ARCHIVUM MATHEMATICUM (BRNO)
Tomus 30 (1994), 277 – 283

NATURAL VECTOR FIELDS ON TANGENT BUNDLES

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Abstract. All natural operators transforming metric fields or connections to the vector fields on tangent bundle are determinated.

1.0 All reasonings in this paper are performed in the category of smooth manifolds of constant dimension with embeddings in the role of morphisms, and in the category of smooth fibre bundles and their morphisms (see [4 pages 42 – 44], or [3]) or in the corresponding real analytic categories. The functors used here are functors from the category of manifolds to the category of fibre bundles. They assign to every manifold the fibre bundle, associated with manifold of frames, which this manifold is the base of. And if the values of one of the functors in question have furthermore the standard fibre Q, we denote it by F_Q. The tangent functor T is one of these. By TTX we always mean the tangent bundle $T(TX)$ with basis TX and projection $\tau_{2,1}(X)$.

If F_1 and F_2 are two functors we will denote:

(1) $C^\infty F_1$ the mapping, which map every manifold to the set of all smooth sections of $F_1(X)$.
(2) $C^\infty(F_1,\ F_2)$ the mapping, which map every manifold to the set of all fibre bundle morphisms $F_1X \longrightarrow F_2X$ over the identity of X.

(The real analytic case is denote in the similar way using C^ω.)

Besides, we shall denote $f_*(g) = f \circ g$. In certain cases, if one of the arguments of a mapping is a manifold, we denote it as a subscript.

1.1 This short note aims to find all natural operators $D : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha(TT))$ for some functors F and $\alpha \in \{\infty, \omega\}$. In such a case, to be natural means: to be regular, and for every embedding related sections to make commutative the square with values of the operator in question on the sections and values of the functors T and TT on this embedding.

1991 *Mathematics Subject Classification*: 58A20, 53A55.
Key words and phrases: natural bundle, natural operator, vector field, connection, metrics.
Received April 3, 1993.

On TTX beside the bundle projection $\tau_{2,1}(X)$ there is a projection $T\tau_{1,0}(X)$ defined to TX, too. For every pair of vector fields α and β, for which besides the upper half of the diagram

(1)
$$\begin{tikzcd}
 & & TX & & \\
TTX \arrow[urr, "\tau_{2,1}(X)"] \arrow[drr, "T\tau_{1,0}(X)"'] & & TX \arrow[ll, "\alpha"'] \arrow[rr, "\beta"'] \arrow[u, "id_{TX}"'] & & TTX \arrow[ull, "\tau_{2,1}(X)"'] \arrow[dll, "T\tau_{1,0}(X)"] \\
 & & TX & &
\end{tikzcd}$$

also the lower one commutes, it holds that all values of the field $\alpha - \beta$ lie in the vertical tangent bundle VTX. This one is canonically isomorphic with $TX \times_X TX$. We denote this isomorphism by $(\tau_{2,1}(X)|_{VTX}^{TX \times_X TX}, i_X)$ and its inversion by j_X. (Of course $\tau_{2,1}(X)|_{VTX}^{TX \times_X TX}$ is not the corestriction of $\tau(X)|_{VTX}$ because $TX \times_X TX$ is not the subset of TTX, but we omit the composition with identification if there is no danger of confusion.) Hence $\alpha - \beta$ can be corestricted on VTM, and the following diagram commutes:

(2)
$$\begin{tikzcd}
TX \arrow[r, "\alpha - \beta|^{VTX}"] \arrow[dr, "\left(id|_{TX}, i_X \circ (\alpha-\beta)|^{VTX}\right)"'] & VTX \arrow[r, "id_{VTX}"] \arrow[d, "(\tau_{2,1}(X)|_{VTX}^{TX \times_X TX}, i_X)"] & VTX \\
 & TX \times_X TX \arrow[ur, "j_X"'] &
\end{tikzcd}$$

1.2 Functors for which we are going to solve the problem in question are assumed to have the two following properties ($\alpha \in \{\infty, \omega\}$):

(1) For all $D : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha(T,T))$ there exists natural function $G : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha(T, F_{\mathbb{R}}))$ (action on $\mathbb{R}$ is trivial), such that $D = G \cdot \widehat{id_T}$, where

$$\widehat{id_T} : \begin{cases} Codom\,(C^\alpha F) & \longrightarrow & Codom\,(C^\alpha(T,T)) \\ (X, s) & \longmapsto & id_{TX} \end{cases}$$

and the dot means the multiplication by a scalar in the tangent bundle.

