ARCHIVUM MATHEMATICUM (BRNO)
Tomus 30 (1994), 85 – 95

ON TERNARY SEMIGROUPS OF HOMOMORPHISMS OF ORDERED SETS

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Abstract. The paper deals with the characterization of ordered sets by means of ternary semigroups of homomorphisms of ordered sets.

1. Introduction

In many areas of mathematics the mutual connections between algebraic, ordered, topological structures and semigroups (groups) of some mappings of these structures are studied (e.g. [2]). In the present paper we introduce the notion of a ternary semigroup of homomorphisms of ordered sets which is the counterpart of a semigroup of endomorphisms of an ordered set. In the main theorem of this paper we give necessary and sufficient conditions for a certain characterization of ordered sets by means of ternary semigroups of homomorphisms of these sets.

2. Basic definitions

Definition 2.1 (cf. [3]). A ternary semigroup is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \to A$ is a ternary operation satisfying the following associative law:

$$\begin{aligned} f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = \\ = f(x_1, x_2, f(x_3, x_4, x_5)) \end{aligned}$$

for all $x_1, \dots, x_5 \in A$.

Definition 2.2 (cf. [3]). A nonempty subset $I \subset A$ is called an ideal of a ternary semigroup (A, f) if $f(I, A, A) \subset I$, $f(A, I, A) \subset I$, $f(A, A, I) \subset I$.

1991 *Mathematics Subject Classification*: 08A62, 20N15.
Key words and phrases: homomorphism and isomorphism of ordered sets, ternary semigroup of homomorphisms of ordered sets.
Received November 22, 1992.

Definition 2.3. An element $x_0 \in A$ is said to be a left zero of a ternary semigroup (A, f) if $f(x_0, x_1, x_2) = x_0$ for all $x_1, x_2 \in A$.

Let X and Y be nonempty sets. Let $T(X, Y)$ be the set of all mappings of X into Y. Put $T[X, Y] = T(X, Y) \times T(Y, X)$. Define the ternary operation $f : T[X, Y]^3 \to T[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$$

for all $p_i, q_i) \in T[X, Y]$ where $i = 1, 2, 3$.

The algebraic structure $(T[X, Y], f)$ is a ternary semigroup.

Definition 2.4. The ternary semigroup $(T[X, Y], f)$ is called the ternary semigroup of mappings of the sets X and Y.

It is easy to check that the ternary semigroups $(T[X, Y], f)$ and $(T[Y, X], f)$ are isomorphic.

A slightly modified argument applied in the proof of Theorem 3 (cf. [1]) yields the following theorem.

Theorem 2.1. *Every ternary semigroup (A, f) is embeddable into a ternary semigroup $(T[X, Y], f)$ of mappings of some sets X and Y.*

Now we give some definitions concerning partially ordered sets. Partially ordered sets we shall simply call ordered sets. Throughout this paper we shall consider nonempty ordered sets.

Definition 2.5. A mapping $p : X \to Y$ is called a homomorphism of ordered sets X and Y if

$$\forall x_1, x_2 \in X[x_1 \leq x_2 \Rightarrow p(x_1) \leq p(x_2)] .$$

A mapping $p : X \to Y$ is called an isomorphism of ordered sets X and Y if

(i) p is a bijection of X onto Y,

(ii) $\forall x_1, x_2 \in X[x_1 \leq x_2 \Longleftrightarrow p(x_1) \leq p(x_2)]$.

Let $H(X, Y)$ be the set of all homomorphisms from the ordered set X to the ordered set Y. Put $H[X, Y] = H(X, Y) \times H(Y, X)$. Define the ternary operation $f : H[X, Y]^3 \to H[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$$

for all $(p_i, q_i) \in H[X, Y]$ where $i = 1, 2, 3$. The algebraic structure $(H[X, Y], f)$ is a ternary semigroup.

Definition 2.6. The ternary semigroup $(H[X, Y], f)$ is called the ternary semigroup of homomorphisms of the ordered sets X and Y.

3. Some properties of the ternary semigroup $H[X,Y]$

The symbol $E(X)$ denotes the set of all equivalence relations on the set X.

Let X and Y be ordered sets. Consider the smallest equivalence relations $\alpha \in E(X)$ and $\beta \in E(Y)$ containing the ordering relations of the sets X and Y, respectively. The relations α and β are called the connectivity relations.

Suppose that $p \in H(X,Y)$. Define the mapping $p^* : X/\alpha \to Y/\beta$ by the rule

$$p^*([x]_\alpha) = [p(x)]_\beta$$

for $[x]_\alpha \in X/\alpha$. If $[x]_\alpha = [x']_\alpha$, then $(x,x') \in \alpha$. Since p is an order-preserving mapping, it follows that $(p(x),p(x')) \in \beta$, and so $[p(x)]_\beta = [p(x')]_\beta$. Therefore the mapping p is well-defined. Suppose that $q \in H(Y,X)$. Define the mapping $q^* : Y/\beta \to X/\alpha$ by the rule

$$q^*([y]_\beta) = [q(y)]_\alpha$$

for $[y]_\beta \in Y/\beta$.

