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FINITE INTERPOLATION ON SEQUENCES IN THE DISC

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Abstract. This note deals with interpolation of values of analytic functions belonging to a given space, on finite sets of consecutive points of sequences in the disc, performed by rational functions and polynomials. Our goal is to identify sequences and spaces whose functions provide a bound of the error at the first uninterpolated point that is as small as desired. For certain sequences, we prove that this happens for bounded functions, Lipschitz functions and those that have derivatives in the disc algebra.

1. Introduction and statement of results

Let $\mathcal{Z} = (z_i)_{i=0}^{\infty}$ denote any sequence in the open unit disc $\mathbb{D}$ of the complex plane. We write $D(\mathcal{Z}) = \sup_{i,j} |z_i - z_j|$ and choose the finite subset $\mathcal{Z}_n = \{z_0, \dots, z_n\}$ without losing generality for our purposes. Let $\mathcal{H}$ denote the space of all analytic functions in $\mathbb{D}$ and let $f^{(k)}$ denote the k-th derivative of a function f in $\mathcal{H}$ ($f^{(0)} := f$). For z, $w \in \mathbb{D}$, we write $\psi(z,w) = (w-z)/(1-\overline{w}z)$, so that $|\psi(z,w)|$ is the pseudo-hyperbolic distance on $\mathbb{D}$. It turns out that $|\psi(z,w)| < 1$ and the diameter of $\mathbb{D}$ for this distance is 1 ([4]). Let $G(\mathcal{Z}) = \sup_{i,j} |\psi(z_i, z_j)|$. We put c for positive constants.

Let H^∞ denote the space of functions $f \in \mathcal{H}$ such that

$$\|f\|_\infty = \sup_{z \in \overline{\mathbb{D}}} |f(z)| < c$$

and let A be the disc algebra of functions in $\mathcal{H}$ that are continuous on $\overline{\mathbb{D}}$. For an integer $p > 0$, let A^p be the algebra of functions f such that $f^{(k)} \in A$, $k = 0, \dots, p$. If $f \in A^p$, then

$$\left| f(z_i) - f(z_j) - f'(z_j)\frac{z_i - z_j}{1!} - \dots - f^{(p)}(z_j)\frac{(z_i - z_j)^p}{p!} \right| = o(|z_i - z_j|^p) \tag{1}$$

uniformly in z_i, z_j. Let A^∞ denote the algebra of functions f such that $f^{(k)} \in A$ for all $k \geq 0$. If $f \in A^\infty$, then the estimate in (1) holds for all p. Let Λ^p be the Lipschitz class of order p consisting of all functions $f \in A^{p-1}$ ($A^0 := A$) such that

$$\left| f^{(p-1)}(z) - f^{(p-1)}(w) \right| = O(|z - w|)\,.$$

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If $f \in \Lambda^p$, then

$$\Big| f(z_i) - f(z_j) - f'(z_j)\frac{z_i - z_j}{1!} - \cdots - f^{(p-1)}(z_j)\frac{(z_i - z_j)^{p-1}}{(p-1)!}\Big| = O(|z_i - z_j|^p)$$

uniformly in z_i, z_j. If $s > 0$ is not an integer and $m = [s]$, let Λ^s be the Lipschitz class of order s consisting of all functions $f \in A^m$ for which

$$|f^{(m)}(z) - f^{(m)}(w)| = O(|z - w|^{s-m}) .$$

In [3], Λ^s is defined as the space of all jets $f = (f^{(0)}, \ldots, f^{(m)})$ of continuous functions such that

$$\begin{aligned}\Big| f^{(k)}(z_i) - f^{(k)}(z_j) - f^{(k+1)}(z_j)\frac{z_i - z_j}{1!} - \cdots - f^{(m)}(z_j)\frac{(z_i - z_j)^{m-k}}{(m-k)!}\Big| \\ = O(|z_i - z_j|^{s-k})\end{aligned} \tag{2}$$

uniformly in z_i, z_j, $k = 0, 1, \ldots, m$.

Given a function f in $\mathcal{H}$ and a sequence $\mathcal{Z}$, we denote by $Q_n(z)$ a rational function of degree at most n ($Q_n = \frac{p}{q}$, where p and q are polynomials and degree$\,(p) =$ degree$\,(q) \le n$) interpolating f on $\mathcal{Z}_n$, namely, $Q_n(z_i) = f(z_i)$, $i = 0, \ldots, n$. We put $E[Q_n]$ for the "error" at the first uninterpolated point of $\mathcal{Z}$, that is,

$$E[Q_n] = |Q_n(z_{n+1}) - f(z_{n+1})|.$$

Let $R_n(z)$ be the rational function defined, as in Newton's divided differences ([2]), by

$$R_n(z) = [z_0] + \sum_{k=1}^{n} [z_0, \ldots, z_k] \prod_{j=0}^{k-1} \psi(z, z_j) , \quad z \in \mathbb{D} , \tag{3}$$

where $[z_i] = f(z_i)$, and

$$[z_i, \ldots, z_j] = \frac{[z_{i+1}, \ldots, z_j] - [z_i, \ldots, z_{j-1}]}{\psi(z_j, z_i)} , \quad i < j . \tag{4}$$

