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ON AN OBLIQUE DERIVATIVE PROBLEM INVOLVING AN INDEFINITE WEIGHT

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Abstract. In this paper we derive results concerning the angular distrubition of the eigenvalues and the completeness of the principal vectors in certain function spaces for an oblique derivative problem involving an indefinite weight function for a second order elliptic operator defined in a bounded region.

1. INTRODUCTION

Although there is a relatively large literature devoted to the spectral theory for linear elliptic boundary value problems involving an indefinite weight function, most of the work to date has been concerned with either selfadjoint problems or non–selfadjoint problems arising from perturbations of selfadjoint ones. We refer to [6], [10, 11], [17, 18], and [19-21] for further information. Thus, even for second order operators, such investigations do not apply to such a typical non–selfadjoint problem as the oblique derivative problem. With this in mind, the author [12, 13] has recently initiated an investigation into the spectral theory of quite general non–selfadjoint problems, but only under the assumption that the weight function and its reciprocal were essentially bounded in the set concerned. In this paper we focus our attention upon an oblique derivative problem for a second order operator and establish some basic facts concerning the spectral theory for such an operator under much more general conditions on the weight function than considered in [12, 13].

Accordingly, in this paper we shall be concerned with the spectral theory for the boundary value problem

$$Lu = \lambda\omega(x)u \quad \text{in} \quad \Omega, \tag{1.1}$$

$$Bu = 0 \quad \text{on} \quad \Gamma, \tag{1.2}$$

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where L is a linear elliptic operator of the second order defined in a bounded region $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary Γ, B is a linear differential operator of the first order defined on Γ, and ω is a real–valued function in $L^\infty(\Omega)$ which assumes both positive and negative values. Our assumptions concerning the problem (1.1–2) will be made precise in §§2 and 3; in particular we mention that, unlike [12, 13], we no longer require that $1/\omega(x) \in L^\infty(\Omega)$. Thus, since it is imperative for the success of our method in developing the spectral theory for the problem (1.1–2) to obtain local a priori estimates for solutions of (1.1) near boundary points of the sets $\{x \in \Omega \mid \omega(x) > 0\}$, $\{x \in \Omega \mid \omega(x) < 0\}$, and $\{x \in \Omega \mid \omega(x) = 0\}$, this means that the results of [12, 13] are in general no longer valid for the problem under consideration here as they were based upon the usual a priori estimate for the solution of a half–space elliptic problem (in the sense of [2, 5]) involving operators with constant coefficients which depend upon a parameter that arises from a local rectification of the boundary concerned. Consequently, we have had to introduce new functions, the so–called generalized parabolic cylinder functions studied in [15], as well as new techniques, in order to arrive at the required a priori estimates. These estimates were established in part in [14, 16] and are further extended in §4 of this paper. By means of these estimates, as well as under certain assumptions concerning the problem (1.1–2) (e.g., we require that the resolvent set of (1.1–2) not be empty – see Assumption 3.1 below), we are able to determine the behaviour along various rays in the complex plane of the modified resolvent of a certain compact operator $K^\dagger$ introduced below whose characteristic values and generalized characteristic vectors are precisely the eigenvalues and principal vectors, respectively, of the problem (1.1–2). With this information we are then able to apply the Phragmén–Lindelhöf principle in order to obtain quite general results concerning the angular distribution of the eigenvalues of the problem (1.1–2) as well as the completeness of the principal vectors in certain function spaces.

Finally, in §2 of this paper we introduce some of our basic assumptions and collect some known facts concerning the problem (1.1–2) which we require in the sequel. In §3 we introduce the last of our basic assumptions, state our main theorem, Theorem 3.1, and then introduce the operator $K^\dagger$ mentioned above. In §4 we establish some results concerning the growth of the modified resolvent of $K^\dagger$ along certain rays in the complex plane and these are used in §5 to prove Theorem 3.1.

2. Preliminaries

In this section we are going to introduce some of our basic assumptions concerning the problem (1.1–2) as well as collect some results from [12] which we require in the sequel. We will also introduce some definitions and notation which will be needed later on. Hence, to begin with, we let $x = (x_1, \dots, x_n) = (x', x_n)$ denote a generic point in $\mathbb{R}^n$ and use the notation $D_j = \partial/\partial x_j$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi–index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. For G an open set in $\mathbb{R}^n$ and k a non–negative integer, we let $H^k(G)$ denote the usual Sobolev space of order k related to $L^2(G)$ and let

$(\ ,\ )_{k,G}$ and $\|\ \|_{k,G}$ denote the inner product and norm, respectively, in $H^k(G)$. We shall at times in the sequel also consider the spaces $H^s(G)$, $s > 0$, $s \notin \mathbb{Z}$, and $H^s(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$, where, with $[s]$ denoting the integer part of s, the first space is defined to be that subspace of $H^{[s]}(G)$ consisting of vectors u for which

$$\|u\|_{s,G} = \left[\|u\|^2_{[s],G} + \sum_{|\alpha|=[s]} \iint\limits_{G\,G} \frac{\left|D^\alpha u(x) - D^\alpha u(y)\right|^2}{|x-y|^{n+2(s-[s])}} \, dx dy \right]^{1/2}$$

is finite, while the second space is defined to be the completion of $C_0^\infty(\mathbb{R}^{n-1})$ with respect to the norm $\|u\|_{s,\mathbb{R}^{n-1}} = \left[\int\limits_{\mathbb{R}^{n-1}} \left(1 + |\xi|^2\right)^s |Fu|^2 d\xi \right]^{1/2}$ and Fu denotes the Fourier transform of u. Lastly, we let $(\ ,\ )$ and $\|\ \|$ denote the inner product and norm, respectively, in $\mathcal{H} = L^2(\Omega)$.

Turning now to the problem (1.1–2), we henceforth suppose that:

Assumption 2.1.

1) Γ is of class $C^{2,1}$;
2) $L(x,D) = \sum_{|\alpha|\le 2} a_\alpha(x) D^\alpha$ is uniformly elliptic in Ω with a_α real–valued if $|\alpha| = 2$ and complex–valued otherwise and such that $a_\alpha \in C^{|\alpha|-1,1}(\bar{\Omega})$ for $|\alpha| \ge 1$, $a_\alpha \in L^\infty(\Omega)$ otherwise, where $^-$ denotes closure;
3) $B(x,D) = \sum_{|\alpha|\le 1} b_\alpha(x) D^\alpha$, with b_α real–valued if $|\alpha| = 1$ and complex–valued otherwise, while $b_\alpha \in C^{|\alpha|,1}(\Gamma)$ for $|\alpha| \ge 0$;
4) Γ is non–characteristic to B at each of its points.

Remark 2.1. By employing a known extension procedure, we may suppose from now on that B is defined in $\bar{\Omega}$ with $b_\alpha \in C^{|\alpha|,1}(\bar{\Omega})$.

Thus we see that apart from certain smoothness conditions, Assumption 2.1 ensures that the boundary value problem:

$$Lu = f \quad \text{in} \quad \Omega, \tag{2.1}$$

together with the boundary condition (1.2) is a regular elliptic problem in the sense of [2, 24]. Note that if $L^\star$ denotes the formal adjoint of L and C denotes a boundary operator adjoint to B with respect to the problem (2.1), (1.2) (see [24, p.121], [26]), then the formal adjoint problem of (2.1), (1.2)

$$\begin{aligned} L^\star u &= f \quad \text{in} \quad \Omega, \\ Cu &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{2.2}$$

is also a regular elliptic problem (see [12, §2]).

In $\mathcal{H}$ we now introduce the operator A (resp. A') with domain $D(A)$ (resp. $D(A')$) as follows: we let $D(A)$ (resp. $D(A')$) denote the closure in $H^2(\Omega)$ of the class of functions in $C^2(\bar{\Omega})$ satisfying the boundary condition (1.2) (resp. (2.2)) and put $Au = Lu$ for $u \in D(A)$ (resp. $A'u = L^\star u$ for $u \in D(A')$). Then we know from [2] that

Theorem 2.1. *If $u \in D(A)$, then $\|u\|_{2,\Omega} \leq c\big[\|Au\| + \|u\|\big]$, where the constant c does not depend upon u.*

It follows immediately that A is semi–Fredholm and $\dim\ker A < \infty$. Analogous results also hold for A'. Moreover, if $A^\star$ denotes the Hilbert space adjoint of A, then we know from [12] that $A^\star = A'$, and hence it follows that A and $A^\star$ are actually Fredholm operators, while we also know from [12] that index $A = -$ index $A^\star = 0$. Lastly, we note from [12] that A and $A^\star$ have non–empty resolvent sets and compact resolvents.

Turning to our assumptions concerning $\omega(x)$, let

$$\Omega^+ = \{x \in \Omega \mid \omega(x) > 0\},\ \Omega^- = \{x \in \Omega \mid \omega(x) < 0\},\ \Omega^0 = \{x \in \Omega \mid \omega(x) = 0\}.$$

Assumption 2.2. In the sequel we suppose that:

1) $|\Omega^\pm| > 0$ and $|\Omega^0| \geq 0$, where $|\ |$ denotes n–dimensional Lebesgue measure;
2) $|\Omega^\pm \setminus \text{ int } \Omega^\pm| = 0$, where int = interior;
3) int Ω^+ (resp. int Ω^-) is the union of a finite number of non–empty disjoint regions, say $\{\Omega_r^+\}$ (resp. $\{\Omega_r^-\}$), in each of which $\omega(x)$ is continuous and such that for at least one r, Ω_r^+ (resp. Ω_r^-) contains a closed ball in which $\omega(x)$ is Lipschitz continuous;
4) each component Γ_{rj}^+ (resp. Γ_{rj}^-) of $\partial\Omega_r^+$ (resp. $\partial\Omega_r^-$), where ∂ = boundary, is either a component of Γ or is contained in Ω and is either a component of $\partial\Omega_s^+$ (resp. $\partial\Omega_s^-$) for some $s \neq r$, or a component of $\partial\Omega_s^-$ (resp. $\partial\Omega_s^+$) for some s, or a component of $\partial\Omega_0$ if $|\Omega^0| > 0$, where $\Omega_0 = \Omega\setminus\bar{\Omega}_1$ and $\Omega_1 = \text{ int } \Omega^+ \cup \text{int}\Omega^-$;
5) for each component $\Gamma_{rj}^\pm$ of $\partial\Omega_r^\pm$ either (i) $\Gamma_{rj}^\pm$ is of class $C^{1,1}$ and there is a neighbourhood of $\Gamma_{rj}^\pm$ such that in the intersection of this neighbourhood with $\Omega_r^\pm$, $\omega(x)$ is uniformly continuous and $|\omega(x)|$ has a positive infimum or (ii) $\Gamma_{rj}^\pm$ is of class $C^{2,1}$ and there is a neighbourhood of $\Gamma_{rj}^\pm$ such that in the intersection of this neighbourhood with $\Omega_r^\pm$, $\omega(x) = \omega_{rj}^\pm(x)\big(d_{rj}^\pm(x)\big)^{\gamma_{rj}^\pm}$, where $\omega_{rj}^\pm(x)$ is uniformly continuous and $|\omega_{rj}^\pm(x)|$ has a positive infimum, $d_{rj}^\pm(x) = \text{ dist } \{x, \Gamma_{rj}^\pm\}$, and $\gamma_{rj}^\pm \geq 2$;
6) $\omega(x)$ has been modified on a set of measure zero, if necessary, so that if $|\Omega^0| > 0$, then $\omega(x) = 1$ for $x \in \Gamma_{rj}^\pm$ if $\Gamma_{rj}^\pm \subset \Omega\setminus\bar{\Omega}_0$ and $\omega(x) = 0$ for $x \in \Omega_0$, while if $|\Omega^0| = 0$, then $\omega(x) = 1$ for $x \in \Gamma_{rj}^\pm$ if $\Gamma_{rj}^\pm \subset \Omega$.

It is an immediate consequence of Assumption 2.2 that if $|\Omega^0| > 0$, then Ω_0 is the union of a finite number of non–empty disjoint regions, say $\{\Omega_{0j}\}$, and each component of $\partial\Omega_{0j}$ is either a component of Γ or is contained in Ω and is either a component of $\partial\Omega_s^+$ or a component of $\partial\Omega_s^-$ for some s. Also fixing our attention upon condition (5) of Assumption 2.2, we shall henceforth define $\gamma_{rj}^\pm = 0$ if alternative (i) is valid and let $\gamma_0 = \max E$, where E denotes the subset of $\mathbb{R}$ consisting of all of the $\gamma_{rj}^\pm$.

In order to arrive at our main results and in order to use the results of [16] we also require

Assumption 2.3. It will henceforth be supposed that:

1) if Γ_{rj}^{+} (resp. Γ_{rj}^{-}) coincides with a $\Gamma_{sk}^{\pm}$, then $\gamma_{rj}^{+} = \gamma_{sk}^{\pm}$ (resp. $\gamma_{rj}^{-} = \gamma_{sk}^{\pm}$);

2) if $|\Omega^0| > 0$ and $\Gamma_{rj}^{\pm}$ coincides with a component of $\partial\Omega_0$, then (i) $\Gamma_{rj}^{\pm}$ is of class $C^{n^\star+1,1}$ if $\gamma_{rj}^{\pm} \geq 2$ and of class $C^{n^\star,1}$ otherwise, where $n^\star$ denotes the integer part of $n/2$, and (ii) there is a neighbourhood of $\Gamma_{rj}^{\pm}$ such that in the intersection $U_{rj}^{\pm}$ of this neighbourhood with Ω_0, $a_\alpha(x) \in C^{n^\star,1}(\overline{U_{rj}^{\pm}})$ for $|\alpha| = 2$, where we refer to [1, pp.9–10] for notation.