Let us define the mapping $F : H[X,Y] \to T[X/\alpha, Y/\beta]$ by the formula

$$f(p,q) = (p^*, q^*)$$

for $(p,q) \in H[X,Y]$. We shall prove that the mapping F is an epimorphism of the ternary semigroups $H[X,Y]$ and $T[X/\alpha, Y/\beta]$. Assume that $(p_i, q_i) \in H[X,Y]$ for $i = 1,2,3$. First we prove that $(p_1 \circ q_2 \circ p_3)^* = p_1^* \circ q_2^* \circ p_3^*$ and $(q_1 \circ p_2 \circ q_3)^* = q_1^* \circ p_2^* \circ q_3^*$. Indeed, $(p_1 \circ q_2 \circ p_3)^*([x]_\alpha) = [p_1(q_2(p_3(x)))]_\beta = p_1^*([q_2(p_3(x))]_\alpha) = p_1^*(q_2^*([p_3(x)]_\beta)) = p_1^*(q_2^*(p_3^*([x]_\alpha))) = (p_1^* \circ q_2^* \circ p_3^*)([x]_\alpha)$ for all $[x]_\alpha \in X/\alpha$, and we have $(p_1 \circ q_2 \circ p_3)^* = p_1^* \circ q_2^* \circ p_3^*$. Similarly $(q_1 \circ p_2 \circ q_3)^* = q_1^* \circ p_2^* \circ q_3^*$. Therefore, $F(f((p_1,q_1),(p_2,q_2),(p_3,q_3))) = F(p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3) = ((p_1 \circ q_2 \circ p_3)^*, (q_1 \circ p_2 \circ q_3)^*) = (p_1^* \circ q_2^* \circ p_3^*, q_1^* \circ p_2^* \circ q_3^*) = f((p_1^*, q_1^*), (p_2^*, q_2^*), (p_3^*, q_3^*)) = f(F(p_1,q_1), F(p_2,q_2), F(p_3,q_3))$. Assume that $(r,s) \in T[X/\alpha, Y/\beta]$. Define the following mappings:

$$\mu_1 : X \to X/\alpha, \quad \mu_1(x) = [x]_\alpha, \quad x \in X\,,$$
$$\mu_2 : Y \to Y/\beta, \quad \mu_2(y) = [y]_\beta, \quad y \in Y\,.$$

Consider the choise functions:

$$\omega_1 : X/\alpha \to X, \quad \omega_1([x]_\alpha) \in [x]_\alpha, \quad [x]_\alpha \in X/\alpha,$$
$$\omega_2 : Y/\beta \to Y, \quad \omega_2([y]_\beta) \in [y]_\beta, \quad [y]_\beta \in Y/\beta\,.$$

Define the mappings $p(r) = \omega_2 \circ r \circ \mu_1$ and $q(s) = \omega_1 \circ s \circ \mu_2$. If $x_1, x_2 \in X$ and $x_1 \le x_2$ then $(x_1,x_2) \in \alpha$, this means that $[x_1]_\alpha = [x_2]_\alpha$. Thus $p(r)(x_1) = p(r)(x_2)$. Consequently $p(r) \in H(X,Y)$. Similarly $q(s) \in H(Y,X)$. Notice that $p(r)^*([x]_\alpha) = [p(r)(x)]_\beta = [\omega_2(r(\mu_1(x)))]_\beta = [\omega_2(r([x]_\alpha))]_\beta = r([x]_\alpha)$ for each $[x]_\alpha \in X/\alpha$. Hence $p(r)^* = r$. Similarly $q(s)^* = s$. Consequently $F(p(r), q(s)) = (p(r)^*, q(s)^*) = (r,s)$. Therefore F is an epimorphism.

We have obtained the following result.

Proposition 3.1. *The ternary semigroups $H[X,Y]/Ker(F)$ and $T[X/\alpha, Y/\beta]$ are isomorphic.*

Assume that $P(X)$ denotes the power-set of X. Put $P_0(X) = P(X)\backslash\{\emptyset\}$.

Definition 3.1. A homomorphism $p \in H(X,Y)$ is called a complete homomorphism of X into Y if the following two conditions are fulfilled:
(i) $\forall A \in P_0(X)[\exists \vee A \Rightarrow (\exists \vee p(A) \wedge p(\vee A) = \vee p(A))]$,
(ii) $\forall A \in P_0(X)[\exists \wedge A \Rightarrow (\exists \wedge p(A) \wedge p(\wedge A) = \wedge p(A))]$.

Notice that every constant homomorphism $p \in H(X,Y)$ is a complete homomorphism. Let $SH(X,Y)$ be the collection of all complete homomorphisms of X into Y. Put $SH[X,Y] = SH(X,Y) \times SH(Y,X)$. It is easy to verify that $SH[X,Y]$ is a ternary subsemigroup of the ternary semigroup $H[X,Y]$.