As a consequence of the Schwarz lemma, if $f \in H^\infty$, then

$$|f(z) - f(w)| \le c\,|\psi(z, w)| , \tag{5}$$

so that $[z_i, z_j]$ is bounded for any i, j, but also all quotients $[z_i, \ldots, z_j]$ in (4) are bounded (see ([6]). Thus,

$$E[R_n] \le c\,|\psi(z_{n+1}, z_0)| + c \sum_{k=1}^{n} \prod_{j=0}^{k-1} |\psi(z_{n+1}, z_j)| < (n+1)c\,|\psi(z_{n+1}, z_0)| . \tag{6}$$

In [5], sequences that provide Q_n with $E[Q_n] \le c(n)\,|z_{n+1} - z_n|$ are given for the Lipschitz class of order 1. This bound and that of (6) facilitate a simple control of the error, but they do not allow it to be as small as desired. The motivation of this paper is to get the error so. Regarding this aim, we pose the following sequences $\mathcal{Z}$ for a given subspace S of $\mathcal{H}$.

Definition 1.1. We say that $\mathcal{Z}$ is S-placed if for each f in S and $\epsilon > 0$, there exists $n \in \mathbb{N}$ and a rational function Q_n, such that Q_n interpolates f on $\mathcal{Z}_n$ and $E[Q_n] < \epsilon$.

Definition 1.2. We say that $\mathcal{Z}$ is S-placed in the strong sense if for each f in S and $\epsilon > 0$, there exists $n \in \mathbb{N}$ and a polynomial P_n of degree n, such that P_n interpolates f on $\mathcal{Z}_n$ and $E[P_n] < \epsilon$.

We distinguish between "placed" and "placed in the strong sense", depending on if the interpolation is performed by a rational function or a polynomial. We determine different function spaces and sequences with these properties.

Definition 1.3. We say that $\mathcal{Z}$ is compatible for the distance d if for each $n \geq 2$,

$$d(z_n, z_i) \leq c\, d(z_i, z_j)\,, \quad 0 \leq i, j < n\,, \tag{7}$$

where the constant c is independent of n.

Our results, which will be proven in Section 3, are the following ones.

Proposition 1.1. *If $\mathcal{Z}$ is compatible for the pseudo-hyperbolic distance and $G(\mathcal{Z}) < \frac{1}{2}$, then $\mathcal{Z}$ is H^∞-placed. Besides, given $f \in H^\infty$ and $\epsilon > 0$,*

(i) *For sufficiently large m, there exists a rational function Q_m that interpolates f on $\mathcal{Z}_m$, satisfies $E[Q_m] < \epsilon$, and, on a subsequence $(z_{n_i})_i$ of $\mathcal{Z}$, also satisfies*

$$|Q_m(z_{n_i}) - f(z_{n_i})| < \epsilon$$

for any $n_i > m$.

(ii) *If Q_m is as in* (i) *for all $m \geq N$, then f is the uniform limit on the compact set $(z_{n_i})_i \cup \{\lim_i z_{n_i}\}$ of a subsequence of $(Q_m)_{m \geq N}$.*

Proposition 1.2. *If $\mathcal{Z}$ is compatible for the Euclidean distance and $D(\mathcal{Z}) < \frac{1}{2}$, then there exists*

a) *$s_0 > 1$ such that $\mathcal{Z}$ is Λ^s-placed in the strong sense for all $s > s_0$.*
b) *$p_0 > 1$ such that $\mathcal{Z}$ is Λ^p-placed in the strong sense for all $p > p_0$.*
c) *$q_0 > 1$ such that $\mathcal{Z}$ is A^q-placed in the strong sense for all $q > q_0$.*

As an immediate consequence of Proposition 1.2 c), we have the following result.

Corollary 1.1. *If $\mathcal{Z}$ is compatible for the Euclidean distance and $D(\mathcal{Z}) < \frac{1}{2}$, then $\mathcal{Z}$ is A^∞-placed in the strong sense.*

For example, $\mathcal{Z} = (3^{-(i+1)})_{i=0}^\infty$ is compatible for distances pseudo-hyperbolic and Euclidean, and $G(\mathcal{Z}) = D(\mathcal{Z}) = \frac{1}{3}$.