In the following sections we will require some further terminology. Accordingly, with this in mind we now introduce the following

Definition 2.1. Let X be a complex Hilbert space and S a linear operator in X. Then the set of all non–zero complex numbers λ for which $I - \lambda S$ has an inverse in $\mathcal{L}(X)$ is called the modified resolvent set of S and denoted by $\rho_m(S)$. For $\lambda \in \rho_m(S)$ we let $S_\lambda = S(I - \lambda S)^{-1}$ and call S_λ the modified resolvent of S. A complex number λ is called a characteristic value of S if there exists a $u \neq 0$ in $D(S)$ such that $(I - \lambda S)u = 0$; u is called a characteristic vector of S corresponding to λ. If λ is a characteristic value of S, then a non–zero vector u is called a generalized characteristic vector of S corresponding to λ if for some $p \in \mathbb{N}$, $u \in D(S^p)$ and $(I - \lambda S)^p u = 0$. The set consisting of all generalized characteristic vectors of S corresponding to λ together with the zero vector in X is a subspace of X which we denote by $\mathcal{G}_\lambda(S, X)$. Lastly, the ray $\arg \lambda = \theta$ in the complex plane is said to be a ray of growth of S_λ of order τ if for all λ on the ray, with $|\lambda|$ sufficiently large, we have $\lambda \in \rho_m(S)$ and $\|S_\lambda\|_X \leq c|\lambda|^{-\tau}$ for some τ satisfying $0 < \tau \leq 1$, where c denotes a positive constant and $\| \ \|_X$ denotes the norm in $\mathcal{L}(X)$.

Finally, let T denote the operator of multiplication in $\mathcal{H}$ induced by ω. Then we observe that when $\mathcal{H}$, considered only as a vector space, is equipped with the inner product $(\ , \ )_T = (T., .)$, it becomes an indefinite inner product space [8, p.4]; and in the sequel we shall denote this latter space by $\mathcal{H}_T$ in order to distinguish it from the Hilbert space $\mathcal{H}$. Let $\mathcal{M}$ and $\mathcal{N}$ denote any two subspaces of $\mathcal{H}$. Then we say that $\mathcal{M}$ and $\mathcal{N}$ form a dual pair of subspaces of $\mathcal{H}_T$ if for each $u \neq 0$ in $\mathcal{M}$ there is a $v \in \mathcal{N}$ such that $(u, v)_T \neq 0$ and for each $v \neq 0$ in $\mathcal{N}$ there is a $u \in \mathcal{M}$ such that $(u, v)_T \neq 0$ [8, p.21]. We note from the definition and [8, Lemma 10.3, p.21] that if $\mathcal{M}$ and $\mathcal{N}$ form a dual pair in $\mathcal{H}_T$, then $\dim \mathcal{M} = \dim T\mathcal{M} = \dim \mathcal{N} = \dim T\mathcal{N}$, where dimension is meant in the algebraic sense (see [8, p.2]).

3. The main theorem

In this section we are going to state the main results of this paper (see Theorem 3.1 below) as well as introduce the operator $K^\dagger$ which will play a vital role in the subsequent analysis. However, we must firstly introduce one further assumption. To this end we are now going to give a precise meaning to the eigenvalue problem (1.1–2). Accordingly, recalling the definition of T given in the last paragraph of §2, it is clear that the problem (1.1–2) can be formulated from a purely operator–theoretic point of view, namely, as the spectral problem for the pencil $S(\lambda) = A - \lambda T$, $\lambda \in \mathbb{C}$. Observe that for each λ, $S(\lambda)$ is a closed operator in $\mathcal{H}$ with domain $D(A)$. Let us recall from [25, pp.56–57 and 102] that a point $\mu \in \mathbb{C}$ is called a regular point of $S(\lambda)$ if $S(\mu)$ has an inverse in $\mathcal{L}(\mathcal{H})$. The set of all regular points of $S(\lambda)$ is called the resolvent set of $S(\lambda)$ and is denoted by $\rho(S)$, while the set $\mathbb{C}\backslash\rho(S)$ is called the spectrum of $S(\lambda)$ and is denoted by $\sigma(S)$. It follows from [23, Problem 5.32, p.242] that $\rho(S)$ is open in $\mathbb{C}$, and hence $\sigma(S)$ is closed. A point $\mu \in \mathbb{C}$ is called an eigenvalue of $S(\lambda)$ if there exists a vector $u \neq 0$ in $D(A)$ such that $S(\mu)u = 0$; such a vector u is called an eigenvector of $S(\lambda)$ corresponding to μ. If μ is an eigenvalue of $S(\lambda)$ and N_μ denotes the set of all eigenvectors of $S(\lambda)$ corresponding to μ together with the zero vector in $\mathcal{H}$, then N_μ is a subspace of $\mathcal{H}$ which we call the eigenspace of $S(\lambda)$ corresponding to μ and dim N_μ is called the geometric multiplicity of μ. If μ is an eigenvalue of $S(\lambda)$ and u_0 a corresponding eigenvector, then there may exist vectors $\{u_j\}_1^r$ in $D(A)$ such that $S(\mu)u_j = Tu_{j-1}$ for $j = 1, \dots, r$. Then the vectors $\{u_j\}_1^r$ are said to be associated with the eigenvector u_0 and the set M_μ consisting of all eigenvectors of $S(\lambda)$ corresponding to μ together with their associated vectors and the zero vector in $\mathcal{H}$ forms a subspace of $\mathcal{H}$ which we call the principal subspace of $S(\lambda)$ corresponding to μ and dim M_μ is called the algebraic multiplicity of μ. Any vector $u \neq 0$ in M_μ is called a principal vector for the eigenvalue μ of $S(\lambda)$.

We have seen in §2 that A (as well as $A^\star$) is a Fredholm operator with index zero, and hence it follows from Theorem 2.1, [23, Theorem 5.26, p.238], and Rellich's theorem [3, p.30] that $S(\lambda)$ is a Fredholm operator with index zero for every $\lambda \in \mathbb{C}$. Thus we conclude from [23, Theorem 5.31, p.241] that for $\lambda \in \mathbb{C}$, nul $S(\lambda)$ = def $S(\lambda)$ = constant, with the possible exception of certain isolated points. For our purposes we require that this constant be zero, and hence this leads us to introduce

Assumption 3.1. We suppose from now on that $\rho(S) \neq \emptyset$.

Before stating the main results of this paper, let us introduce the following
Terminology. In the sequel, when we speak of regular points, resolvent set, spectrum, eigenvalues, eigenvectors, associated vectors, principal vectors, eigenspaces, or principal subspaces of the problem (1.1–2), then this will always be meant with respect to the pencil $S(\lambda)$.

The following theorem contains the main results of this paper (we refer to Assumption 2.2 for terminology).

Theorem 3.1. *The spectrum of the problem (1.1–2) consists solely of eigenvalues of finite algebraic multiplicity which form a denumerably infinite subset of $\mathbb{C}$ having no finite points of accumulation. Moreover, for any ϵ satisfying $0 < \epsilon < \pi/2$, there are infinitely many eigenvalues in each of the sectors $|\arg \lambda| < \epsilon$ and $|\arg \lambda - \pi| < \epsilon$, while there are at most a finite number of eigenvalues in each of the sectors $\epsilon \leq \arg \lambda < \pi - \epsilon$ and $-\pi + \epsilon \leq \arg \lambda \leq -\epsilon$. Finally the principal vectors of the problem (1.1–2) are complete in each of the function spaces $L^2(\Omega^+ \cup \Omega^-)$ and $L^2(\Omega^+ \cup \Omega^-; |\omega(x)|dx)$.*

Of course when we speak of the completeness of the principal vectors of (1.1–2) in the spaces just cited, we mean that the restrictions of the principal vectors to the set $\Omega^+ \cup \Omega^-$ are complete (in this vein see Proposition 3.2 below as well as the remarks following Assumption 3.2.).

Since the proof of the theorem depends upon the results of §4, it will be deferred until §5. However, for our purposes we have to analyse Assumption 3.1 in greater detail. To this end let us firstly observe that $0 \in \sigma(S)$ (and hence 0 must be an eigenvalue of $S(\lambda)$) if and only if $0 \in \sigma(A)$, and if $0 \in \sigma(S)$, then $N_0 = \ker A$. Now let us fix our attention upon the boundary value problem

$$L^\star u = \lambda\omega(x)u \text{ in } \Omega \tag{3.1}$$

together with the boundary condition (2.2). We define the eigenvalues, eigenspaces, principal subspaces, spectrum, et cetera of the problem (3.1), (2.2) in an analogous manner to those for the problem (1.1–2) (see Terminology above, where now we are to replace $S(\lambda)$ by its adjoint $S^\star(\lambda)$); and for λ an eigenvalue, we let $N_\lambda^\star$ and $M_\lambda^\star$ denote the eigenspace and principal subspace, respectively, of (3.1), (2.2) corresponding to λ. It is clear that $0 \in \sigma(S^\star)$ (and hence 0 must be an eigenvalue of $S^\star(\lambda)$) if and only if $0 \in \sigma(A^\star)$, and if $0 \in \sigma(S^\star)$, then $N_0^\star = \ker A^\star$. Referring to the last paragraph of §2 for terminology, we have next.

Proposition 3.1. *In order that $\rho(S) \neq \phi$, it is necessary and sufficient that either $0 \in \rho(A)$ or $0 \in \sigma(A)$ and: either (i) N_0 and $N_0^\star$ form a dual pair of subspaces of $\mathcal{H}_T$ or (ii) N_0 and $N_0^\star$ do not form a dual pair in $\mathcal{H}_T$, but* $\dim M_0 < \infty$.

Proof. To begin with, let us prove the sufficiency part of the proposition. Accordingly, suppose firstly that $0 \in \sigma(A)$ and that N_0 and $N_0^\star$ form a dual pair in $\mathcal{H}_T$. Then we know from [12] that $N_0 = M_0$, $N_0^\star = M_0^\star$, and we have the decomposition $\mathcal{H} = M_0 \dot{+} (TM_0^\star)^\perp$, where $\dot{+}$ denotes the direct sum of subspaces of $\mathcal{H}$ and $X^\perp$ denotes the orthogonal complement of the subspace X of $\mathcal{H}$. Suppose next that $0 \in \sigma(A)$, that N_0 and $N_0^\star$ do not form a dual pair in $\mathcal{H}_T$, and that $\dim M_0 < \infty$. Then we assert that M_0 and $M_0^\star$ form a dual pair in $\mathcal{H}_T$ and $\mathcal{H} = M_0 \dot{+} (TM_0^\star)^\perp$. Indeed, this assertion was proved in [12, Theorem 3.3] under the hypothesis that $0 \in \rho(T)$ and this hypothesis was only used in proving the linear independence of certain sets of vectors in $\mathcal{H}$. It follows from a scrutiny of the proof just cited that the assertion remains perfectly valid for the problem under consideration here provided that $|\Omega^0| = 0$. In order to indicate how the proof given in [12] is to be modified in order to prove the assertion when $|\Omega^0| > 0$, it is enough to demonstrate that if $\{p_j\}_1^\ell$ is a sequence of non–negative integers satisfying $1 \leq p_1 \geq p_2 \geq \cdots \geq p_\ell \geq 0$ and $\{z_{ji}\}$, $j = 1, \ldots, \ell$, $i = 0, \ldots, p_j$, is a sequence of vectors in $D(A)$ satisfying $Az_{ji} = Tz_{j,i-1}(z_{j,-1} = 0)$ and such that $\{z_{j0}\}_1^\ell$ is a basis of N_0, then the z_{ji} form a linearly independent set in $\mathcal{H}$. Indeed, if this is not the case, then there is a non–trivial linear combination of these vectors, say u_1, such that $u_1 = 0$. Hence it follows that there is a non–trivial linear combination of the z_{ji}, $i < p_j$, say u_2, such that $u_2 = 0$ almost everywhere in a non–empty open subset $\Omega^\#$ of Ω. If u_2 is a linear combination of only the z_{j0}, then, since L has the unique continuation property [22, Theorem 2.4], we arrive at the contradiction that u_2 is the zero vector in $\mathcal{H}$. If u_2 is not a linear combination of only the z_{j0}, then there is a non–trivial linear combination of the z_{ji}, $i < p_j - 1$, say u_3, such that $u_3 = 0$ almost everywhere in $\Omega^{\#}$. By arguing with u_3 as we did with u_2, and by repeating the steps indicated if necessary, we finally arrive at the contradiction that there is a non–trivial linear combination of the z_{j0} which is equal to zero almost everywhere in $\Omega^{\#}$.

If $0 \in \sigma(A)$, then let $\mathcal{H}_0 = (TM_0^\star)^\perp$ and $A_0 = A|\mathcal{H}_0$. Then it is clear that $D(A_0) = D(A) \cap \mathcal{H}_0$, $D(A) = M_0 \dot{+} D(A_0)$, and $A_0 : D(A_0) \subset \mathcal{H}_0 \to R(A_0)$ (= range of A_0) is densely defined in $\mathcal{H}_0$ and closed, with ker $A_0 = 0$. Moreover, as in [12] we can show that: (**1**) AM_0 and $R(A_0)$ are closed, linearly independent subspaces of $\mathcal{H}$ such that $R(A) = AM_0 \dot{+} R(A_0)$, (**2**) $\mathcal{H} = TM_0 \dot{+} R(A_0)$ and $R(A_0) = (M_0^\star)^\perp$, and (**3**) the mapping $A_0^{-1} : R(A_0) \to \mathcal{H}_0$ is compact. It follows from these results that if $T_0 = T|\mathcal{H}_0$, then $T_0\mathcal{H} \subset R(A_0)$, and hence in $\mathcal{H}_0$ we may now introduce the compact operator $K = A_0^{-1}T_0$. If $0 \in \rho(A)$, then let us write $\mathcal{H}_0$ for $\mathcal{H}$, A_0 for A, T_0 for T, and in $\mathcal{H}_0$ let us introduce the compact operator $K = A_0^{-1}T_0$. Now it was shown in [12] that λ is a non–zero eigenvalue of the problem (1.1–2) if and only if λ is a characteristic value of K (see Definition 2.1 for terminology), and if λ is a non–zero eigenvalue of (1.1–2), then $M_\lambda = \mathcal{G}_\lambda(K, \mathcal{H}_0)$. In light of this last result, the proof of the sufficiency part of the proposition is complete.