Assume that $(r,s) \in T[X/\alpha, Y/\beta]$. We shall show that $(p(r), q(s)) \in SH[X,Y]$. Suppose that $A \in P_0(X)$ and there exists a join $\vee A$. Since $\forall x \in A : x \leq \vee A$, it follows that $\forall x \in A : (x, \vee A) \in \alpha$. Hence $\mu_1(x) = \mu_1(\vee A)$ for every $x \in A$. Consequently $p(r)(A) = \{p(r)(\vee A)\}$ and $\vee p(r)(A) = p(r)(\vee A)$. Similarly for the meet $\wedge A$. Therefore $p(r) \in SH(X,Y)$. Analogously $q(s) \in SH(Y,X)$. Thus $(p(r), q(s)) \in SH[X,Y]$.

We have obtained the following

Proposition 3.2. *The epimorphism $F : H[X,Y] \to T[X/\alpha, Y/\beta]$ restricted to the set $SH[X,Y]$ is an epimorphism of the ternary semigroup $SH[X,Y]$ onto the ternary semigroup $T[X/\alpha, Y/\beta]$.*

As a consequence, we have

Corollary 3.1. *The ternary semigroups $SH[X,Y]/Ker(F)$ and $T[X/\alpha, Y/\beta]$ are isomorphic.*

Let W and Z be any sets for which there exist surjections $f_1 : X \to W$ and $f_2 : Y \to Z$ such that $\mathrm{Ker}(f_1) = \alpha$ and $\mathrm{Ker}(f_2 = \beta)$. Consider the bijections $\varphi_1 : X/\alpha \to W$ and $\varphi_2 : Y/\beta \to Z$ defined by the formulas:

$$\varphi_1([x]_\alpha) = f_1(x), \quad [x]_\alpha \in X/\alpha\,,$$
$$\varphi_2([y]_\beta) = f_2(y), \quad [y]_\beta \in Y/\beta\,.$$

Define the mapping $G : T[X/\alpha, Y/\beta] \to T[W,Z]$ by the rule:

$$G(r,s) = (\varphi_2 \circ r \circ \varphi_1^{-1}, \varphi_1 \circ s \circ \varphi_2^{-1})$$

for all $(r,s) \in T[X/\alpha, Y/\beta]$.

It is easy to check that G is an isomorphism of the ternary semigroups $T[X/\alpha, Y/\beta]$ and $T[W,Z]$. Put $K = G \circ F$. Thus, we conclude that K is an epimorphism of $H[X,Y]$ onto $T[W,Z]$.

Therefore we have obtained the following

Proposition 3.3. *Let X and Y be ordered sets. Let $\alpha \in E(X)$ and $\beta \in E(Y)$ be connectivity relations. Consider the surjections $f_1 : X \to W$ and $f_2 : Y \to Z$ such that $\mathrm{Ker}(f_1) = \alpha$ and $\mathrm{Ker}(f_2) = \beta$. Then the ternary semigroups $T[X/\alpha, Y/\beta]$ and $T[W, Z]$ are isomorphic. Moreover, the ternary semigroup $T[W, Z]$ is an epimorphic image of the ternary semigroup $H[X, Y]$ by the epimorphism K.*

Assume that $(g, h) \in T[W, Z]$. Consider the choice functions:

$$\omega_1 : X/\alpha \to X, \quad \omega_1([x]_\alpha) \in [x]_\alpha, \quad [x]_\alpha \in X/\alpha$$
$$\omega_2 : Y/\beta \to Y, \quad \omega_2([y]_\beta) \in [y]_\beta, \quad [y]_\beta \in Y/\beta$$

Define the mappings $p(g) = \omega_2 \circ \varphi_2^{-1} \circ g \circ f_1$ and $q(h) = \omega_1 \circ \varphi_1^{-1} \circ h \circ f_2$. Similarly as for the mappings $p(r)$ and $q(s)$ we can show that $(p(g), q(h)) \in SH[X, Y]$. We shall prove that $K(p(g), q(h)) = (g, h)$. Indeed, $K(p(g), q(h)) = (G \circ H)(p(g), q(h)) = (\varphi_2 \circ p(g)^* \circ \varphi_1^{-1}), \varphi_1 \circ q(h)^* \circ \varphi_2^{-1})$. If $w \in W$, then there exists an $x \in X$ such that $f_1(x) = w$. Therefore, $(\varphi_2 \circ p(g)^* \circ \varphi_1^{-1})(w) = \varphi_2(p(g)^*(\varphi_1^{-1}(f_1(x)))) = \varphi_2(p(g)^*([x]_\alpha)) = \varphi_2([p(g)(x)]_\beta) = \varphi_2([\omega_2(\varphi_2^{-1}(g(f_1(x))))]_\beta) = f_2(\omega_2(\varphi_2^{-1}(g(f_1(x))))) = g(f_1(x)) = g(w)$. Thus $\varphi_2 \circ p(g)^* \circ \varphi_1^{-1} = g$. Similarly $\varphi_1 \circ q(h)^* \circ \varphi_2^{-1} = h$. Therefore $K(p(g), q(h)) = (g, h)$.

