\end{aligned}$$

Hence all assumptions of the theorem 1 of [8] are fulfilled. According to this theorem the equation (p) has the property A on J, what means that every solution of this equation is oscillatory on J if n is even, and every solution is either oscillatory on J or

$$\lim_{t\to\infty} z^{(i)}(t) = 0 \qquad \textit{for } i = 0, 1, \dots, n-1, \tag{4}$$

if n is odd.

With regard to the relation (T) this implies that a) is true, as well as the part of b) which concerns of the oscillatory properties of solutions of the equation (q) on I.

Let n be odd and y be a non-oscillatory solution of the equation (q) on I. By the lemma for y we have

$$y(x) = C[u_1(x)]^{n-1} z[t(x)], \qquad x \in I; \tag{5}$$

where z is a solution of the equation (p) on J, $C \in \mathbb{R}, C \neq 0$. It follows from (5) that z is non-oscillatory on J and from the theorem 1 of [8] we get (4). With regard to the assumption that u_1 is bounded on I, we have

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} C[u_1(x)]^{n-1} z[t(x)] = 0.$$

Further we obtain

$$y'(x) = C[u_1(x)]^{n-3}\{\dot{z}[t(x)] + (n-1)u_1(x)u_1'(x)z[t(x)]\},$$

where $\dot{z}$ means $\frac{dz}{dt}$. With regard to our assumptions this implies:
$\lim_{x\to\infty} y'(x) = 0$.

This completes the proof. □

Theorem 2. *Let equation (a) have on I a solution without zeros and u_1, u_2 be the normed principal base of solutions of this equation on I. Let $q(x) > 0$ for all $x \in I$, (2) holds and the following condition be fulfilled*

$$\begin{aligned}&\limsup_{x\to\infty} \Big(\frac{u_1(x)}{u_2(x)} \int_{x_0}^{x} q(s)[u_1(s)]^{n-2}[u_2(s)]^n ds + \\ &+ \left[\frac{u_2(x)}{u_1(x)}\right]^{1-\epsilon} \int_x^\infty q(s)[u_1(s)]^{n-\epsilon}[u_2(s)]^{n+\epsilon-2} ds\Big) > (n-1)!, \qquad \epsilon \in (0, \alpha_1).\end{aligned}$$

Then the equation (q) has the property A_1.

Proof. The method of the proof is similar to the previous theorem. By using of results of the theory of the global equivalence we obtain that all assumptions of the theorem 2 of the work [8] are fulfilled and this implies that our theorem is true. □

In the next part of the work we use the following notation (according to [1]):

M_n^* is the maximal and m_n^* is the minimum of the local maxima of the polynomial

$$P_n{}^*(x) = -x(x-1)\dots(x-n+1)$$

and M_{*n} is the maximal one and m_{*n} is the minimum of the local maxima of the polynomial

$$P_{*n}(x) = x(x-1)\dots(x-n+1).$$

Theorem 3. *Let the equation (a) have on I a solution without zeros and u_1, u_2 be the normed principal base of solutions of this equation on I. Let $q(x) \geq 0$ for all $x \in I$ and the following condition be fulfilled:*

$$\liminf_{x\to\infty} \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \int_x^\infty q(s)[u_1(s)]^{2(n-1)}ds > \frac{M_n^*}{n-1}.$$

Then the equation (q) has the property A_1.

Proof. We proceed again similarly as in the proof of the theorem 1. We prove that all assumptions of the theorem 2.1 of [1] are fulfilled and this implies that our theorem holds. □

Theorem 4. *Let the equation (a) have on I a solution without zeros and u_1, u_2 be the normed principal base of solutions of this equation on I. Let for a certain $\epsilon \in (0,1]$ the following conditions be fulfilled:*

$$(6)\quad \int_a^\infty [u_1(x)u_2(x)]^{-1}[\ln u_2(x) - \ln u_1(x)]^{1-\epsilon}[q(x)(u_1(x)u_2(x))^n - m_n^*]_+ dx = \infty$$

$$(7)\quad \int_a^\infty [u_1(x)u_2(x)]^{-1}[\ln u_2(x) - \ln u_1(x)]^{2}[q(x)(u_1(x)u_2(x))^n - m_n^*]_- dx < \infty,$$

where $[f(x)]_+ = \max\{f(x);0\}, [f(x)]_- = \max\{-f(x);0\}$.

Then the equation (q) is oscillatory on I.