Finally, if $\rho(S) \neq \phi$, then by introducing a shift in the spectral parameter λ, if necessary, there is no loss of generality in assuming that $0 \in \rho(S)$, and hence that $0 \in \rho(A)$. Putting $K = A^{-1}T$, the necessity part of the proposition is an immediate consequence of the properties of K cited in the previous paragraph.

□

In the course of proving Proposition 3.1 it was shown that

Corollary 3.1. *The spectrum of the problem (1.1–2) consists of at most isolated eigenvalues of finite algebraic multiplicity.*

In light of Corollary 3.1., we see that there is no loss of generality in supposing that $0 \in \rho(S)$, and hence that $0 \in \rho(A)$, since this situation can always be achieved, if necessary, by means of a shift in the spectral parameter λ.

Assumption 3.2. It will henceforth be supposed that $0 \in \rho(A)$.

For the remainder of this paper we let $K = A^{-1}T$, so that K is a compact operator in $\mathcal{H}$. We recall from the proof of Proposition 3.1 that λ is an eigenvalue of the problem (1.1–2) if and only if λ is a characteristic value of K, and if λ is an eigenvalue of (1.1–2), then $M_\lambda = \mathcal{G}_\lambda(K, \mathcal{H})$.

Suppose next that $|\Omega^0| > 0$ and let $\Omega^\dagger = \Omega \backslash \bar{\Omega}_0$, $\mathcal{H}^\dagger = L^2(\Omega^\dagger)$, where we refer to Assumption 2.2 for terminology. Then let us introduce the extension operator $\mathcal{E} : \mathcal{H}^\dagger \to \mathcal{H}$ by putting $(\mathcal{E}f)(x) = f(x)$ in $\Omega^\dagger$, $(\mathcal{E}f)(x) = 0$ in $\Omega \backslash \Omega^\dagger$ for $f \in \mathcal{H}^\dagger$. Let us also introduce the restriction operator $\mathcal{R}$ mapping $\mathcal{H}$ onto $\mathcal{H}^\dagger$ by putting $\mathcal{R}f = f|\Omega^\dagger$ for $f \in \mathcal{H}$ and in $\mathcal{H}^\dagger$ introduce the compact operator $K^\dagger = \mathcal{R}K\mathcal{E}$.

Proposition 3.2. *Suppose that $|\Omega^0| > 0$. Then λ is a characteristic value of K if and only if λ is a characteristic value of $K^\dagger$. Moreover, if λ is a characteristic value of K, then $\mathcal{R}$ maps $\mathcal{G}_\lambda(K, \mathcal{H})$ onto $\mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$ injectively.*

Proof. Before the beginning the proof let us observe that $Tu = T\mathcal{E}\mathcal{R}u$ for $u \in \mathcal{H}$, and hence $Ku = K\mathcal{E}\mathcal{R}u$ for $u \in \mathcal{H}$. Now suppose firstly that λ is a characteristic value of K. If $u \neq 0$ and $(I - \lambda K)u = 0$, then it follows that $\lambda K^\dagger \mathcal{R}u = \mathcal{R}u \neq 0$, and so λ is a characteristic value of $K^\dagger$ and $\mathcal{R}u$ a corresponding characteristic vector. Suppose next that $u \in \mathcal{G}_\lambda(K, \mathcal{H})$ and that for some integer $p > 1$, $(I - \lambda K)^p u = 0$, $(I - \lambda K)^{p-1} u \neq 0$. Then there exist the vectors $\{u_j\}_0^{p-1}$ in $\mathcal{G}_\lambda(K, \mathcal{H})$, where $u_{p-1} = u$ and $u_0 \neq 0$, such that $(K - \lambda^{-1} I)u_j = u_{j-1}$, where $u_{-1} = 0$. Hence $(K^\dagger - \lambda^{-1} I)\mathcal{R}u_j = \mathcal{R}u_{j-1}$, and so $(K^\dagger - \lambda^{-1} I)^p \mathcal{R}u = 0$, $(K^\dagger - \lambda^{-1} I)^{p-1} \mathcal{R}u \neq 0$. Thus it follows that $\mathcal{R}$ maps $\mathcal{G}_\lambda(K, \mathcal{H})$ into $\mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$, and since a simple argument shows that this mapping is injective, we also have $\dim \mathcal{G}_\lambda(K, \mathcal{H}) \leq \dim G_\lambda(K^\dagger, \mathcal{H}^\dagger)$.

Suppose next that λ is a characteristic value of $K^\dagger$. If $u \neq 0$ and $(I - \lambda K^\dagger)u = 0$, then $\lambda \mathcal{R} K \mathcal{E} u = u = \mathcal{R}\mathcal{E}u$, $T(\lambda K \mathcal{E} u - \mathcal{E}u) = 0$, and hence $\lambda K(K\mathcal{E}u) = K\mathcal{E}u \neq 0$. Thus λ is a characteristic value of K and $K\mathcal{E}u$ a corresponding characteristic vector. Now let $u \in \mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$ and assume that for some integer $p > 1$, $(I - \lambda K^\dagger)^p u = 0$, $(I - \lambda K^\dagger)^{p-1} u \neq 0$. Then there exist the vectors $\{u_j\}_0^{p-1}$ in $\mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$, where $u_{p-1} = u$ and $u_0 \neq 0$, such that $(K^\dagger - \lambda^{-1} I)u_j = u_{j-1}$, where $u_{-1} = 0$. Hence $(K - \lambda^{-1} I)K\mathcal{E}u_j = K\mathcal{E}u_{j-1}$, and so $(K - \lambda^{-1} I)^p K\mathcal{E}u = 0$, $(K - \lambda^{-1} I)^{p-1} K\mathcal{E}u \neq 0$. Thus $K\mathcal{E}$ maps $\mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$ into $\mathcal{G}_\lambda(K, \mathcal{H})$, and since it is easy to show that this mapping is injective, we also have $\dim \mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger) \leq \dim \mathcal{G}_\lambda(K, \mathcal{H})$. Thus we conclude from these results that $\mathcal{R}$ maps $\mathcal{G}_\lambda(K, \mathcal{H})$ onto $\mathcal{G}_\lambda(K^\dagger, \mathcal{H}^\dagger)$ injectively, which completes the proof of the proposition. □

If $|\Omega^0| = 0$, then we will henceforth write $\Omega^\dagger$ for Ω, $\mathcal{H}^\dagger$ for $\mathcal{H}$, $K^\dagger$ for K, and put $\mathcal{E} = \mathcal{R} = I$.

Proposition 3.3. *The range of $K^\dagger$ is dense in $\mathcal{H}^\dagger$.*

Proof. Let $f \in \mathcal{H}^\dagger$ and let ϵ be an arbitrary positive number. Then there exists a $\phi \in C_0^\infty(\Omega)$ such that $\operatorname{supp} \phi \subset \Omega_1$ and $\|\phi - \mathcal{E}f\| < \epsilon$, where supp = support and where we refer to Assumption 2.2 for terminology. Let g denote the element of $\mathcal{H}^\dagger$ defined by $g(x) = \omega(x)^{-1}(L\phi)(x)$ for $x \in \Omega_1$ and $g(x) = 0$ otherwise. Then $A\phi = T\mathcal{E}g$, $u = \mathcal{R}\phi = K^\dagger g$, and $\|u - f\|_{0,\Omega^\dagger} < \epsilon$. Since ϵ is arbitrary, the proof is complete. □

4. The modified resolvent of $K^\dagger$

In this section we are going to investigate the growth of the modified resolvent (see Definition 2.1) of the operator $K^\dagger$ introduced in §3 along certain rays emanating from the origin in $\mathbb{C}$ and the results so obtained will then be used in §5 to prove Theorem 3.1. Accordingly, referring to Assumption 2.2 and to the ensuing statements for terminology, let us henceforth put $\sigma_{rj}^\pm = (\gamma_{rj}^\pm + 2)/2$ and $\sigma_0 = (\gamma_0 + 2)/2$. Then the main result of this section is contained in the following

Proposition 4.1. *If $\theta \in \mathbb{R}$ and $\theta \neq k\pi$ for $k \in \mathbb{Z}$, then the ray* $\arg \lambda = \theta$ *is a ray of growth of $K_\lambda^\dagger$ of order $1/\sigma_0$.*

The proof of the proposition will depend upon certain lemmas which will be presented below, and in order to state these lemmas we require the following definitions. Accordingly, let $x^0 \in \Gamma_{rj}^{\pm}$. Then by hypothesis there is an open set $U \subset \mathbb{R}^n$ and a real–valued function ϕ of $n-1$ variables such that the following conditions hold: (**1**) there is a Cartesian coordinate system $(y_1, \ldots, y_n)$ (in short (y', y_n)) in $\mathbb{R}^n$ about x^0, where the y_n–axis is directed along the inward normal to $\Gamma_{rj}^{\pm}$ at x^0 (i.e., pointing into $\Omega_r^{\pm}$) and the $y_1-, \ldots, y_{n-1}-$ axes lie in the tangent plane to $\Gamma_{rj}^{\pm}$ at x^0 such that $U = \{(y', y_n) | y' \in U',\ |y_n - \phi(y')| < \rho_1\}$, where U' is the open ball $|y'| < \rho_0$ and ρ_0, ρ_1 are positive constants, (**2**) $\phi \in C^{1,1}(\overline{U'})$ (see [1, pp.9 and 10] for notation), and (**3**) $U \cap \Omega_r^{\pm} = \{(y', y_n) \in U | y_n > \phi(y')\}$, $U \cap \Gamma_{rj}^{\pm} = \{(y', y_n) \in U | y_n = \phi(y')\}$, and $U \cap (\mathbb{R}^n \backslash \overline{\Omega_r^{\pm}}) = \{(y', y_n) \in U | y_n < \phi(y')\}$. We call U a neighbourhood and (y', y_n) a system of coordinates connected with the point x^0; and by choosing ρ_1 sufficiently small, if necessary, we shall assume henceforth that U is contained in that neighbourhood of $\Gamma_{rj}^{\pm}$ whose existence is asserted in alternatives (i), (ii) of condition (5) of Assumption 2.2. Moreover, if we let $\mathcal{U} = \{\eta | \eta = (\eta_1, \ldots, \eta_n) = (\eta', \eta_n) \in \mathbb{R}^n,\ |\eta'| < \rho_0,\ |\eta_n| < \rho_1\}$, then U can be mapped onto $\mathcal{U}$ by means of the mapping $\eta_j = y_j$ for $j = 1, \ldots, (n-1)$, $\eta_n = y_n - \phi(y')$, and we refer to (η', η_n) as local coordinates of ordinary type of $\Gamma_{rj}^{\pm}$ at the point x^0. When $\gamma_{rj}^{\pm} \geq 2$, it will however be more convenient for us to work with local coordinates of a different kind than that just defined, and which we introduce in the following way. Let us relabel the $y_1, \ldots, y_{n-1}$ coordinates by $\eta_1, \ldots, \eta_{n-1}$, respectively, let $\eta' = (\eta_1, \ldots, \eta_{n-1})$, denote that portion of $\Gamma_{rj}^{\pm}$ described by $(y', \phi(y'))$, $|y'| < \rho_0$, by $(\eta', \psi(\eta'))$, $|\eta'| < \rho_0$, and let $\nu(\eta')$ denote the interior unit normal to $\Gamma_{rj}^{\pm}$ at $(\eta', \psi(\eta'))$. If we now let $\mathcal{U} = \{\eta | \eta = (\eta_1, \ldots, \eta_n) = (\eta', \eta_n) \in \mathbb{R}^n,\ |\eta'| < \rho_0/2,\ |\eta_n| < \rho_2\}$, where $\rho_2 < 1/4$ is sufficiently small, then $\mathcal{U}$ is diffeomorphic to a subset $U^{\#}$ of U under the mapping $y = (y_1, \ldots, y_n) = (\eta', \psi(\eta')) + \eta_n \nu(\eta')$. We henceforth refer to (η', η_n), as just defined, as local coordinates of normal type of $\Gamma_{rj}^{\pm}$ at the point x^0. Note that if we let e_k denote the unit vector in $\mathbb{R}^n$ parallel to and pointing in the direction of the positive y_k–axis, then in terms of the local coordinates at x^0 we have $D_i = \sum_{k=1}^n e_{ki} (\mathcal{D}_k + \sum_{s=1}^n c_{ks}(\eta)\mathcal{D}_s)$, where $\mathcal{D}_k = \partial/\partial\eta_k$, e_{ki} is the i–th component of e_k with respect to the standard basis of $\mathbb{R}^n$, and $c_{ks}(0) = 0$. Hence if we pass to local coordinates at x^0 and restrict ourselves to the set $\{\eta \in \mathcal{U} | \eta_n > 0\}$, then in this set (1.1) becomes

$$\mathcal{L}(\eta, \mathcal{D})v - \lambda \mathrm{w}(\eta) v = 0 \tag{4.1}$$

where $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$, $\mathcal{L}(\eta, \mathcal{D}) = \sum_{|\alpha| \leq 2} a'_\alpha(\eta) \mathcal{D}^\alpha$, $\mathrm{w}(\eta) = \omega(x(\eta))$ if $\gamma_{rj}^{\pm} = 0$, $\mathrm{w}(\eta) = \omega_{rj}^{\pm}(x(\eta)) \eta_n^{\gamma_{rj}^{\pm}}$ if $\gamma_{rj}^{\pm} \geq 2$, and $v(\eta) = u(x(\eta))$, while if $\Gamma_{rj}^{\pm}$ is also a component of Γ, then, still restricting ourselves to the set $\mathcal{U}$, (1.2) becomes

$$\mathcal{B}(\eta, \mathcal{D})v = 0 \ \text{ on } \ \eta_n = 0, \tag{4.2}$$

where $\mathcal{B}(\eta,\mathcal{D}) = \sum_{|\alpha|\le 1} b'_\alpha(\eta)\mathcal{D}^\alpha$. Lastly, let us note that similar definitions and results hold for the case where x^0 is a point of Γ which does not belong to any of the $\Gamma^\pm_{rj}$ (this situation could arise when $|\Omega^0| > 0$).