From the foregoing it follows the following

Proposition 3.4. *The epimorphism $K : H[X, Y] \to T[W, Z]$ restricted to the set $SH[X, Y]$ is an epimorphism of the ternary semigroup $SH[X, Y]$ onto the ternary semigroup $T[W, Z]$.*

As a consequence, we have

Corollary 3.2. *The ternary semigroups $SH[X, Y]/Ker(K)$ and $T[W, Z]$ are isomorphic.*

Summarizung we get the following

Corollary 3.3. *The ternary semigroups $H[X, Y]/Ker(F)$, $H[X, Y]/Ker(K)$, $SH[X, Y]/Ker(F)$, $SH[X, Y]/Ker(K)$, $T[X/\alpha, Y/\beta]$, $T[W, Z]$ are isomorphic.*

4. Main result

The main result of this paper is contained in Theorem 4.1. Let X and Y be ordered sets. Consider the following sets:

$$H_0(X, Y) = \{p \in H(X, Y) : \exists y_0 \in Y \forall x \in X : p(x) = y_0\},$$
$$H_0(Y, X) = \{q \in H(Y, X) : \exists x_0 \in X \forall y \in Y : q(y) = x_0\}.$$

The homomorphisms $p \in H_0(X, Y)$ and $q \in H_0(Y, X)$ such that their single values are $y_0 \in Y$ and $x_0 \in X$ we denote by p_{y_0} and q_{x_0}, respectively. Define the partial order on the set $H_0(X, Y)$ by the rule:

$$p_{y_0} \le p_{y_0'} \Longleftrightarrow y_0 \le y_0'$$

for $p_{y_0}, p_{y'_0} \in H_0(X, Y)$.
Define the partial order on the set $H_0(Y, X)$ by the rule:

$$q_{x_0} \leq q_{x'_0} \Longleftrightarrow x_0 \leq x'_0$$

for $q_{x_0}, q_{x'_0} \in H_0(Y, X)$.

Put $H_0[X, Y] = H_0(X, Y) \times H_0(Y, X)$. It is easy to notice that $H_0[X, Y]$ is a ternary subsemigroup of the ternary semigroup $H[X, Y]$.

Lemma 4.1. *Let X and Y be ordered sets. A pair of homomorphisms (p, q) is a left zero of the ternary semigroup $H[X, Y]$ if and only if $(p, q) \in H_0[X, Y]$.*

Proof. Let (p, q) be a left zero of $H[X, Y]$. By Definition 2.3 we have $f((p, q), (p_1, q_1), (p_2, q_2)) = (p, q)$ for all $(p_1, q_1), (p_2, q_2) \in H[X, Y]$. Put $(p_1, q_1) = (p_{y_0}, q_{x_0})$ for some $x_0 \in X$ and $y_0 \in Y$. Hence $f((p, q), (p_{y_0}, q_{x_0}), (p_2, q_2)) = (p, q)$ and $p = p \circ q_{x_0} \circ p_2$, $q = q \circ p_{y_0} \circ q_2$. Therefore, $\forall x \in X : p(x) = p(x_0)$ and $\forall y \in Y : q(y) = q(y_0)$, and so $(p, q) \in H_0[X, Y]$.

Suppose that $(p, q) \in H_0[X, Y]$. Consequently $p = p_{y_0}$ and $q = q_{x_0}$ for some $x_0 \in X$ and $y_0 \in Y$. For any $(p_1, q_1), (p_2, q_2) \in H[X, Y]$ we have: $f((p, q), (p_1, q_1), (p_2, q_2)) = f((p_{y_0}, q_{x_0}), (p_1, q_1), (p_2, q_2)) = (p_{y_0} \circ q_1 \circ p_2, q_{x_0} \circ p_1 \circ q_2) = (p_{y_0}, q_{x_0}) = (p, q)$. Therefore, the pair (p, q) is a left zero of the ternary semigroup $H[X, Y]$. $\square$

Proposition 4.1. *The set $H_0[X, Y]$ is the smallest ideal of the ternary semigroup $H[X, Y]$.*

Proof. It is easy to check that $H_0[X, Y]$ is an ideal of $H[X, Y]$. Put $I_0 = H_0[X, Y]$. Let $I \subset H[X, Y]$ be an ideal of $H[X, Y]$. By Lemma 4.1 $f(I_0, I, I) = I_0$. On the other hand, $f(I_0, I, I) \subset I$. Hence $I_0 \subset I$. $\square$