(2) There exists a natural operator $B : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha TT)$ so that $T\tau_{1,0*}\,(Codom(B)) \circ B = \widehat{id_T}$.

1.3 For all $A : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha TT)$ the mapping $T\tau_{1,0*}Codom(A)\circ A : Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha((T,T)))$ is a natural operator. If the assumption (1.2.1) holds, there exists a natural function G^A such that

$$T\tau_{1,0*}\left(Codom(A)\right)\circ\ A\ = G^A\cdot\widehat{id_T} = T\tau_{1,0*}\left(Codom(A)\right)\circ\ G^A\cdot B$$

where B is an operator whose existence is postulated by (1.2.2). Hence for each manifold X, for all sections s and for $\alpha = A_X(s)$, $\beta = G^A\cdot B_X(s)$, (1.1.1) commutes and consequently (1.1.2) commutes. And since (1.2.1) implies that there exists a natural function H^A such that (we do not write corestrictions explicitly)

$$i_*\left(Codom(A-G^AB)\right)\circ(A-G^AB) = H^A\cdot\widehat{id_T} :$$
$$Codom\,(C^\alpha F)\longrightarrow Codom\,(C^\alpha(T,T))$$

it follows:

$$A = G^A\cdot B + j_*\left(Codom(H^A)\right)\circ\left(\widehat{id_T}, H^A\cdot\widehat{id_T}\right)$$

Conversely, if G^A and H^A are natural functions, then the mapping defined in this way is a natural operator.

1.4 As regards the order of the operator in question, its finiteness is assumed here.

1.5 One natural operator $Codom\,(C^\alpha F) \longrightarrow Codom\,(C^\alpha TT)$ exists for any F. His value is the Liouville's field

$$\xi : \begin{cases} TX & \longrightarrow \qquad TTX \\ j_0^1\alpha & \longmapsto \quad j_0^1\Big(k\mapsto\big(k\cdot j_0^1(\alpha)\big)\Big)\end{cases}$$

(dot is multiplying by a scalar). In the standard local coordinates on TTX its coordinate expression is:

$$\xi = \dot{x}^i\frac{\partial}{\partial\dot{x}^i}$$

However if the value of F are bundles of linear connections or bundles of metrics there exists natural operator the value of which is geodesical semispray. Semispray in general is a vector field ξ on TX, satisfying $T\tau_{1,0}\circ\xi = id_{TX}$. On the manifold with connection any jet $j_0^1z\in TX$ can be represented by a geodetic curve $\alpha_z\in j_0^1z$ The geodesical semispray is a vector field

$$\xi : \begin{cases} TX & \longrightarrow \qquad TTX \\ j_0^1z & \longmapsto \quad j_0^1\big(x\mapsto j_0^1(\alpha_z\circ t(x,-)\big)\end{cases}$$

where $t(x,y) = y-x$ is a translation. In the canonical local coordinates on TTX, it has the form:

$$\xi = \dot{x}^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}\dot{x}^j\dot{x}^k\frac{\partial}{\partial\dot{x}^i}$$

If the value of F is bundle of metrics then the respective connection is the Levi – Civita's one.

Geodesical semispray fulfils (1.2.2). We are going to show that condition (1.2.1) is fulfilled by:

(1) Analytical operators defined on connections.
(For analytical mapping it holds [2] :
Theorem (*Luna*): If f is a real analytic G-invariant function on $\mathbb{R}^n$ and if action of G is completely reducible then there exists a real analytic function g defined on a neighborhood of $p(\mathbb{R}^n) \subset \mathbb{R}^s$ such that $f = g \circ p$, where $p : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ is a projection the components of which are generators of the ring of invariant polynomials on $\mathbb{R}^n$).

(2) First order operators defined on pseudoriemannian metrics.
(On other hand, this does not hold for operators of higher order defined on metrics: if W^i_{jkl} is the curvature of Levi – Civita's connection $W^{ij}_{j\ k}\dot{x}^k$ is the coordinate expression of a vector field which does not satisfy (1.2.2)).

In accordance with the general theory [3, 4] there is a bijection between finite order natural operators and the equivariant mappings of the related standard fibres of the respective fibrations; therefore it is sufficient to study them.