In proving Proposition 4.1 we wish to make use of the results of [14] and to this end we require some further terminology. Accordingly, for $\theta \in \mathbb{R}$, let $\Xi(\theta)$ denote the ray in the complex plane emanating from the origin which makes an angle θ with the positive real axis and suppose that $x^0 \in \Gamma^\pm_{rj}$ with $\gamma^\pm_{rj} \ge 2$. Then in terms of the local coordinates of normal type at x^0, (1.1) goes over into (4.1) for $\eta \in \mathcal{U},\ \eta_n > 0$, while (1.2) goes over into (4.2) if $\Gamma^\pm_{rj}$ is a component of Γ. Assuming now that $0 \ne \lambda \in \Xi(\theta)$ and that θ satisfies the hypothesis of Proposition 4.1, let us fix our attention upon the equation

$$\Big[\mathcal{L}_0(0,\mathcal{D}) - q^2\chi(\eta_n)\Big] v = 0 \quad \text{for} \quad \eta_n \ge 0, \tag{4.3}$$

where $\mathcal{L}_0(0,\mathcal{D}) = \sum_{|\alpha|=2} a'_\alpha(0)\mathcal{D}^\alpha$, $q^2 = \lambda\omega_0$, ω_0 denotes the limit as $x \to x^0$, $x \in \Omega^\pm_r$, of $\omega^\pm_{rj}(x)$, and putting $\eta_n = t$, $\chi(t)$ is a real–valued function defined in $t \ge 0$ which satisfies the following conditions: (i) $\chi(t)$ is of class C^2 in $0 \le t \le 1$ and of class C^1 in $0 \le t < \infty$, (ii) $\chi(t) > 0$ for $t > 0$, $\chi(t) = 1$ for $t \ge 1$, and $\chi(t) = t^\gamma$ for $0 \le t \le 1/2$, where $\gamma = \gamma^\pm_{rj}$. If in (4.3) we make a Fourier transformation with respect to $\eta'\big(\eta' \to \xi' = (\xi_1,\dots,\xi_{n-1})\big)$ and replace η_n by t, then we arrive at the differential equation

$$\begin{aligned}\Big[\mathcal{L}_0(0,i\xi',d/dt) - q^2\chi(t)\Big] V \qquad\qquad & \\ = p_0V'' + ip_1(\xi')V' - \big(p_2(\xi') + q^2\chi(t)\big)V = 0 & \\ 0 \le t < \infty, \quad ' = d/dt, &\end{aligned} \tag{4.4}$$

where $p_0 \ne 0$ and $p_1(\xi')$ (resp. $p_2(\xi')$) is a homogeneous polynomial of degree 1 (resp. 2) in the ξ_j. Let $\gamma = \gamma^\pm_{rj}$, $\sigma = \sigma^\pm_{rj}$, $\mu^2 = q^2/p_0$, $a(\xi') = \big(p_1(\xi')^2 - 4p_0p_2(\xi')\big)/4p_0^2$, $\nu + \gamma^{-1} = \gamma^{-2/\sigma}\mu^{-2/\sigma}a(\xi')$, and $\tau = (\tau_1,\dots,\tau_{n-1})$, where $\tau_k = \xi_k\mu^{-1/\sigma}$ for $k = 1,\dots,(n-1)$ and where here and in the sequel we always assign to $\arg\mu^2$ its principal value when $\mu^2 \ne 0$ and adopt the convention that $z^\alpha = \exp\big\{\alpha(\log|z| + i\arg z)\big\}$ for $\alpha, z \in \mathbb{C}$, $z \ne 0$, and for which $\arg z$ has been specified. Then we know from [14] that there exists the constant $\kappa(x^0,\theta) > 1$ such that when $|\lambda|$ exceeds a certain positive number not depending upon ξ', then a fundamental set of solutions of (4.4) in the interval $0 \le t \le 1/2$ is given by

$$\begin{aligned} v_+(t,\tau,\mu) &= \exp\big\{-itp_1(\xi')/2p_0\big\}D_{\nu,\gamma}(\gamma^{1/\sigma}\mu^{1\sigma}t),\\ v_-(t,\tau,\mu) &= \exp\big\{-itp_1(\xi')/2p_0\big\}D^\dagger_{\nu,\gamma}(\gamma^{1/\sigma}\mu^{1\sigma}t)\end{aligned} \tag{4.5}$$

for $0 < |\tau| < \kappa(x^0,\theta)$, where $D_{\nu,\gamma}(z)$ and $D^\dagger_{\nu,\gamma}(z)$ are the generalized parabolic cylinder functions defined in [15] if $\gamma > 2$, while $D_{\nu,2}(z)$ and $D^\dagger_{\nu,2}(z)$ are the parabolic cylinder functions $D_\nu(z)$ and $D_{-\nu-1}(iz)$, respectively, defined in [27], and by

$$v_\pm(t) = G(t)^{-1/4}\exp\Big\{i\int_0^t \lambda_\pm(s)ds\Big\}\big(1+u_\pm(t)\big) \tag{4.6}$$

for $|\tau| \geq \kappa(x^0,\theta)$, where $G(t) = G(t,\xi',\lambda) = a(\xi') - \mu^2 x(t)$, $\lambda_\pm(s) = \lambda_\pm(s,\xi',\lambda) = \big(-p_1(\xi')/2p_0\big) \pm G(t,\xi',\lambda)^{1/2}$, $\big|u_\pm(t)\big| < 1$ independently of t, ξ', and λ, and where $0 < \arg G(t) < 2\pi$, $Im\lambda_+(s) > 0$, and $Im\lambda_-(s) < 0$. Finally, if $\Gamma_{rj}^\pm$ is also a component of Γ, then we let

$$\mathcal{B}_0(0,\mathcal{D}) = \sum_{|\alpha|=1} b'_\alpha(0)\mathcal{D}^\alpha. \tag{4.7}$$

We are now in a position to deal with the proof of Proposition 4.1. Accordingly, for $0 \neq \lambda \in \Xi(\theta)$, where θ is the angle of the proposition, and for $s > 0$, $\epsilon > 0$, let $|||u|||_{s,\epsilon,G} = \|u\|_{s,G} + |\lambda|^{s/2\epsilon}\|u\|_{0,G}$ for every open set $G \subset \mathbb{R}^n$ and vector $u \in H^s(G)$. Suppose next that the component $\Gamma_{rj}^\pm$ of $\partial\Omega_r^\pm$ is also a component of Γ, that $x^0 \in \Gamma_{rj}^\pm$, and that U is a neighbourhood connected with the point x^0. Then for $u \in H^2(\Omega)$, with supp $u \subset U$ if $\gamma_{rj}^\pm = 0$ and supp $u \subset U^\#$ otherwise, let us put

$$|||Bu|||'_{s,\epsilon} = |\lambda|^{(s-2)/2\epsilon}\|\mathcal{B}v\|_{1/2,\mathbb{R}^{n-1}} + |\lambda|^{(s-3/2)/2\epsilon}\|\mathcal{B}v\|_{0,\mathbb{R}^{n-1}}, \tag{4.8}$$

where in terms of the local coordinates η of x^0 (we henceforth suppose, unless otherwise stated, that the local coordinates are of ordinary type if $\gamma_{rj}^\pm = 0$ and of normal type if $\gamma_{rj}^\pm \geq 2$, whether $\Gamma_{rj}^\pm$ is a component of Γ or not), $v(\eta) = u\big(x(\eta)\big)$, $\mathcal{B}(\eta,\mathcal{D})$ is defined in the statement following (4.2), and $\mathcal{B}v$ is to be interpreted in the sense of trace on the hyperplane $\eta_n = 0$. For the case where x^0 is a point of Γ which does not belong to any of the $\Gamma_{rj}^\pm$ (a situation which could arise when $|\Omega^0| > 0$) and U is a neighbourhood connected with the point x^0, we put $\|Bu\|' = \|\mathcal{B}v\|_{1/2,\mathbb{R}^{n-1}}$ for $u \in H^2(\Omega)$ with supp $u \subset U$, where all terms are defined just as before and the local coordinates are taken to be of ordinary type. Hence referring to Assumption 2.2 and §3 for terminology, we now have

Lemma 4.1. *Suppose that the hypothesis of Proposition 4.1 is satisfied. Then for each point $x^0 \in \Omega_1$ there exists a neighbourhood $X \subset\subset \Omega_1$ of this point and positive numbers c_0, c_1 such that for $\lambda \in \Xi(\theta)$ and $|\lambda| \geq c_1$,*

$$|||u|||_{2,1,\Omega^\dagger} \leq c_0\|(L-\lambda\omega)u\|_{0,\Omega^\dagger}, \tag{4.9}$$

$$|||u|||_{3/2,1,\Omega^\dagger} \leq c_0|\lambda|^{-1/4}\|(L-\lambda\omega)u\|_{0,\Omega^\dagger} \tag{4.10}$$

for every $u \in H^2(\Omega)$ with supp $u \subset X$.

Proof. That part of the lemma concerning (4.9) has been proved in [12, Lemma 4.1], while (4.10) follows from (4.9) by interpolation [24, Proposition 2.3, p.19]. □

Lemma 4.2. *Suppose that the hypothesis of Proposition 4.1 is satisfied and that $\Gamma_{rj}^{\pm}$ is a component of Γ. Then for each point $x^0 \in \Gamma_{rj}^{\pm}$ there exists a neighbourhood X of this point, with $X \cap \Omega \subset \Omega^{\dagger}$, and positive numbers c_0, c_1 such that for $\lambda \in \Xi(\theta)$ and $|\lambda| \geq c_1$,*

$$|||u|||_{2,\sigma_{rj}^{\pm},\Omega^{\dagger}} \leq c_0\Big[\|(L-\lambda\omega)u\|_{0,\Omega^{\dagger}} + |||Bu|||'_{2,\sigma_{rj}^{\pm}}\Big], \tag{4.11}$$

$$|||u|||_{3/2,\sigma_{rj}^{\pm},\Omega^{\dagger}} \leq c_0\Big[|\lambda|^{-1/4\sigma_{rj}^{\pm}}\|(L-\lambda\omega)u\|_{0,\Omega^{\dagger}} + |||Bu|||'_{3/2,\sigma_{rj}^{\pm}}\Big] \tag{4.12}$$

for every $u \in H^2(\Omega)$ with supp $u \subset X$.

Proof. If $\gamma_{rj}^{\pm} = 0$, then that part of the lemma concerning (4.11) has been proved in [12, Lemma 4.1], while (4.12) follows from (4.11) by interpolation. Hence we suppose from now on that $\gamma_{rj}^{\pm} \geq 2$; and to simplify the proof we shall also suppose that we are dealing with the case $x^0 \in \Gamma_{rj}^{+}$. Now let U be a neighbourhood connected with the point x^0, let $X \subset\subset U^{\#}$ be a neighbourhood of x^0, and let $u \in H^2(\Omega)$ such that supp $u \subset X$. Then passing to local coordinates at x^0, we know from [14] that there exist positive constants k_0, k_1 such that for $|\lambda| \geq k_1$,

$$|||v|||_{2,\sigma,\mathbb{R}_+^n} \leq k_0\Big[\|(\mathcal{L}_0 - q^2\chi)v\|_{0,\mathbb{R}_+^n} + |||B_0 u|||'_{2,\sigma}\Big], \tag{4.13}$$

where k_0 does not depend upon X, λ, nor u, $v(\eta) = u\big(x(\eta)\big)$, $\sigma = \sigma_{rj}^{+}$, $\mathbb{R}_+^n = \big\{(\eta', \eta_n) \in \mathbb{R}^n | \eta_n > 0\big\}$, $|||B_0 u|||'_{2,\sigma}$ is given by the right side of (4.8) with $\mathcal{B}(\eta, \mathcal{D})$ replaced by $\mathcal{B}_0(0, \mathcal{D})$ (see (4.7)), s by 2, and ϵ by σ, and all remaining terms are defined above (see (4.3) in particular). Note that in Theorem 2.1 of [14] an estimate similar to (4.13) was given and which differed only from (4.13) in that the term $|||B_0 u|||'_{2,\sigma}$ was replaced by $|||B_0 u|||'_{2,1}$. However, in view of the equations (8.3–4), (8.7–8), (9.7) of [14] and the fact that the right side of (9.5) of [14] can be replaced by $C\Big[|||f|||^2_{\ell-2,q,1} + \|h\|^2_{\ell-m_B-1/2,\mathbb{R}^{n-1}}\Big]$, we see that (4.13) is actually a sharper version of the estimate established in [14]. A standard argument involving an extension of v to $\mathbb{R}^n$ and the use of the Poincaré inequality [3, p.73] shows that

$$\big\|(\mathcal{L}_0 - q^2\chi)v\big\|_{0,\mathbb{R}_+^n} \leq \Phi(d)\Big[k_2\|v\|_{2,\mathbb{R}_+^n} + |\lambda|\,\|\chi v\|_{0,\mathbb{R}_+^n}\Big] + \big\|(\mathcal{L} - \lambda \mathrm{w})v\big\|_{0,\mathbb{R}_+^n},$$

where d denotes the diameter of X, $\Phi(d) \to 0$ and $d \to 0$ and the constant k_2 does not depend upon X, λ, nor u, while we can argue as in [5, §4] to show that

$$|||B_0 u|||'_{2,\sigma} \leq k_3\Big(\Phi(d) + |\lambda|^{-1/2\sigma}\Big)|||v|||_{2,\sigma,\mathbb{R}_+^n} + |||Bu|||'_{2,\sigma},$$

where the constant k_3 does not depend upon X, λ nor u. Since

$$|\lambda|\,\|\chi v\|_{0,\mathbb{R}_+^n} \leq k_4\|v\|_{2,\mathbb{R}_+^n} + \big\|(\mathcal{L}_0 - q^2\chi)v\big\|_{0,\mathbb{R}_+^n},$$

where the constant k_4 does not depend upon X, λ, nor u, it follows that we can choose d sufficiently small and $|\lambda|$ sufficiently large to complete the proof of that part of the lemma concerning (4.11). Finally, (4.12) follows from (4.11) by interpolation. □