Lemma 4.2. *Let X_i and Y_i for $i = 1, 2$ be ordered sets. Let $F : H[X_1, Y_1] \to H[X_2, Y_2]$ be an epimorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$. Then $F(H_0[X_1, Y_1]) = H_0[X_2, Y_2]$.*

Proof. Suppose that $(p, q) \in H_0[X_1, Y_1]$. By Lemma 4.1 we have $f((p, q), (p_1, q_1), (p_2, q_2)) = (p, q)$ for all $(p_1, q_1), (p_2, q_2) \in H[X_1, Y_1]$. Therefore, $f(F(p, q), F(p_1, q_1), F(p_2, q_2)) = F(p, q)$ for all (p_1, q_1), $(p_2, q_2) \in H[X_1, Y_1]$. Again by Lemma 4.1 $F(p, q) \in H_0[X_2, Y_2]$.

Suppose that $(r, s) \in H_0[X_2, Y_2]$. This implies that $r = r_{y_2}$ and $s = s_{x_2}$ for some $x_2 \in X_2$ and $y_2 \in Y_2$. There exists a pair $(p', q') \in H[X_1, Y_1]$ such that $F(p', q') = (r_{y_2}, s_{x_2})$. Assume that $(p_{y'_1}, q_{x'_1}) \in H_0[X_1, Y_1]$ is an arbitrary fixed pair and $(p_1, q_1) \in H[X_1, Y_1]$. Put $(p, q) = f((p', q'), (p_{y'_1}, q_{x'_1}), (p_1, q_1))$. Hence $p = p' \circ q_{x'_1} \circ p_1$ and $q = q' \circ p_{y'_1} \circ q_1$. Set $y_1 = p'(x'_1)$ and $x_1 = q'(y'_1)$. Thus $p = p_{y_1}$ and $q = q_{x_1}$, hence $(p, q) \in H_0[X_1, Y_1]$. We have $F(p, q) = f(F(p', q'), F(p_{y'_1}, q_{x'_1}), F(p_1, q_1)) = f((r_{y_2}, s_{x_2}), F(p_{y'_1}, q_{x'_1}), F(p_1, q_1)) = (r_{y_2}, s_{x_2}) = (r, s)$. Therefore, there exists a pair $(p, q) \in H_0[X_1, Y_1]$ such that $F(p, q) = (r, s)$.

Notice that a mapping $F_0 : H_0[X_1, Y_1] \to H_0[X_2, Y_2]$ is an isomorphism of the ternary semigroups $H_0[X_1, Y_1]$ and $H_0[X_2, Y_2]$ if and only if F_0 is a bijection.

Let X_i and Y_i for $i = 1, 2$ be ordered sets. Suppose that $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ are isomorphisms of ordered sets. Define the mapping $F : H[X_1, Y_1] \to H[X_2, Y_2]$ by the rule:

$$F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1}) \tag{1}$$

for all $(p, q) \in H[X_1, Y_1]$.

It is easy to check that F is an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$. □

Definition 4.1. The mapping F defined by the formula (1) is called the isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ induced by the pair of isomorphisms (f_1, f_2).

An isomorphism $F : H[X_1, Y_1] \to H[X_2, Y_2]$ of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ need not imply the existence of isomorphisms $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ of the ordered sets X_1, X_2, Y_1, Y_2.

Example. Consider the following sets:

$$X_1 = \{x_{11}, x_{12}, x_{13}\}, \quad Y_1 = \{y_1\},$$
$$X_2 = \{x_{21}, x_{22}, x_{32}\}, \quad Y_2 = \{y_2\}.$$

Assume that X_2, Y_1, Y_2 are trivially ordered sets. Define the partial order $\leq$ in the set X_1 by the following rules: $x_{11} \leq x_{11}$, $x_{12} \leq x_{12}$, $x_{13} \leq x_{13}$, $x_{12} \leq x_{13}$. Thus we have:

$$H(X_1, Y_1) = \{p_{y_1}\}, \quad H(Y_1, X_1) = \{q_{x_{11}}, q_{x_{12}}, q_{x_{13}}\},$$
$$H(X_2, Y_2) = \{p_{y_2}\}, \quad H(Y_2, X_2) = \{q_{x_{21}}, q_{x_{22}}, q_{x_{23}}\}.$$

Hence

$$H[X_1, Y_1] = \{(p_{y_1}, q_{x_{11}}), (p_{y_1}, q_{x_{12}}), (p_{y_1}, q_{x_{13}})\},$$
$$H[X_2, Y_2] = \{(p_{y_2}, q_{x_{21}}), (p_{y_2}, q_{x_{22}}), (p_{y_2}, q_{x_{23}})\}.$$

Thus $H[X_1, Y_1] = H_0[X_1, Y_1]$ and $H[X_2, Y_2] = H_0[X_2, Y_2]$. Let us take any bijection $F : H[X_1, Y_1] \longrightarrow H[X_2, Y_2]$. The mapping F is an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$. Notice that the ordered sets X_1 and X_2 are not isomorphic.