G^r_n is the r-th order differential group in dimension n with canonical coordinates $(a^i_{j_1\dots j_p})_{p\le r}$ and with mappings $(b^i_{j_1\dots j_p})_{p\le r}$ which represents the composition of coordinates with the group inversion. The standard fibre of connections will be denoted $Q = \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ with the accustomed action. Coordinates on it are denoted (Γ^i_{jk}). The standard fibre of metrics will be denoted P; $P \subset \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ with coordinates g_{ij}. $\mathbb{R}^{3n}$ is standard fibre of $TTX \longrightarrow X$ with the coresponding action of G^2_n and with coordinates $(\dot{x}^i, x'^j, \dot{x}'^k)_{i,j,k}$.

If X is a manifold, $T^r_n X$ denotes the manifold of r-jets with source at $0 \in \mathbb{R}^n$ and target in X.

If x are coordinates of any point p on a manifold with a left action of Lie group and point q is a value of this action at (p, g) $g \in G$, we write $q = \bar{p}$ and denote coordinates of q by $\bar{x}$.

2.1 Lemma. *Let* $W^i = \mathbb{R}^n \otimes \big(\mathbb{R}^{n*} \wedge \mathbb{R}^{n*} \otimes (\otimes^{k_i}\mathbb{R}^{n*})\big)$ *with tensor action of* G^1_n. *For every analytic invariant vector* $F = (f^i) : \mathbb{R}^n \times W^1 \times \cdots \times W^r \longrightarrow \mathbb{R}^n$ *there exists a scalar invariant (=invariant function)* g *such that* $f^i = g \cdot x^i$.

Proof. Let z^i be coordinates on R^{n*}, let x^i be coordinates on $\mathbb{R}^n$ and $W^i_{j_1\dots j_k}$ be coordinates on W^{k-2}. The mapping $F^* = z_i f^i : \mathbb{R}^{n*} \times Dom(F) \longrightarrow \mathbb{R}$, is a scalar invariant. Every non zero elementary invariant polynomial on $Dom(F^*)$ contains z_i only in components $z_i x^i$, because every elementary polynomial containing the product $z_i W^i_{j_1\dots j_n}$ must be, by virtue of antisymmetry, zero. The Luna's theorem gives: $F^* = f^i z_i = h(x^i z_i, u_l)$, where u_l are invariant elementary polynomials which do not depend on z_i, consequently $f_i = D_1 h(0, u_l) \cdot x^i = g(u_l) \cdot x^i$

2.2 Theorem. *For every analytic equivariant mapping $F : \mathbb{R}^n \times T^r_n Q \longrightarrow \mathbb{R}^{3n}$ there exist analytic scalar invariants G and H such that the coordinate expression of F is: $F = (\dot{x}^i, G \cdot \dot{x}^j, G \cdot \Gamma^k_{lm} \dot{x}^l \dot{x}^m - H \cdot \dot{x}^k)^n_{i,j,k=1}$*

Proof. Well known theorems about basis of invariants [3,5] say that every invariant vector depends on curvature, its covariant derivative, torsion, and its covariant derivative. Theorem follows from 2.1 and 1.3.

2.3 Corollary. *The only vector fields on TX depending analytically and naturally on connection are linear combination of geodesical semispray and Liouville's vector field coefficients of which are scalar invariants. It has this coordinate expression:*

$$G\dot{x}^i \frac{\partial}{\partial x^i} - (G\Gamma^i_{jk}\dot{x}^j \dot{x}^k - H\dot{x}^i)\frac{\partial}{\partial \dot{x}}$$

If one wants to obtain semispray one must set G equal to constant mapping with value 1: $\left(\dot{x}^i \frac{\partial}{\partial x^i} - (\Gamma^i_{jk}\dot{x}^j \dot{x}^k - H\dot{x}^i)\frac{\partial}{\partial \dot{x}^i}\right)$.

2.4 Remark. One can easily see, after derivating the equations of action by higher order elements of the differential group, that for zero order natural operators the theorem holds in smooth case, too.

2.5 Theorem. *For any invariant vector $F : \mathbb{R}^n \times T^1_n P \longrightarrow \mathbb{R}^n$ there exists an invariant function G such that $F = G\dot{x}$*

Proof. Using well known theorems about invariants, one obtains that every invariant vector $F = (F^s)$ depends on (x^i, g_{ij}) only.

Let us denote $O^s = \{x \in \mathbb{R}^n \times T^1_n X,\ \dot{x}^s(x) = 0, i \neq s \implies g_{is}(x) = 0\}$. Choose $h_1 \in G^1_n$ $a^i_j(h_1) = \delta^i_j - 2\delta^s_j\delta^s_i$. Hence for all $x_0 \in O^s$ it holds $F^s(x_0) = F^s(\bar{x}_0) = \bar{F}^s(x_0) = -F^s(x_0)$, and consequently $F^s|_{O^s} = \widehat{0}$ (constant mapping with value 0).