Lemma 4.3. *Suppose that the hypothesis of Proposition 4.1 is satisfied and that $\Gamma^{\pm}_{rj}$ is contained in Ω. Suppose also that $\Gamma^{\pm}_{rj}$ coincides with a $\Gamma^{\pm}_{sk}$. Then for each point $x^0 \in \Gamma^{\pm}_{rj}$ there exists a neighbourhood $X \subset\subset \Omega^{\dagger}$ of this point and positive numbers c_0, c_1 such that for $\lambda \in \Xi(\theta)$ and $|\lambda| \geq c_1$,*

$$|||u|||_{2,\sigma^{\pm}_{rj},\Omega^{\dagger}} \leq c_0 \big\|(L-\lambda\omega)u\big\|_{0,\Omega^{\dagger}}, \tag{4.14}$$

$$|||u|||_{3/2,\sigma^{\pm}_{rj},\Omega^{\dagger}} \leq c_0 |\lambda|^{-1/4\sigma^{\pm}_{rj}} \big\|(L-\lambda\omega)u\big\|_{0,\Omega^{\dagger}}, \tag{4.15}$$

for every $u \in H^2(\Omega)$ with supp $u \subset X$.

Proof. If $\gamma^{\pm}_{rj} = 0$, then that part of the proof concerning (4.14) has been proved in [12, Lemma 4.1], while (4.15) follows from (4.14) by interpolation. Hence we suppose from now on that $\gamma^{\pm}_{rj} \geq 2$; and to simplify the proof we shall also suppose that we are dealing with the case $x^0 \in \Gamma^{+}_{rj}$. Furthermore, fixing our attention upon the terms $\chi(t)$, ω_0, μ^2, and $\kappa(x^0,\theta)$ defined in the statements following (4.3) and (4.4) for the case $x^0 \in \Gamma^{\pm}_{rj}$, we shall now denote these terms by $\chi^{\pm}_{rj}(t)$, $\omega^{\pm}_{0rj}$, $(\mu^{\pm}_{rj})^2$, and $\kappa^{\pm}_{rj}(x^0,\theta)$, respectively, in order to demonstrate the dependence of their definitions on the $\Gamma^{\pm}_{rj}$. Then without loss of generality we can henceforth suppose that $\kappa^{+}_{rj}(x^0,\theta)|\mu^{+}_{rj}|^{1/\sigma} = \kappa^{\pm}_{sk}(x^0,\theta)|\mu^{\pm}_{sk}|^{1/\sigma}$, where $\sigma = \sigma^{+}_{rj}$, since we know from [14] that $\kappa^{+}_{rj}(x^0,\theta)$ or $\kappa^{\pm}_{sk}(x^0,\theta)$ can always be increased if necessary to achieve this end.

Next let U be a neighbourhood and (y', y_n) a system of coordinates connected with the point x^0 (with respect to Γ^{+}_{rj}). It is clear that there is no loss of generality in assuming henceforth that $\{(y',y_n) \in U | y_n < \phi(y')\} = U \cap \Omega^{\pm}_{s}$ and that $U \cap \Omega^{\pm}_{s}$ is contained in that neighbourhood of $\Gamma^{\pm}_{sk}$ whose existence is asserted in condition (5) of Assumption 2.2. Now let $X \subset\subset U^{\#}$ be a neighbourhood of x^0 and let $u \in C^{\infty}(\Omega)$ such that supp $u \subset X$. Then passing to local coordinates at x^0 and assuming henceforth that $\lambda \in \Xi(\theta)$ with $|\lambda| > 0$, let us put $v(\eta) = u\big(x(\eta)\big)$, $\chi_{+}(\eta_n) = \chi^{+}_{rj}(\eta_n)$, $\chi_{-}(\eta_n) = \chi^{\pm}_{sk}(\eta_n)(\eta_n \geq 0)$, $\omega_{+} = \omega^{+}_{0rj}$, $w_{-} = \omega^{\pm}_{0sk}$, $\mu^2_{+} = (\mu^{+}_{rj})^2$, $\mu^2_{-} = (\mu^{\pm}_{sk})^2$, $\kappa_{+} = \kappa^{+}_{rj}(x^0,\theta)$, $\kappa_{-} = \kappa^{\pm}_{sk}(x^0,\theta)$, and

$$\big(\mathcal{L}_{+}(\mathcal{D},\lambda)v\big)(\eta) = f_{+}(\eta,\lambda) \quad \text{for} \quad \eta \in \mathbb{R}^n_{+}, \tag{4.16}$$

$$\big(\mathcal{L}_{-}(\mathcal{D},\lambda)v\big)(\eta) = f_{-}(\eta,\lambda) \quad \text{for} \quad \eta \in \mathbb{R}^n_{-}, \tag{4.17}$$

where $\mathcal{L}_{+}(\mathcal{D},\lambda) = \mathcal{L}_0(0,\mathcal{D}) - \lambda\omega_{+}\chi_{+}(\eta_n)$, $\mathcal{L}_{-}(\mathcal{D},\lambda) = \mathcal{L}_0(0,\mathcal{D}) - \lambda\omega_{-}\chi_{-}(-\eta_n)$, $\mathbb{R}^n_{-} = \{(\eta',\eta_n) \in \mathbb{R}^n | \eta_n < 0\}$, and we refer to (4.3–4) for the remaining definitions.

We are now going to use (4.16–17) to establish a priori bounds for v. To this end, let us suppose firstly that $v \in C^{\infty}_0(\mathcal{U})$ and that the $f_{\pm}$ are defined according to (4.16–17). Writing t for η_n and letting $V(\xi',t) = (\mathcal{F}v)(\xi',t)$, $F_{\pm}(\xi',t,\lambda) = (\mathcal{F}f_{\pm})(\xi',t,\lambda)$, where $\mathcal{F}$ denotes the Fourier transformation introduced in (4.4), it follows immediately from (4.16–17) that $V(\xi',t)$ is the unique solution of each of the initial–value problems,

$$\begin{gathered} \mathcal{L}_{+}\big(i\xi', d/dt, \lambda\big)y = F_{+}(\xi',t,\lambda) \quad \text{in} \quad 0 \leq t \leq 1/4, \\ y^{(r)}(1/4) = 0 \quad \text{for} \quad r = 0,1, \end{gathered} \tag{4.18}$$

and

$$\mathcal{L}_-\big(i\xi', d/dt, \lambda\big)y = F_-(\xi', t, \lambda) \ \text{ in } \ -1/4 \le t \le 0, \quad y^{(r)}(-1/4) = 0 \ \text{ for } \ r = 0,1, \tag{4.19}$$

where $y^{(r)} = d^r y/dt^r$. Then, not showing explicitly the dependence of the functions concerned on ξ' and λ, we already know that for $|\lambda|$ exceeding a certain positive number not depending upon ξ', a fundamental set of solutions for $\mathrm{L}_+ y = 0$ in the interval $0 \le t \le 1/4$ is given by the $v_\pm(t)$ of (4.5) for $0 < |\xi'| < \kappa_+|\mu_+|^{1/\sigma}$ and by the $v_\pm(t)$ of (4.6) for $|\xi'| \ge \kappa_+|\mu_+|^{1/\sigma}$, where $\sigma = \sigma^+_{rj}$ (of course in (4.5–6) we are now to take $\gamma = \gamma^+_{rj}$, $\sigma = \sigma^+_{rj}$, $\mu = \mu_+$, $\chi(t) = \chi_+(t)$, and let ν be determined as before). Minor modifications of the results of [14] give us analogous results for the equation $\mathcal{L}_- y = 0$ in the interval $-1/4 \le t \le 0$; and we denote the corresponding fundamental set of solutions in this case by $v^\dagger_\pm(t)$, where here and below we again for brevity refrain from showing explicitly the dependence of the functions concerned on ξ' and λ. Assuming henceforth that $|\lambda|$ is sufficiently large and $\xi' \ne 0$, it follows immediately from [9, Theorem 6.4, p.87] and (4.18) that

$$V^{(r)}(0) = v_+^{(r)}(0)I_1(0) - v_-^{(r)}(0)I_2(0) \ \text{ for } \ r = 0,1, \tag{4.20}$$

while it follows from [9] and (4.19) that

$$V^{(r)}(0) = -(v^\dagger_+)^{(r)}(0)I^\dagger_1(0) + (v^\dagger_-)^{(r)}(0)I^\dagger_2(0) \ \text{ for } \ r = 0,1, \tag{4.21}$$

where $I_1(0) = \int_0^{1/4}\big(v_-(s)F_+(s)/p_0W(s)\big)\,ds$, $I_2(0) = \int_0^{1/4}\big(v_+(s)F_+(s)/p_0W(s)\big)\,ds$, $I^\dagger_1(0) = \int_{-1/4}^{0}\big(v^\dagger_-(s)F_-(s)/p_0W^\dagger(s)\big)\,ds$, $I^\dagger_2(0) = \int_{-1/4}^{0}\big(v^\dagger_+(s)F_-(s)/p_0W^\dagger(s)\big)\,ds$, $W(s)$ denotes the Wronskian of $\mathcal{L}_+ y = 0$ with respect to v_+ and v_-, and $W^\dagger(s)$ denotes the Wronskian of $\mathcal{L}_- y = 0$ with respect to $v^\dagger_+$ and $v^\dagger_-$. The equations (4.20–21), as they now stand, are not adequate for our purposes since the absolute values of $I_1(0)$ and $I^\dagger_1(0)$ may not remain less than some bound not depending upon ξ' and λ (see [14, Equations (4.11)), (7.4), and (7.6)]). Hence in order to eliminate these terms, we equate the expressions in the right sides of (4.20–21) for $r = 0,1$ to arrive at a linear system of equations in the "unknowns" $I_1(0)$ and $I^\dagger_1(0)$, which on solving gives

$$I_1(0) = c_{11}I^\dagger_2(0) + c_{12}I_2(0), \quad I^\dagger_1(0) = c_{21}I^\dagger_2(0) + c_{22}I_2(0), \tag{4.22}$$

where $\rho c_{11} = W^\dagger(0)$, $\rho c_{12} = v_-^{(1)}(0)v^\dagger_+(0) - v_-(0)(v^\dagger_+)^{(1)}(0)$, $\rho c_{21} = v_+^{(1)}(0)v^\dagger_-(0) - v_+(0)(v^\dagger_-)^{(1)}(0)$, $\rho c_{22} = -W(0)$, and $\rho = v_+^{(1)}(0)v^\dagger_+(0) - v_+(0)(v^\dagger_+)^{(1)}(0)$.

Let us firstly fix our attention upon the case $|\xi'| \ge \kappa_+|\mu_+|^{1/\sigma}$. Then by appealing to the results of [14, §§3, 4 and 6] and arguing as in the proofs of Theorems 8.1–2 of [14], it is not difficult to verify that $|\rho| \ge C\big|G(0)^{-1/2}\big(\lambda_+(0) - \lambda_-(0)\big)\big|$, $|c_{pq}| \le$

C, $|I_2(0)| \le C|\xi'|^{-1}\big(\int_0^\infty |F_+|^2 dt\big)^{1/2}$, and $|I_2^\dagger(0)| \le C|\xi'|^{-1}\big(\int_{-\infty}^0 |F_-|^2 dt\big)^{1/2}$, where here and below, C denotes a generic constant which may vary from inequality to inequality, but does not depend upon ξ' nor λ. Consequently, by applying these estimates to (4.22) and then substituting back into (4.20–21), we obtain

$$|V(0)| \le C|\xi'|^{-3/2}\Big[\Big(\int_0^\infty |F_+|^2 dt\Big)^{1/2} + \Big(\int_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}\Big], \tag{4.23}$$

and hence

$$\int_{|\xi'|\ge\kappa_+|\mu_+|^{1/\sigma}} \Big[\big(1+|\xi'|^2\big)^{3/2} + |\lambda|^{3/2\sigma}\Big]\big|V(\xi',0)\big|^2 d\xi' \le C\Big[\|f_+\|^2_{0,\mathbb{R}^n_+} + \|f_-\|^2_{0,\mathbb{R}^n_-}\Big]. \tag{4.24}$$