Let X_i and Y_i for $i = 1, 2$ be ordered sets. Let $F : H[X_1, Y_1] \to H[X_2, Y_2]$ be an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ induced by the pair of isomorphisms (f_1, f_2). Assume that p_{y_1}, $p_{y_1'} \in H_0(X_1, Y_1)$ and $q, q' \in H_0(X_1, Y_1)$. We have

$$F(p_{y_1}, q) = (f_2 \circ p_{y_1} \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1}),$$
$$F(p_{y_1'}, q') = (f_2 \circ p_{y_1'} \circ f_1^{-1}, f_1 \circ q' \circ f_2^{-1}).$$

Notice that $f_2 \circ p_{y_1} \circ f_1^{-1} = r_{f_2(y_1)}$ and $f_2 \circ p_{y_1'} \circ f_1^{-1} = r_{f_2(y_1')}$, this means that $r_{f_2(y_1)}$, $r_{f_2(y_1')} \in H_0(X_2, Y_2)$. If $p_{y_1} \leq p_{y_1'}$, then $y_1 \leq y_1'$. Since $f_2(y_1) \leq f_2(y_1')$, it follows that $r_{f_2(y_1)} \leq r_{f_2(y_1')}$. Conversely, suppose that $r_{f_2(y_1)} \leq r_{f_2(y_1')}$. Hence $f_2(y_1) \leq f_2(y_1')$, and so $y_1 \leq y_1'$. This means that $p_{y_1} \leq p_{y_1'}$. Let us denote by π_1 and π_2 the projections of Cartesian product. From the foregoing we have obtained the following condition:

$$(\mathrm{W}_1) \quad \forall p, p' \in H_0(X_1, Y_1) \forall q, q' \in H_0(Y_1, X_1)[p \leq p' \Longleftrightarrow \pi_1(F(p, q)) \leq \\ \leq \pi_1(F(p', q'))] \,.$$

A similar argument yields the following condition:

$$(\mathrm{W}_2) \quad \forall p, p' \in H_0(X_1, Y_1) \forall q, q' \in H_0(Y_1, X_1)[q \leq q' \Longleftrightarrow \pi_2(F(p, q)) \leq \\ \leq \pi_2(F(p', q'))] \,.$$

Notice that the isomorphism $F : H[X_1, Y_1] \to H[X_2, Y_2]$ defined in the former example does not satisfy the condition (W_2).

Theorem 4.1. *Let X_i and Y_i for $i = 1, 2$ be ordered sets such that the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ are isomorphic. The isomorphism $F : H[X_1, Y_1] \to H[X_2, Y_2]$ of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ is induced by the pair of isomorphisms (f_1, f_2) if and only if the isomorphism F satisfies the conditions (W_1) and (W_2).*

Proof. We have proved that the isomorphism F induced by the pair of isomorphisms (f_1, f_2) satisfies the conditions (W_1) and (W_2).

Let us assume that $F : H[X_1, Y_1] \to H[X_2, Y_2]$ is an isomorphism of the ternary semigroups $H[X_1, Y_1]$ and $H[X_2, Y_2]$ such that the conditions (W_1) and (W_2) are satisfied.
In view of Lemma 4.2 we can define the mapping $F^* : X_1 \times Y_1 \to X_2 \times Y_2$ by the formula:

$$F^*(x_1, y_1) = (x_2, y_2) \Longleftrightarrow F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2}) \tag{2}$$

for $(x_1, y_1) \in X_1 \times Y_1$, $(x_2, y_2) \in X_2 \times Y_2$.
It is easy to notice that $F^*$is a bijection. Let $y_0 \in Y_1$ be an arbitrary fixed element. We define the mapping $f_1 : X_1 \to X_2$ by the rule:

$$f_1(x_1) = x_2 \Longleftrightarrow \pi_1(F^*(x_1, y_0)) = x_2 \tag{3}$$

for $x_1 \in X_1$, $x_2 \in X_2$. We will prove that

$$f_1(x_1) = x_2 \Longleftrightarrow \forall y_1 \in Y_1 : \pi_1(F^*(x_1, y_1)) = x_2$$

for $x_1 \in X_1, x_2 \in X_2$. Suppose that $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(x_1, y_1) = (x_2', y_2')$ for an arbitrary fixed element $y_1 \in Y_1$, and $x_1 \in X_1$, x_2, $x_2' \in X_2$, y_2, $y_2' \in Y_2$. Thus we have

$$F^*(x_1, y_0) = (x_2, y_2) \Longleftrightarrow F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2}) \,,$$
$$F^*(x_1, y_1) = (x_2', y_2') \Longleftrightarrow F(p_{y_1}, q_{x_1}) = (r_{y_2'}, s_{x_2'}) \,.$$