Let us indicate $M^s = \{x \in \mathbb{R}^n \times T^1_n X,\ \dot{x}^s(x) = 0\}$. Now we will show that the orbit of any point from M^s with respect to subgroup of G^1_n action of which does not change value of F^s intersects O^s. The consequence is that $F^s|_{M^s} = 0$. Choose $x_1 \in M^s$, and let us reason about the system of $n-1$ equations with $n-1$ variable h_i:

$$\left(\sum_{j \neq s} g_{ij}(x_1)\dot{h}_j = -g_{is}(x_1)\right)_{i \neq s}$$

The matrix of this system is a submatrix of a regular matrix; then it has the solution $(h_i)_{i \neq s}$. Let us define $h_s = 0$. The matrix

$$(\delta^i_j + \delta^s_j \delta^i_j \delta^i_p h_p)^n_{i,j=1} = \begin{pmatrix} 1 & 0 & \dots & h_1 & \dots & 0 \\ 0 & 1 & \dots & h_2 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & h_n & \dots & 1 \end{pmatrix} = H$$

is regular, that is why there exists $h_2 \in G^2_n$ such that $b^i_j(h) = H$ (then $a^i_j(h_2) = ((\delta^i_j - \delta^s_j\delta^i_j\delta^i_p h_p)^n_{i,j=1})$. All these matrices H form the mentioned subgroup. We obtain: For all components F^s of an invariant vector there exists function G^s such that $F^s = G \cdot \dot{x}^s$.

Now let us suppose that there exists a point, in which for some $s \neq z$ it holds $G^s = G^z$. Since G^s and G^z are continuous, we can choose a point x_2 satisfying $\dot{x}^s(x_2) \neq 0 \neq \dot{x}^z(x_2)$. Let us compute an action of $h_3 \in G^2_n$ on this point.

$$b^i_j(h_3) = \begin{cases} \delta^i_j & i \neq s \\ 0 & i = s,\ j \neq s, z \\ \frac{1}{x^s(x_2)} & i = s,\ j = s \\ \frac{-1}{x^z(x_2)} & i = s,\ j = z \end{cases}$$

$$\begin{aligned} 0 = F^s(x^1, \dots, x^{s-1}, 0, x^{s+1}, \dots, x^n, g_ij)(x_2) = \\ = F^s(\bar{x}_2) = \bar{F}^s(\dot{x}_2) = \left(\frac{\dot{x}^s G^s}{x^s} - \frac{\dot{x}^z G^z}{x^z}\right)(x_2) = \\ = (G^s - G^z)(x^2) \end{aligned}$$

This is a contradiction, consequently $G^s = G^z = G$ and $F^s = G \cdot x^s$.

Since

$$\begin{aligned} (a^s_p \dot{x}^p G)(a^i_j \dot{x}^j, b^p_i b^q_j g_{pq}) = \\ = \dot{F}^s(a^i_j x^j, b^p_i b^q_j g_{pq}) = \dot{\bar{F}}^s(\dot{x}^i, g_{ij}) = a^s_p \dot{F}^p(\dot{x}^i, g_{ij}) = \\ = a^s_p(\dot{x}^p G)(\dot{x}^i, g_{ij}) \end{aligned}$$

it follows that $G(\dot{\bar{x}}^i, \bar{g}_{ij}) = \bar{G}(\dot{x}^i, g_{ij})$ at every point x satisfying $a^s_j x^j(x) \neq 0$ and because G is continuous, we have: G is a scalar invariant. ☐

2.6 Corollary. *The only vector fields on TX depending smoothly and naturally on some metric field and his first derivative are linear combination of geodesical semispray and Liouville's vector field. At the point $(x^i, \dot{x}^j)^n_{i,j=1}$ it has the coordinate expression:*

$$G\dot{x}^i \frac{\partial}{\partial x^i} - (G\Gamma^i_{jk}\dot{x}^j \dot{x}^k - H\dot{x}^i)\frac{\partial}{\partial \dot{x}}$$

where $\Gamma^i_{jk} = \frac{1}{2}(g_{pj,k} + g_{pk,j} - g_{jk,p})g^{ip}$

The only semispray one obtains setting G equal to constant mapping with value 1 $\left(\text{in coordinates: } \dot{x}^i \frac{\partial}{\partial x^i} - (\Gamma^i_{jk}\dot{x}^j \dot{x}^k - H\dot{x}^i)\frac{\partial}{\partial \dot{x}^i}\right)$.

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