Suppose next that $0 < |\xi'| < \kappa_+|\mu_+|^{1/\sigma}$. Then writing γ for γ_{rj}^+, we know from [14, §7] that

$$v_+(0) = d_+(\zeta)f_+(\zeta),\ v_+^{(1)}(0) = \mu_+^{1/\sigma}d_+(\zeta)\Big[g_+(\zeta) + \mu_+^{-1/\sigma}\big(-ip_1(\xi')/2p_0\big)f_+(\zeta)\Big],$$

where for $\gamma = 2$, $d_+(\zeta) = \pi^{1/2}2^{(\zeta-1/2)/2}$, $f_+(\zeta) = 1/\Gamma\big(\frac{3}{4}-\zeta/2\big)$, and $g_+(\zeta) = -2/\Gamma\big(\frac{1}{4}-\zeta/2\big)$ (here Γ denotes the Gamma function), while for $\gamma > 2$, $d_+(\zeta) = \pi^{1/2}(2\gamma\sigma)^{-\gamma/4\sigma}/\sin\{\pi/2\sigma\}$, $f_+(\zeta)$ and $g_+(\zeta)$ are entire functions of exponential type whose zeros lie on the positive real axis, $\zeta = \gamma^{-2/\sigma}\mu_+^{-2/\sigma}a(\xi')$ lies in a closed sector with vertex at the origin which, with the exception of the origin, is contained in the left–half of the complex plane (i.e., that half consisting of numbers with negative real parts), $|\zeta| \le \gamma^{-2/\sigma}\kappa_+^2\delta_2^2$, and δ_2 is a constant defined in [14, §3]. Similarly, we can show that $v_+^\dagger(0) = d_+(\zeta^\dagger)f_+(\zeta^\dagger)$, $(v_+^\dagger)^{(1)}(0) = -\mu_-^{1/\sigma}d_+(\zeta^\dagger)\Big[g_+(\zeta^\dagger) + \mu_-^{-1/\sigma}\big(ip_1(\xi')/2p_0\big)f_+(\zeta^\dagger)\Big]$, where $\zeta^\dagger = \gamma^{-2/\sigma}\mu_-^{-2/\sigma}a(\xi')$ has properties analogous to those of ζ. Observing that $\mu_\pm^2 \notin \mathbb{R}$ and that $\mu_-^2 = \epsilon\mu_+^2$, where $\epsilon = \omega_-/\omega_+$, we conclude from these results that $|\rho| \ge C|\mu_+|^{1/\sigma}\Big|m_2(\zeta) + \epsilon^{1/2\sigma}m_2\big(\epsilon^{-1/\sigma}\zeta\big)\Big|$, where $m_2(\zeta) = -\gamma^{-1/\sigma}g_+(\zeta)/f_+(\zeta)$ and where we take $\arg\epsilon = 0$ if $\epsilon > 0$, $\arg\epsilon = -\pi$ if $\epsilon < 0$ and $\arg\mu_+^2 > 0$, and $\arg\epsilon = \pi$ if $\epsilon < 0$ and $\arg\mu_+^2 < 0$. The function $m_2(\zeta)$ has been studied in [15] for the case $\gamma > 2$ (note that the f_1, f_2 of [15] are just f_+ and $\gamma^{-1/\sigma}g_+$, respectively), where the following results were established: (1) $-Im\, m_2(\zeta)/Im\,\zeta > 0$ for $Im\,\zeta \ne 0$, (2) $m_2(\zeta) > 0$ for ζ real and non–positive, and (3) if $0 < |\alpha| < \pi/2$ and $\arg\zeta = \alpha + \pi$, then $Re\, m_2(\zeta) \ge m_2(0) > 0$, $0 < \arg m_2(\zeta) < \alpha$ if $\alpha > 0$, $\alpha < \arg m_2(\zeta) < 0$ if $\alpha < 0$. It follows immediately from these results that $|\rho| \ge C|\mu_+|^{1/\sigma}$ if $\gamma > 2$, and similarly we can show that this is also the case if $\gamma = 2$. By appealing to the results of [14, §7] and by arguing as in the proof of Theorem 8.3 of [14], it is now not difficult to verify that $|c_{pq}| \le C$, $|I_2(0)| \le C|\mu_+|^{-3/2\sigma}\Big(\int_0^\infty |F_+|^2 dt\Big)^{1/2}$, and $I_2^\dagger(0)| \le C|\mu_+|^{-3/2\sigma}\Big(\int_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}$.

Consequently, by applying these estimates to (4.22) and then substituting back into (4.20–21), we obtain

$$\big|V(0)\big| \le C|\mu_+|^{-3/2\sigma}\Big[\Big(\int_0^\infty |F_+|^2 dt\Big)^{1/2} + \Big(\int_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}\Big],$$

and hence

$$\int\limits_{|\xi'|<\kappa_+|\mu_+|^{1/\sigma}} \Big[\big(1+|\xi'|^2\big)^{3/2} + |\lambda|^{3/2\sigma}\Big]\big|V(\xi',0)\big|^2 d\xi' \le C\Big[\|f_+\|^2_{0,\mathbb{R}^n_+} + \|f_-\|^2_{0,\mathbb{R}^n_-}\Big]. \tag{4.25}$$

Let us now fix our attention upon the boundary value problem: (4.16) together with the boundary condition $v(\eta) = h(\eta')$ on $\eta_n = 0$, where $(\mathcal{F}h)(\xi') = V(\xi', 0)$. Then it follows immediately from (4.24–25) and [14, Theorem 2.1] (here we use the sharper verison as explained in the proof of Lemma 4.2) that

$$|||v|||_{2,\sigma,\mathbb{R}^n_+} \le C\Big[\|f_+\|_{0,\mathbb{R}^n_+} + \|f_-\|_{0,\mathbb{R}^n_-}\Big],$$

and similarly we can show that

$$|||v|||_{2,\sigma,\mathbb{R}^n_-} \le C\Big[\|f_+\|_{0,\mathbb{R}^n_+} + \|f_-\|_{0,\mathbb{R}^n_-}\Big].$$

These latter results have been established for the case $v \in C_0^\infty(\mathcal{U})$, and it is clear that they also remain valid for v as defined in the statements preceding (4.16). Thus fixing our attention upon this latter v, we now have

$$|||v|||_{2,\sigma,\mathbb{R}^n} \le C\left[\Big\|\big(\mathcal{L}_0 - \lambda\omega_+\chi_+(\eta_n)\big)v\Big\|_{0,\mathbb{R}^n_+} + \Big\|\big(\mathcal{L}_0 - \lambda\omega_-\chi_-(-\eta_n)\big)v\Big\|_{0,\mathbb{R}^n_-}\right],$$

and hence we may argue as we did in the proof of Lemma 4.2 to complete the proof of that part of the lemma concerning (4.14) for the case $u \in C^\infty(\Omega)$. The proof for the case $u \in H^2(\Omega)$ then follows from a standard approximation procedure. Finally, (4.15) follows from (4.14) by interpolation. $\square$

We now turn to the case where $|\Omega^0| > 0$, $\Gamma^\pm_{rj} \subset \Omega$, and $\Gamma^\pm_{rj}$ coincides with a component of $\partial\Omega_0$. Then if, for $x^0 \in \Gamma^\pm_{rj}$ and $u \in H^2(\Omega)$, we recall that when passing to the local coordinates η at x^0 the expression Lu goes over into $\mathcal{L}v$, we have

Lemma 4.4. *Suppose that the hypothesis of Proposition 4.1 is satisfied and that $\Gamma^\pm_{rj}$ is contained in Ω. Suppose also that $|\Omega^0| > 0$ and that $\Gamma^\pm_{rj}$ coincides with a component of $\partial\Omega_0$. Then for each point $x^0 \in \Gamma^\pm_{rj}$ there exists a neighbourhood*

$X \subset\subset \Omega$ of this point, with $X\backslash\Gamma^{\pm}_{rj} \subset \Omega^{\dagger} \cup \Omega_0$, and positive numbers c_0, c_1 such that for $\lambda \in \Xi(\theta)$ and $|\lambda| \geq c_1$,

$$(4.26)\quad |||u|||_{2,\sigma^{\pm}_{rj},\Omega^{\dagger}} + \|u\|_{2,\Omega_0} + |\lambda|^{1/4\sigma^{\pm}_{rj}}\|u\|_{3/2,\Omega_0}$$
$$\leq c_0\Big[\big\|(L-\lambda\omega)u\big\|_{0,\Omega^{\dagger}} + \|Lu\|_{0,\Omega_0} + |\lambda|^{1/4\sigma^{\pm}_{rj}}\Big(\int_{-\infty}^{0} \|\mathcal{L}v\|^2_{-1/2,\mathbb{R}^{n-1}}d\eta_n\Big)^{1/2}\Big],$$

$$(4.27)\quad |||u|||_{3/2,\sigma^{\pm}_{rj},\Omega^{\dagger}} + \|u\|_{3/2,\Omega_0}$$
$$\leq c_0\Big[|\lambda|^{-1/4\sigma^{\pm}_{rj}}\big\|(L-\lambda\omega)u\big\|_{0,\Omega^{\dagger}} + \|Lu\|_{0,\Omega_0}\Big]$$

for every $u \in H^2(\Omega)$ with supp $u \subset X$.

Proof. If $\gamma^{\pm}_{rj} = 0$, then that part of the lemma concerning (4.26) has been proved in [16], while (4.27) follows from (4.26) by interpolation. Hence we suppose from now on that $\gamma^{\pm}_{rj} \geq 2$; and to simplify the proof we will also suppose that $x^0 \in \Gamma^{+}_{rj}$.

Let U be a neighbourhood and (y', y_n) a system of coordinates connected with the point x^0 (with respect to Γ^{+}_{rj}). It is clear that there is no loss of generality in assuming henceforth that $\{(y', y_n) \in U | y_n < \phi(y')\} = U \cap \Omega_0 \subset U^{+}_{rj}$, where we refer to condition (2) of Assumption 2.3 for the definition of U^{+}_{rj}. Now let $X \subset\subset U^{\#}$ be a neighbourhood of x^0 and let $u \in C^{\infty}(\Omega)$ such that supp $u \subset X$. Then passing to local coordinates at x^0 and assuming henceforth that $\lambda \in \Xi(\theta)$ with $|\lambda| > 0$, we again arrive at the equations (4.16–17), where v and $\mathcal{L}_+$ are defined as before, but now $\mathcal{L}_- = \mathcal{L}_0(0, \mathcal{D})$. Supposing for the moment that $v \in C_0^{\infty}(\mathcal{U})$ and that the $f_{\pm}$ are defined according to (4.16–17), we may, as before, write t for η_n and make a Fourier transformation $\mathcal{F}$ with respect to $\eta'(\eta' \to \xi')$ in (4.16–17) to arrive at the initial–value problems (4.18–19). Assuming henceforth that $|\lambda|$ is sufficiently large, a fundamental set of solutions for $\mathcal{L}_+ y = 0$ in the interval $0 \leq t \leq 1/4$ is given by $v_{\pm}(t)$, defined precisely as in the proof of Lemma 4.3 (again for brevity we do not show explicitly the dependence of the functions concerned on ξ' and λ), while a fundamental set of solutions of $\mathcal{L}_- y = 0$ in the interval $-1/4 \leq t \leq 0$ is given by $v^{\dagger}_{\pm}(t)$, where now $v^{\dagger}_{+}(t) = \exp\{it\lambda_-(0)\}$, $v^{\dagger}_{-}(t) = \exp\{it\lambda_+(0)\}$, where the $\lambda_{\pm}(s)$ are defined precisely as in the proof just cited, and with the restriction $|\xi'| \geq \kappa_+|\mu_+|^{1/\sigma}$ given there being replaced by $|\xi'| > 0$. Using these fundamental solutions, we again arrive at the formulae (4.20–21) for $V^{(r)}(0)$, and since as before, we wish to eliminate the terms $I_1(0)$ and $I^{\dagger}_1(0)$, we may argue as we did in the proof of Lemma 4.3 to arrive at (4.22) and then, with the same terminology as in that proof, prove that for the problem under consideration here, (4.23–24) remain valid for $|\xi'| \geq \kappa_+|\mu_+|^{1\sigma}$. For $|\xi'| < \kappa_+|\mu_+|^{1/\sigma}$, we now have $\rho = -\gamma^{1/\sigma}d_+(\zeta)f_+(\zeta)\mu_+^{1/\sigma}\times [m_2(\zeta) - i\zeta^{1/2}]$, where we assign to arg ζ its value in $(0, 2\pi)$ (here and below we always employ the terminology of the proof of Lemma 4.3). From the results given in the proof of Lemma 4.3 and in [14] it now follows that $|\rho| \geq$

$C|\mu_+|^{1/\sigma}$, $|c_{11}| \leq C|\xi'||\mu_+|^{-1/\sigma}$, $|c_{pq}| \leq C$ for the remaining p,q, $\big|I_2(0)\big| \leq C|\mu_+|^{-3/2\sigma}\Big(\int_0^\infty |F_+|^2 dt\Big)^{1/2}$, $\big|I_2^\dagger(0)\big| \leq C\alpha(\xi')|\xi'|^{-1}\Big(\int_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}$, where $\alpha(\xi') = 1$ if $|\xi'| \leq 1$, $\alpha(\xi') = |\xi'|^{-1/2}$ otherwise, and hence we conclude from (4.22) and (4.20) that

$$|V(0)| \leq C\Big[|\mu_+|^{-3/2\sigma}\Big(\int\limits_0^\infty |F_+|^2 dt\Big)^{1/2} + \alpha(\xi')|\mu_+|^{-1/\sigma}\Big(\int\limits_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}\Big], \tag{4.28}$$

while (4.22) and (4.21) give

$$|V(0)| \leq C\Big[|\mu_+|^{-3/2\sigma}\Big(\int\limits_0^\infty |F_+|^2 dt\Big)^{1/2} + \alpha(\xi')|\xi'|^{-1}\Big(\int\limits_{-\infty}^0 |F_-|^2 dt\Big)^{1/2}\Big]. \tag{4.29}$$