Proof. By the lemma the relation (e) holds and the coefficient p of the equation (p) is determined by (1) on I. From this, (6) and (7) it holds:

$$\begin{aligned}
&\int_c^t t^{n-1}\ln^{1-\epsilon} t \left[p(t) - \frac{m_n^*}{t^n}\right]_+ dt = \int_a^\infty \left[\frac{u_2(x)}{u_1(x)}\right]^{n-1} \ln^{1-\epsilon}\frac{u_2(x)}{u_1(x)} \times \\
&\times \left[q(x)(u_1(x))^{2n} - m_n^* \left(\frac{u_1(x)}{u_2(x)}\right)^n\right]_+ (u_1(x))^{-2}dx = \\
&= \int_a^\infty (u_1(x)u_2(x))^{-1}[\ln u_2(x) - \ln u_1(x)]^{1-\epsilon} \times \\
&\times [q(x)(u_1(x)u_2(x))^n - m_n^*]_+ dx = \infty,
\end{aligned}$$

$$\int_c^\infty t^{n-1}\ln^2 t\left[p(t)-\frac{m_n^*}{t^n}\right]_- dt=\int_a^\infty [u_1(x)u_2(x)]^{-1}[\ln u_2(x)-\ln u_1(x)]\times$$
$$\times\,[q(x)(u_1(x)u_2(x))^n-m_n^*]_- dx<\infty.$$

Hence all assumptions of the theorem 3.1 of [1] are fulfilled. According to this theorem the equation (p) is oscillatory on J. With regard to (T) it implies that the equation (q) is oscillatory on I, too. □

Theorem 5. *Let the equation (a) have a solution without zeros on I and u_1, u_2 be the normed principal base of solutions of this equation on I. Let the following conditions be fulfilled for a certain $\epsilon \in (0,1]$:*

$$\int_a^\infty [u_1(x)u_2(x)]^{-1}[\ln u_2(x)-\ln u_1(x)]^{1-\epsilon}[q(x)(u_1(x)u_2(x))^n+m_{*n}]_- dx=\infty,$$

$$\int_a^\infty [u_1(x)u_2(x)]^{-1}[\ln u_2(x)-\ln u_1(x)]^{2}[q(x)(u_1(x)u_2(x))^n+m_{*n}]_+ dx<\infty.$$

Then the equation (q) is oscillatory on I.

Proof. We prove similarly as in the proof of the previous theorem that all assumptions of the theorem 3.2 of [1] are fulfilled and this implies that our theorem is valid. □

Theorem 6. *Let the equation (a) have a solution without zeros on I and u_1, u_2 be the normed principal base of solutions of this equation on I. Let $n \geq 3$ be odd, $q(x) \leq 0$ for all $x \in I$ and*

$$\liminf_{x\to\infty}\left[\frac{u_2(x)}{u_1(x)}\right]^{n-1}\int_x^\infty |q(x)|[u_1(x)]^{2(n-1)}dx>\frac{M_{*n}}{n-1}.$$

Then either the equation (q) is strictly oscillatory on I or for every non-oscillatory solution y of this equation it holds

$$\lim_{x\to\infty}|y(x)|=+\infty$$

if $\liminf_{x\to\infty} u_1(x)>0$.

Proof. By the lemma, equations (p) and (q) are equivalent and the coefficient p of the equation (p) is determined by (1) on I. From this and from the assumption that $q(x) \leq 0$ on I it follows that $p(t) \leq 0$ on J. Further it holds:

$$\liminf_{t\to\infty} t^{n-1}\int_t^\infty |p(s)|ds=$$
$$=\liminf_{x\to\infty}\left[\frac{u_2(x)}{u_1(x)}\right]^{n-1}\int_{t(x)}^\infty |q(x(s))|[u_1(x(s))]^{2n}ds=$$
$$=\liminf_{x\to\infty}\left[\frac{u_2(x)}{u_1(x)}\right]^{n-1}\int_x^\infty |q(x)|[u_1(x)]^{2n}[u_1(x)]^{-2}dx=$$
$$=\liminf_{x\to\infty}\left[\frac{u_2(x)}{u_1(x)}\right]^{n-1}\int_x^\infty |q(x)|[u_1(x)]^{2(n-1)}dx>\frac{M_{*n}}{n-1}.$$

Hence the assumptions of the theorem 2.2 of [1] are fulfilled. If n is odd then, according to this theorem the equation (p) on J has the property B — what means that either this equation is strictly oscillatory on J or every non-oscillatory solution z of this equation fulfills the following condition

$$\lim_{t\to\infty} |z^{(i)}(t)| = \infty, \qquad i = 0, 1, ..., n-1.$$

By the lemma to any solution y of the equation (q) on I there exists a solution z of the equation (p) on J such that for $x \in I$ it holds

$$y(x) = C[u_1(x)]^{n-1} z[t(x)],$$

where $C \in \mathbb{R}, C \neq 0$.

From this follows that the proof is complete. □

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