In view of the condition (W_2) we infer that $s_{x_2} = s_{x'_2}$, and so $x_2 = x'_2$. Therefore,

$$f_1(x_1) = x_2 \iff \forall y_1 \in Y_1 : \pi_1(F^*(x_1, y_1)) = x_2 \quad \text{for} \quad x_1 \in X_1, x_2 \in X_2\,, \tag{4}$$

$$f_1(x_1) = x_2 \iff \exists y_1 \in Y_1 : \pi_1(F^*(x_1, y_1)) = x_2 \quad \text{for} \quad x_1 \in X_1, x_2 \in X_2\,. \tag{5}$$

Next we will prove that $f_1 : X_1 \to X_2$ is a bijection. Suppose that $x_2 \in X_2$. Let us take an arbitrary fixed element $y_2 \in Y_2$. Thus there exists a pair $(x_1, y_1) \in X_1 \times Y_1$ such that $F^*(x_1, y_1) = (x_2, y_2)$. Therefore using the condition (5) we obtain $f_1(x_1) = x_2$, and so f_1 is a surjection. Suppose that $f_1(x_1) = f_1(x'_1)$ for $x_1, x'_1 \in X_1$. Hence $f_1(x_1) = x_2$ and $f_1(x'_1) = x_2$ for some $x_2 \in X_2$. By (3) it follows that $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(x'_1, y_0) = (x_2, y'_2)$ for some $y_2, y'_2 \in Y_2$. Hence we have $F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2})$ and $F(p_{y_0}, q_{x'_1}) = (r_{y'_2}, s_{x_2})$. Using ($W_2$) we get $q_{x_1} = q_{x'_1}$, and so $x_1 = x'_1$. Therefore f_1 is an injection. We shall prove that

$$\forall x_1, x'_1 \in X_1 [x_1 \leq x'_1 \iff f_1(x_1) \leq f_1(x'_1)]\,. \tag{6}$$

Suppose that $x_1 \leq x'_1$ for $x_1, x'_1 \in X_1$. Set $f_1(x_1) = x_2$ and $f_1(x'_1) = x'_2$ where x_2, $x'_2 \in X_2$. Hence

$$f_1(x_1) = x_2 \iff \pi_1(F^*(x_1, y_0)) = x_2\,,$$
$$f_1(x'_1) = x'_2 \iff \pi_1(F^*(x'_1, y_0)) = x'_2\,.$$

Thus $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(x'_1, y_0) = (x'_2, y'_2)$ for some y_2, $y'_2 \in Y_2$. We have

$$F^*(x_1, y_0) = (x_2, y_2) \iff F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2})\,,$$
$$F^*(x'_1, y_0) = (x'_2, y'_2) \iff F(p_{y_0}, q_{x'_1}) = (r_{y'_2}, s_{x'_2})\,.$$

Since $q_{x_1} \leq q_{x'_1}$, the condition (W_2) yields $s_{x_2} \leq s_{x'_2}$, that is $x_2 \leq x'_2$, and so $f_1(x_1) \leq f_1(x'_1)$. It is easy to notice that $f_1(x_1) \leq f_1(x'_1)$ implies $x_1 \leq x'_1$ for each x_1, $x'_1 \in X_1$. Therefore we have proved the condition (6).

Summarizing, the mapping $f_1 : X_1 \longrightarrow X_2$ is an isomorphism of the ordered sets X_1 and X_2.

Let $x_0 \in X_1$ be an arbitrary fixed element. We define the mapping $f_2 : Y_1 \to Y_2$ by the formula:

$$f_2(y_1) = y_2 \iff \pi_2(F^*(x_0, y_1)) = y_2 \tag{7}$$

for $y_1 \in Y_1$ and $y_2 \in Y_2$. The analogous argument applied to the mapping f_2 allows to prove that

$$f_2(y_1) = y_2 \iff \forall x_1 \in X_1 : \pi_2(F^*(x_1, y_1)) = y_2\,, \tag{8}$$

$$f_2(y_1) = y_2 \iff \exists x_1 \in X_1 : \pi_2(F^*(x_1, y_1)) = y_2 \tag{9}$$

for $y_1 \in Y_1$ and $y_2 \in Y_2$. We can similarly show that the mapping $f_2 : Y_1 \to Y_2$ is an isomorphism of the ordered sets Y_1 and Y_2. By the conditions (4) and (8) we get $F^*(x_1,y_1) = (\pi_1(F^*(x_1,y_1)),\ \pi_2(F^*(x_1,y_1))) = (f_1(x_1), f_2(y_1))$ for each $(x_1,y_1) \in X_1 \times Y_1$. Consequently,

$$F^* = (f_1, f_2)\,. \tag{10}$$

We shall prove that the isomorphism F is induced by the pair of isomorphisms (f_1, f_2). First, we shall show that the following condition is satisfied:

$$\begin{aligned}&\forall x_1 \in X_1 \forall y_1 \in Y_1 \forall (p,q) \in H[X_1,Y_1] : F(p,q)(f_1(x_1), f_2(y_1)) = \\ &\qquad (f_2(p(x_1)), f_1(q(y_1)))\,.\end{aligned} \tag{11}$$