Finally, from (4.28) we obtain

$$\begin{aligned} &\int\limits_{|\xi'|<\kappa_+|\mu_+|^{1/\sigma}} \Big[\big(1+|\xi'|^2\big)^{3/2} + |\lambda|^{3/2\sigma}\Big]\big|V(\xi',0)\big|^2 d\xi' \\ &\qquad \leq C\Big[\|f_+\|^2_{0,\mathbb{R}^n_+} + |\mu_+|^{1/\sigma}\int\limits_{-\infty}^0 \|f_-\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big], \end{aligned} \tag{4.30}$$

while from (4.29) we obtain

$$\begin{aligned} &\int\limits_{|\xi'|<\kappa_+|\mu_+|^{1/\sigma}} \Big[|\xi'|^3 + |\lambda|^{1/2\sigma}|\xi'|^2\Big]\big|V(\xi',0)\big|^2 d\xi' \\ &\qquad \leq C\Big[\|f_+\|^2_{0,\mathbb{R}^n_+} + |\mu_+|^{1/\sigma}\int\limits_{-\infty}^0 \|f_-\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big]. \end{aligned} \tag{4.31}$$

Let us now fix our attention upon the boundary value problem: (4.16) together with the boundary condition $v(\eta) = h(\eta')$ on $\eta_n = 0$, where $(\mathcal{F}h)(\xi') = V(\xi',0)$. Then it follows from (4.24), (4.30), and [14, Theorem 2.1] (in the sharpened form as explained before) that

$$\begin{aligned} |||v|||_{2,\sigma,\mathbb{R}^n_+} \leq C\Big[&\Big\|\big(\mathcal{L}_0 - \lambda\omega_+\chi_+\big)v\Big\|_{0,\mathbb{R}^n_+} + \|\mathcal{L}_0 v\|_{0,\mathbb{R}^n_-} + \\ &+ |\mu_+|^{1/2\sigma}\Big(\int\limits_{-\infty}^0 \|\mathcal{L}_0 v\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big)^{1/2}\Big]. \end{aligned} \tag{4.32}$$

Turning next to the boundary value problem: (4.17) together with the boundary condition $v(\eta) = h(\eta')$ on $\eta_n = 0$, where $h(\eta')$ is defined above, we know from the proof of Theorem 2.1 of [16] that for this problem we have the a priori estimates

$$\|v\|_{2,\mathbb{R}^n_-} \le C\Big[\|f\|_{0,\mathbb{R}^n_-} + \Big(\int\limits_{\mathbb{R}^{n-1}} |\xi'|^3 \big|V(\xi',0\big|^2 d\xi'\Big)^{1/2}\Big]$$

and

$$\|v\|_{3/2,\mathbb{R}^n_-} \le C\Big[\Big(\int\limits_{-\infty}^{0} \|f_-\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big)^{1/2} + \Big(\int\limits_{\mathbb{R}^{n-1}} |\xi'|^2 \big|V(\xi',0)\big|^2 d\xi'\Big)^{1/2}\Big],$$

and hence it follows from (4.24) and (4.31) that the inequality (4.32) persists when $|||v|||_{2,\sigma,\mathbb{R}^n_+}$ is replaced by $|||v|||^{\dagger}_{2,\sigma,\mathbb{R}^n_-} = \|v\|_{2,\mathbb{R}^n_-} + |\lambda|^{1/4\sigma}\|v\|_{3/2,\mathbb{R}^n_-}$. Thus we have established that

$$\begin{aligned} |||v|||_{2,\sigma,\mathbb{R}^n_+} + |||v|||^{\dagger}_{2,\sigma,\mathbb{R}^n_-} \le C\Big[&\Big\|(\mathcal{L}_0 - \lambda\omega_+\chi_+)v\Big\|_{0,\mathbb{R}^n_+} \\ &+ \|\mathcal{L}_0 v\|_{0,\mathbb{R}^n_-} + |\lambda|^{1/4\sigma}\Big(\int\limits_{-\infty}^{0} \|\mathcal{L}_0 v\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big)^{1/2}\Big] \end{aligned} \tag{4.33}$$

for $v \in C_0^\infty(\mathcal{U})$, and it is clear that this inequality persists for $v(\eta) = u\big(x(\eta)\big)$. Then with this latter v in (4.33), we can argue with $(\mathcal{L}_0 - \lambda\omega_+\chi_+)v$ and $\mathcal{L}_0 v$ as we did in the proof of Lemma 4.2 and with $\Big(\int_{-\infty}^{0} \|\mathcal{L}_0 v\|^2_{-1/2,\mathbb{R}^{n-1}} dt\Big)^{1/2}$ as we did in the proof of Theorem 2.1 of [16], to show that (4.33) remains valid when $(\mathcal{L}_0 - \lambda\omega_+\chi_+)v$ is replaced by $(\mathcal{L} - \lambda \mathrm{w})v$ and $\mathcal{L}_0 v$ by $\mathcal{L}v$, provided that the diameter of X is sufficiently small. This proves that part of the lemma concerning (4.26) for the case $u \in C^\infty(\Omega)$, while the proof for the case $u \in H^2(\Omega)$ follows from a standard approximation procedure. Finally, (4.27) follows from (4.26) by interpolation. $\square$

Before commencing with the proof of Proposition 4.1, we state the following standard results (cf. [4], [16]).

Lemma 4.5. *Suppose that $|\Omega^0| > 0$. Then for each point $x^0 \in \Omega_0$ there exists a neighbourhood $X \subset\subset \Omega_0$ of this point and a positive number c_0 such that $\|u\|_{2,\Omega_0} \le c_0\|Lu\|_{0,\Omega_0}$ for every $u \in H^2(\Omega)$ with $\operatorname{supp} u \subset X$.*

Lemma 4.6. *Suppose that $|\Omega^0| > 0$ and that Γ_0 is a component of $\partial\Omega_0$ which is also a component of Γ. Then for each point $x^0 \in \Gamma_0$ there exists a neighbourhood X of this point, with $X \cap \Omega \subset \Omega_0$, and a positive number c_0 such that $\|u\|_{2,\Omega_0} \le c_0\big[\|Lu\|_{0,\Omega_0} + \|Bu\|'\big]$ for every $u \in H^2(\Omega)$ with $\operatorname{supp} u \subset X$.*

Proof of Proposition 4.1. Let $\mathcal{V} = \big\{u | u \in D(A),\ \|\omega^{-1}Lu\|_{0,\Omega^\dagger} < \infty\big\}$ if $|\Omega^0| = 0$ and $\mathcal{V} = \Big\{u | u \in D(A),\ \|\omega^{-1}Lu\|_{0,\Omega^\dagger} < \infty,\ \|Lu\|_{0,\Omega_0} = 0\Big\}$ if $|\Omega^0| > 0$. Then by considering a suitable covering of $\bar{\Omega}$ by means of a finite number of open balls, each of which is contained in one of the sets X of Lemmas 4.1–6, we may appeal to these lemmas and argue as in [5,§4] to show that there exist positive constants k_0, k_1 such that if $\lambda \in \Xi(\theta)$ and $|\lambda| \geq k_1$, then

$$\begin{aligned} &(4.34) \qquad |||u|||_{3/2,\sigma_0,\Omega^\dagger} \leq k_0|\lambda|^{-1/4\sigma_0}\|T\mathcal{E}f\|_{0,\Omega^\dagger} \ \text{ if } \ |\Omega^0| = 0, \\ &(4.35) \qquad |||u|||_{3/2,\sigma_0,\Omega^\dagger} + \|u\|_{3/2,\Omega_0} \\ &\qquad\qquad \leq k_0\Big[|\lambda|^{-1/4\sigma_0}\|T\mathcal{E}f\|_{0,\Omega^\dagger} + \|u\|_{0,\Omega_0}\Big] \ \text{ if } \ |\Omega^0| > 0 \end{aligned}$$

for every pair $u \in \mathcal{V}$, $f \in \mathcal{H}^\dagger$ for which $(A - \lambda T)u = T\mathcal{E}f$.

Referring to (4.35), let us show that when $|\Omega^0| > 0$, then there exists the constant $k_1^\dagger \geq k_1$ such that if $\lambda \in \Xi(\theta)$ and $|\lambda| \geq k_1^\dagger$, then $|||u|||_{3/2,\sigma_0,\Omega^\dagger} \leq 2k_0|\lambda|^{-1/4\sigma_0}\|T\mathcal{E}f\|_{0,\Omega^\dagger}$ for every pair u, f as defined above. Indeed, if this is not the case, then it follows from (4.35) that there exists a $u \in H^{3/2}(\Omega)$, and sequences $\{u_i\}_1^\infty$ in $\mathcal{V}$ and $\{\lambda_i\}_1^\infty$ in $\Xi(\theta)$, where $\|u\| = \|u_i\| = 1$ and $|\lambda_i| \to \infty$ as $i \to \infty$, such that $|\lambda_i|^{-1/4\sigma_0}\big\|(A - \lambda_i T)u_i\big\| < 1$ for each i, $u_i \to u$ weakly in $H^{3/2}(\Omega)$ and $u_i \to u$ strongly in $H^1(\Omega)$ as $i \to \infty$, while $\|u\|_{0,\Omega^\dagger} = 0$. We are now going to prove that $Lu = 0$ in the sense of distributions on Ω; and since it is easy to show that $Lu(\phi) = 0$ for $\phi \in C_0^\infty(\Omega_0)$ and for $\phi \in C_0^\infty(\Omega^\dagger)$, it is clear that in order to achieve our goal we need only fix our attention upon the case where Γ_{rj}^+ is a component of $\partial\Omega_0$, $x^0 \in \Gamma_{rj}^+$, U is a neighbourhood connected with the point x^0, $\phi \in C_0^\infty(\Omega)$ with $\operatorname{supp}\phi \subset U$, and show that $Lu(\bar{\phi}) = 0$. To this end, let us observe from the Banach–Saks theorem [7, p.181] that there exists the subsequence $\{u_{i(p)}\}_{p=1}^\infty$ of the u_i such that $\mathrm{w}_p = p^{-1}\sum_{s=1}^p u_{i(s)} \to u$ strongly in $H^{3/2}(\Omega)$ as $p \to \infty$. Hence

$$(4.36)\qquad \begin{aligned} Lu(\bar{\phi}) &= -\sum_{|\alpha|=2}\big(D^{\alpha-\alpha'}u, D^{\alpha'}\bar{a}_\alpha\phi\big)_{0,\Omega_0} + \sum_{|\alpha|\leq 1}\big(a_\alpha D^\alpha u, \phi\big)_{0,\Omega_0} \\ &= \lim_{p\to\infty}\Big[-\sum_{|\alpha|=2}\big(D^{\alpha-\alpha'}\mathrm{w}_p, D^{\alpha'}\bar{a}_\alpha\phi\big)_{0,\Omega_0} + \sum_{|\alpha|\leq 1}\big(a_\alpha D^\alpha \mathrm{w}_p, \phi\big)_{0,\Omega_0}\Big], \end{aligned}$$

where α' is a multi–index satisfying $\alpha' \leq \alpha$, $|\alpha'| = 1$. By passing to local coordinates of ordinary type at x^0 and integrating by parts, it is not difficult to verify that the modulus of the expression in square brackets on the right side of (4.36) does not exceed $c\sum_{i=1}^n\Big(\int_{|\eta'|<\rho_0}\big|tr\mathcal{D}_i v_p\big|^2 d\eta'\Big)^{1/2}$, where, referring to the paragraph following the statement of Proposition 4.1 for notation, $v_p(\eta) = \mathrm{w}_p\big(x(\eta)\big)$, tr denotes trace on the hyperplane $\eta_n = 0$, the constant c does not depend upon p, and all the remaining terms are defined as before. Since $\|\mathrm{w}_p\|_{3/2,\Omega^\dagger} \to 0$ as $p \to \infty$, it follows immediately that $Lu(\bar{\phi}) = 0$. Thus we have shown that $Lu = 0$, and hence by unique continuation [22, Theorem 2.4] we arrive at the contradiction that $u = 0$.

For $a \geq 0$ let $\Xi(\theta, a) = \big\{\lambda \in \Xi(\theta)\,\big|\,|\lambda| \geq a\big\}$. Then it follows from (4.34–35) and from what we have just shown that there exist the positive constants c_0, c such that for $\lambda \in \Xi(\theta, c)$,

$$|||u|||_{3/2,\sigma_0,\Omega^\dagger} \leq c_0|\lambda|^{-1/4\sigma_0}\big\|T\mathcal{E}f\big\|_{0,\Omega^\dagger} \tag{4.37}$$

for every pair $u \in \mathcal{V}$, $f \in \mathcal{H}^\dagger$ for which $\big(A - \lambda T\big)u = T\mathcal{E}f$. Consequently, if $\lambda \in \Xi(\theta, c)$, $f \in \mathcal{H}^\dagger$, and $(I - \lambda K^\dagger)f = 0$, then a simple argument shows that there exists a $u \in \mathcal{V}$ such that $\mathcal{R}u = f$ and $(A - \lambda T)u = 0$, and so we see from (4.37) that $f = 0$. Thus we conclude that $\Xi(\theta, c) \subset \rho_m(K^\dagger)$. Moreover, if $\lambda \in \Xi(\theta, c)$, $f \in \mathcal{H}^\dagger$, and $K^\dagger_\lambda f = u$, then it is not difficult to verify that there exists a $v \in \mathcal{V}$ such that $\mathcal{R}v = u$ and $(A - \lambda T)v = T\mathcal{E}f$, and hence it follows from (4.37) that

$$\big\|K^\dagger_\lambda f\big\|_{0,\Omega^\dagger} \leq c^\dagger|\lambda|^{-1/\sigma_0}\|f\|_{0,\Omega^\dagger} \quad \text{for} \quad \lambda \in \Xi(\theta, c), \tag{4.38}$$

where $c^\dagger = c_0\|\omega\|_{L^\infty(\Omega)}$. This completes the proof of the proposition. $\square$

Remark 4.1. We assert that we can choose the constants c and $c^\dagger$ in (4.38) so that this equation remains valid along every ray $\arg\lambda = \theta'$ for which $\theta' \in \Sigma_\epsilon = \big\{\theta' \mid |\theta' - \theta| \leq \epsilon\big\}$ for some suitable positive number ϵ. Indeed, since the proofs of Lemmas 4.1–4 depended upon the results of [5, 14, 16] which were actually established under the assumption that $\arg\lambda$ varied in a closed sector with vertex at the origin, and since it is clear that under our assumptions the conditions for the validity of these latter results at the point x^0 in question (see Conditions I, II of [5, §§2, 3], Assumption 2.1 and Theorem 10.1 of [14], and Assumption 2.1 of [16]) also hold for $\arg\lambda = \theta'$ for $\theta' \in \Sigma_\epsilon$ if they hold for $\arg\lambda = \theta$, and with ϵ depending only upon θ and not upon x^0, it follows immediately that in Lemmas 4.1–4 we may choose the constants c_0, c_1 so that (4.9–10), (4.11–12), (4.14–15), and (4.26–27) remain valid along every ray $\arg\lambda = \theta'$ for $\theta' \in \Sigma_\epsilon$ and with ϵ depending only upon θ. Arguing as in the proof of Proposition 4.1, we can now show that the constants k_0, k_1 of (4.34–35) can be chosen so that the inequalities shown remain valid along every ray $\arg\lambda = \theta'$ for $\theta' \in \Sigma_\epsilon$, and hence by arguing with (4.35) as we did before for the case $|\Omega^0| > 0$ we can complete the justification of our assertion.

Recalling the definition of $\Xi(\theta, a)$ given in the proof of Proposition 4.1, we now extend this definition to arbitrary $\theta \in \mathbb{R}$.

Proposition 4.2. Suppose that $\theta \in \mathbb{R}$ and there exist positive constants c_0, c such that for $\lambda \in \Xi(\theta, c)$, (4.37) holds for every pair $u \in \mathcal{V}$, $f \in \mathcal{H}^\dagger$ for which $(A - \lambda T)u = T\mathcal{E}f$. Then $\theta \neq k\pi$ for $k \in \mathbb{Z}$.

Proof. We shall only prove the proposition under the assumption that $L_0(x, i\xi) > 0$ for $x \in \bar{\Omega}$ and $0 \neq \xi \in \mathbb{R}^n$, where $L_0(x, D)$ denotes the principal part of $L(x, D)$; the remaining case can be dealt with similarly. We also let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \phi(x) \leq 1$, $\phi(x) = 1$ for $|x| < 1$, and $\phi(x) = 0$ for $|x| > 2$.

Now suppose that the proposition is false and that $\theta = 2k\pi$. Then recalling from Assumption 2.2 that for some r, Ω_r^+ contains a closed ball with centre x^0 in which $\omega(x)$ is Lipschitz continuous, we see that there is a non–zero vector $\xi^0 \in \mathbb{R}^n$ such that $L_0(x^0, i\xi^0) = e^{i\theta}$. Hence if for $\lambda \in \Xi(\theta, c)$ sufficiently large we put $v_\lambda(x) = \phi(\lambda^{1/4}(x - x^0))$, $u_\lambda(x) = \exp\{i(\lambda\omega(x^0))^{1/2}\xi^0 \cdot x\}v_\lambda(x)$, where $\cdot$ denotes inner product, and let $u(x) = u_\lambda(x)$ in (4.37), then we may appeal to Theorem 10.2 of [24, p.52] to show that the expression on the left side of (4.37) is not less than $C\lambda^{\frac{3}{4}-\frac{n}{8}}$ for all λ sufficiently large, where C denotes a positive constant, while the expression on the right side of (4.37) is $O(\lambda^{\frac{3}{4}-\frac{n}{8}-\frac{1}{4\sigma_0}})$ as $\lambda \to \infty$. Thus we arrive at a contradiction. Similarly we can show that the supposition $\theta = (2k+1)\pi$ leads to a contradiction, and this completes the proof of the proposition. □

5. Proof of Theorem 3.1

Let us firstly show that the generalized characteristic vectors of $K^\dagger$ are complete in $\mathcal{H}^\dagger$. To this end let us divide the complex plane into sectors by means of the distinct rays $\arg\lambda = \theta_j$, $j = 1, \ldots, p$, where: (1) the angular opening of each sector is less than $2\pi/n$ and (2) $\theta_j \neq k\pi$ for $j = 1, \ldots, p$ and $k \in \mathbb{Z}$. Let us also observe from [2] that for any $\delta > 0$ there exists the increasing sequence $\{r_j\}_1^\infty$ of positive numbers, tending to ∞ with j, such that for each j, the circle $|\lambda| = r_j$ is contained in $\rho_m(K^\dagger)$ and $\left[\|(I - \lambda K^\dagger)^{-1}\|_{\mathcal{H}^\dagger} + \|K_\lambda^\dagger\|_{\mathcal{H}^\dagger}\right] \leq \exp\{|\lambda|^{\frac{n}{2}+\delta}\}$ for $|\lambda| = r_j$, where $\|\ \|_{\mathcal{H}^\dagger}$ denotes the norm in $\mathcal{L}(\mathcal{H}^\dagger)$. Now let $\mathcal{G}$ denote the closed span of all the generalized characteristic vectors of $K^\dagger$ and suppose that $\mathcal{G} \neq \mathcal{H}^\dagger$. Then there exists an $h \neq 0$ in $\mathcal{H}^\dagger$ such that $(g, h)_{0,\Omega^\dagger} = 0$ for every $g \in \mathcal{G}$, and hence it follows that for any $f \in \mathcal{H}^\dagger$, $(K_\lambda^\dagger f, h)_{0,\Omega^\dagger}$ is an entire function. Thus we see that if we fix our attention upon any one of the open sectors in the complex plane determined by the rays $\arg\lambda = \theta_j$ and restrict ourselves here to a particular branch of λ^{1/σ_0}, then $(\lambda^{1/\sigma_0}K_\lambda^\dagger f, h)_{0,\Omega^\dagger}$ is analytic in and continuous on the closure of this sector. In light of Proposition 4.1 and the above estimates for the norms of $K_\lambda^\dagger$ on the circles $|\lambda| = r_j$, we may appeal to the Phragmén–Lindelhöf principle to deduce that $(K_\lambda^\dagger f, h)_{0,\Omega^\dagger}$ tends uniformly to zero as $|\lambda| \to \infty$. Thus $(K_\lambda^\dagger f, h)_{0,\Omega^\dagger} \equiv 0$, and hence $(K^\dagger f, h)_{0,\Omega^\dagger} = 0$. Since f is arbitrary, we conclude from Proposition 3.3 that $h = 0$, which is a contradiction.

In light of the results of §3, all the assertions of the theorem, except those concerning completeness in $\mathcal{H}_\omega = L^2(\Omega^+ \cup \Omega^-;\ |\omega(x)|dx)$ and the angular distribution of the eigenvalues, now follow. On the other hand, if we bear in mind that $C_0^\infty(\operatorname{int}\Omega^+ \cup \operatorname{int}\Omega^-)$ is dense in $\mathcal{H}_\omega$, then the assertion concerning completeness in $\mathcal{H}_\omega$ follows easily from the foregoing results.

We are now going to show that there are infinitely many eigenvalues of the problem (1.1–2) lying in the sector $|\arg\lambda| < \epsilon$; and in proving this result, we can, without loss of generality, suppose that $\epsilon < \pi/n$. Accordingly, let us suppose that there are at most a finite number of eigenvalues of (1.1–2) lying in the sector just cited. Then it follows from the results of §3 that there is an $r > 0$ such that if Σ_r' denotes the region in the λ–plane defined by the inequalities $\theta_1 < \arg\lambda < \theta_2$, $|\lambda| > r$, where $\theta_j = (-1)^j\epsilon$, then $\overline{\Sigma_r'} \subset \rho_m(K^\dagger)$ and (4.37) is valid for

$\lambda \in \Xi(\theta_j, r),\ j = 1,2$. For $0 \neq f \in \mathcal{H}^\dagger$ and $\lambda \in \overline{\Sigma'_r}$, let $u = K^\dagger_\lambda f$. Then we know from the final paragraph of the proof of Proposition 4.1 that there exists a $v \in \mathcal{V}$ such that $\mathcal{R}v = u$ and $(A - \lambda T)v = T\mathcal{E}f$, and so we see from (4.37) that

$$\|\lambda^{1/\sigma_0} K^\dagger_\lambda f\|_{0,\Omega^\dagger} \leq C\|T\mathcal{E}f\|_{0\,\Omega^\dagger}, \tag{5.1}$$

$$\|\lambda^{1/4\sigma_0} K^\dagger_\lambda f\|_{3/2,\Omega^\dagger} \leq C\|T\mathcal{E}f\|_{0,\Omega^\dagger} \tag{5.2}$$

for $\lambda \in \Xi(\theta_j, r),\ j = 1,2$, where here and below C denotes a generic constant which does not depend upon f nor λ. Observing that $(A - \lambda T)^{-1} : \lambda \to \mathcal{L}(\mathcal{H})$ is anlytic in $\rho_m(K^\dagger)$ and that for $\lambda \in \Sigma''_r = \{\lambda | \lambda \in \overline{\Sigma'_r},\ |\lambda| = r\}$, $\|K^\dagger_\lambda f\|_{0,\Omega^\dagger} / \|T\mathcal{E}f\|_{0,\Omega^\dagger} \leq \|v\| / \|T\mathcal{E}f\|_{0,\Omega^\dagger} \leq \sup_{\Sigma''_r} \|(A - \lambda T)^{-1}\|_{\mathcal{H}}$, where $\|\ \|_{\mathcal{H}}$ denotes the norm in $\mathcal{L}(\mathcal{H})$, we also see that (5.1) holds for $\lambda \in \Sigma''_r$. Thus, since $\lambda^{1/\sigma_0} K^\dagger_\lambda f : \overline{\Sigma'_r} \to \mathcal{H}^\dagger$ is analytic in Σ'_r and continuous in $\overline{\Sigma'_r}$, and bearing in mind the estimates for the norms of $K^\dagger_\lambda$ on the circles $|\lambda| = r_j$ given above, we can now appeal to the Phragmén–Lindelhöf principle to deduce that (5.1) persists for $\lambda \in \overline{\Sigma'_r}$. Furthermore, we may argue with the closed graph theorem as in the proof of Lemma 13.4 of [3, p.210] to deduce that $K^\dagger \in \mathcal{L}(\mathcal{H}^\dagger, H^{3/2}(\Omega^\dagger))$ and that $(A - \lambda T)^{-1} \in \mathcal{L}(\mathcal{H}, H^{3/2}(\Omega))$ for $\lambda \in \rho_m(K^\dagger)$. Thus $(A - \lambda T)^{-1} : \lambda \to \mathcal{L}(\mathcal{H}, H^{3/2}(\Omega))$ is analytic in $\rho_m(K^\dagger)$, while we also see that for $\lambda \in \Sigma''_r$, $\|K^\dagger_\lambda f\|_{3/2,\Omega^\dagger} / \|T\mathcal{E}f\|_{0,\Omega^\dagger} \leq \|v\|_{3/2,\Omega} / \|T\mathcal{E}f\|_{0,\Omega^\dagger} \leq \sup_{\Sigma''_r} \|(A - \lambda T)^{-1}\|^\dagger_{\mathcal{H}}$, where $\|\ \|^\dagger_{\mathcal{H}}$ denotes the norm in $\mathcal{L}(\mathcal{H}, H^{3/2}(\Omega))$, and so we conclude that (5.2) persists for $\lambda \in \Sigma''_r$. Lastly, it follows from the remark just made about $K^\dagger$ that $\lambda^{1/4\sigma_0} K^\dagger_\lambda f : \overline{\Sigma'_r} \to H^{3/2}(\Omega^\dagger)$ is analytic in Σ'_r and continuous in $\overline{\Sigma'_r}$, and hence bearing in mind the estimates for the norms of $(I - \lambda K^\dagger)^{-1}$ on the circles $|\lambda| = r_j$ given above, we can now appeal to the Phragmén–Lindelhöf principle to deduce that (5.2) persists for $\lambda \in \overline{\Sigma'_r}$.

We conclude from the foregoing results that (5.1–2) hold for any $f \in \mathcal{H}^\dagger$ and $\lambda \in \Xi(0, r)$. On the other hand, since it is easy to show that if $\lambda \in \Xi(0, r)$ and $u \in \mathcal{V}$, $f \in \mathcal{H}^\dagger$ are any pair for which $(A - \lambda T)u = T\mathcal{E}f$, then we must have $\mathcal{R}u = K^\dagger_\lambda f$, it follows that the hypothesis of Proposition 4.2 is satisfied for $c_0 = C,\ c = r$, and $\theta = 0$. Hence, in view of Proposition 4.2, we arrive at a contradiction.

Similarly, we can show that there are infinitely many eigenvalues of (1.1–2) lying in the sector $|\arg\lambda - \pi| < \epsilon$. Furthermore, as a consequence of Remark 4.1 we see that there exists a positive number r_ϵ such that the sets $\{\lambda \in \mathbb{C} | \epsilon \leq \arg\lambda \leq \pi - \epsilon,\ |\lambda| \geq r_\epsilon\}$ and $\{\lambda \in \mathcal{C} | -\pi + \epsilon \leq \arg\lambda \leq -\epsilon,\ |\lambda| \geq r_\epsilon\}$ are both contained in $\rho_m(K^\dagger)$, and hence it follows that there are at most a finite number of eigenvalues of (1.1–2) lying in each of the sectors $\epsilon \leq \arg\lambda \leq \pi - \epsilon$ and $-\pi + \epsilon \leq \arg\lambda \leq -\epsilon$. This completes the proof of the theorem. $\square$

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