Suppose that $x_1 \in X_1$, $y_1 \in Y_1$, and $(p,q), (p_1,q_1) \in H[X_1,Y_1]$. Hence $f((p,q), (p_{y_1}, q_{x_1}), (p_1,q_1)) = (p \circ q_{x_1} \circ p_1, q \circ p_{y_1} \circ q_1) = (p_{p(x_1)}, q_{q(y_1)})$. We have $F(p_{p(x_1)}, q_{q(y_1)}) = F(f((p,q),(p_{y_1},q_{x_1}),(p_1,q_1))) = f(F(p,q), F(p_{y_1},q_{x_1}), F(p_1,q_1))$. Set $F(p,q) = (r,s)$ and $F(p_1,q_1) = (r_1,s_1)$. By Lemma 4.2 we get $F(p_{y_1},q_{x_1}) = (r_{y_2}, s_{x_2})$ for some $x_2 \in X_2$, $y_2 \in Y_2.By(10)$ $F(p_{y_1},q_{x_1}) = (r_{y_2},s_{x_2}) \Longleftrightarrow F^*(x_1,y_1)$ $(x_2,y_2) \Longleftrightarrow (f_1(x_1), f_2(y_1)) = (x_2,y_2) \Longleftrightarrow (x_2 = f_1(x_1) \wedge y_2 = f_2(y_1))$. Therefore, $F(p_{p(x_1)}, q_{q(y_1)}) = f((r,s),(r_{f_2(y_1)}, s_{f_1(x_1)}),(r_1,s_1)) = (r \circ s_{f_1(x_1)} \circ r_1, s \circ r_{f_2(y_1)} \circ s_1) = (r_{r(f_1(x_1))}, s_{s(f_2(y_1))})$. On the other hand, $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{y_2}, s_{x_2})$ for some $x_2 \in X_2$, $y_2 \in Y_2$. By (10) $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{y_2}, s_{x_2}) \Longleftrightarrow F^*(q(y_1), p(x_1)) = (x_2,y_2) \Longleftrightarrow (f_1(q(y_1)), f_2(p(x_1))) = (x_2,y_2) \Longleftrightarrow (x_2 = f_1(q(y_1)) \wedge y_2 = f_2(p(x_1)))$. Therefore, $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{f_2(p(x_1))}, s_{f_1(q(y_1))})$. Consequently, $r(f_1(x_1)) = f_2(p(x_1))$ and $s(f_2(y_1)) = f_1(q(y_1))$. Thus, $F(p,q)(f_1(x_1), f_2(y_1)) = (r,s)(f_1(x_1), f_2(y_1)) = (r(f_1(x_1)), s(f_2(y_1))) = (f_2(p(x_1)), f_1(q(y_1)))$. Therefore we have obtained the formula (11).

For $x_2 \in X_2$ and $y_2 \in Y_2$ there exist $x_1 \in X_1$ and $y_1 \in Y_1$ such that $f_1(x_1) = x_2$ and $f_2(y_1) = y_2$. Hence $x_1 = f_1^{-1}(x_2)$ and $y_1 = f_2^{-1}(y_2)$. Using the formula (11) we obtain $F(p,q)(x_2,y_2) = ((f_2 \circ p \circ f_1^{-1})(x_2), (f_1 \circ q \circ f_2^{-1})(y_2)) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})(x_2,y_2)$ for any pair $(p,q) \in H[X_1,Y_1]$. Therefore, $F(p,q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$ for each $(p,q) \in H[X_1,Y_1]$. Finally, we conclude that the isomorphism F is induced by the pair of isomorphisms (f_1,f_2) defined by the formulas (3) and (7). The proof of Theorem 4.1 is completed. □

Definition 4.2. Let X_i and Y_i for $i = 1,2$ be ordered sets. The ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ are called W-isomorphic if there exists an isomorphism $F : H[X_1,Y_1] \to H[X_2,Y_2]$ of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ fulfilling the conditions (W_1) and (W_2).

From Theorem 4.1 we deduce the following two corollaries.

Corollary 4.1. *Let X_i and Y_i for $i = 1,2$ be ordered sets. The ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$ are W-isomorphic if and only if the ordered sets X_1 and X_2 are isomorphic and the ordered sets Y_1 and Y_2 are isomorphic.*

Corollary 4.2. *Let X_i and Y_i for $i = 1,2$ be ordered sets such that there exists an isomorphism $G : H[X_1,Y_1] \to H[X_2,Y_2]$ of the ternary semigroups $H[X_1,Y_1]$ and $H[X_2,Y_2]$. The ordered sets X_1 and X_2 are isomorphic and the ordered sets*

Y_1 and Y_2 are isomorphic if and only if there exists an automorphism μ of the ternary semigroup $H[X_1, Y_1]$ such that the isomorphism $F = G \circ \mu$ satisfies the conditions (W_1) and (W_2).

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