2. Crucial Lemma

We prove an estimate for a function in H^∞ and all its derivatives up to any order, which will be used in the proof of Proposition 1.1. For a fixed $w \in \mathbb{D}$, let $\tau_w(z) := \psi(z, w)$ be the automorphism of $\mathbb{D}$ interchanging w and 0. We recall that the Bell polynomials are

$$\beta_N^j(x_1, \dots, x_{N-j+1}) = \sum \frac{N!}{m_1! \cdots m_{N-j+1}!} \Big(\frac{x_1}{1!}\Big)^{m_1} \cdots \Big(\frac{x_{N-j+1}}{(N-j+1)!}\Big)^{m_{N-j+1}},$$

where the sum is extending over all sequences $m_1, \dots, m_{N-j+1}$ of non-negative integers such that

$$\begin{cases} m_1 + \cdots + m_{N-j+1} = j \\ m_1 + \cdots + (N-j+1)m_{N-j+1} = N\,, \end{cases}$$

and the Stirling numbers of the second kind are $S(N,j) = \beta_N^j(1, \dots, 1)$.

Lemma 2.1. *Let*

$$A(f,N,w) = \frac{\sum_{j=1}^N S(N,j)\overline{w}^{N-j}(|w|^2-1)^j f^{(j)}(w)}{N!}\,.$$

If $f \in H^\infty$, then for all $k \in \mathbb{N}$,

$$|f(z) - f(w) - \sum_{N=1}^k A(f,N,w)\psi(w,z)^N| = O(|\psi(w,z)|^{k+1})\,, \quad w, z \in \mathbb{D}\,. \tag{8}$$

Proof. We write

$$T_k f(0,z) = f(0) + f'(0)z + \cdots + \frac{f^{(k)}(0)}{k!} z^k$$

and define the function

$$g_k(z) = \begin{cases} \frac{f(z) - T_k f(0,z)}{z^{k+1}}\,, & \text{if } z \neq 0\,; \\ \frac{f^{(k+1)}(0)}{(k+1)!}\,, & \text{if } z = 0\,. \end{cases}$$

It is immediate to check that g_k is analytic and

$$\|g_k\|_\infty \le \|f\|_\infty + |f(0)| + |f'(0)| + \cdots + \frac{|f^{(k)}(0)|}{k!} \le (k+2)\|f\|_\infty\,.$$

Then,

$$|f(z) - T_k f(0,z)| \le c\,|z|^{k+1}\,.$$

Replacing f by $f \circ \tau_w$,

$$|(f \circ \tau_w)(z) - T_k(f \circ \tau_w)(0,z)| \le c\,|z|^{k+1}\,. \tag{9}$$

We have

$$T_k(f \circ \tau_w)(0,z) = f(w) + \sum_{N=1}^k \frac{\sum_{j=1}^N \beta_N^j(\tau_w'(0), \dots, \tau_w^{N-j+1)}(0)) f^{(j)}(\tau_w(0))}{N!} z^N\,.$$

Replacing z by $\tau_w(z)$ in (9) and taking into account that

$$\tau_w^{(k)}(z) = \frac{k!\,\overline{w}^{k-1}(|w|^2-1)}{(1-\overline{w}z)^{k+1}},$$

(9) becomes

$$\Big| f(z) - f(w) - \sum_{N=1}^{k} \frac{\sum_{j=1}^{N} \beta_N^j((|w|^2-1), \dots, (N-j+1)!\,\overline{w}^{N-j}(|w|^2-1)) f^{(j)}(w)}{N!} \psi(w,z)^N \Big| \leq c\,|\psi(w,z)|^{k+1}$$

and (8) follows. □

3. Proof of results

Proof of Proposition 1.1. Let $f \in H^\infty$ and $\epsilon > 0$. We define

$$\alpha_n(z) = \sum_{N=1}^{n} A(f,N,z)\psi(z,z_{n+1})^N\,, \quad z \in \mathbb{D}\,,$$

where n will be determined later. We take $[z_i] := f(z_i) - \alpha_n(z_i)$, $[z_i, \dots, z_j]$ as in (4), and $q_n(z)$ as $R_n(z)$ in (3). Since $|f^{(j)}(w)| = O((1-|w|^2)^{-j})$ for all $f \in$ BMOA ([1]) and $H^\infty \subset$ BMOA, note that $|A(f,N,z)| = O(1)$ and then, $|\alpha_n(z)| = O(|\psi(z,z_{n+1})|)$. We have that $Q_n = q_n + \alpha_n$ is a rational function of degree at most n interpolating f on $\mathcal{Z}_n$. On the other hand,

$$E[Q_n] = E[q_n] < |f(z_0) - \alpha_n(z_0) - f(z_{n+1})| + |\psi(z_{n+1},z_0)| \sum_{k=1}^{n}[z_0, \dots, z_k]. \tag{10}$$

By (8), the first term in (10) is $O(|\psi(z_{n+1},z_0)|^{n+1})$. By (8) and (7),

$$\begin{aligned} |[z_i,z_j]| &\leq \frac{|f(z_i) - \alpha_n(z_i) - f(z_{n+1})| + |f(z_j) - \alpha_n(z_j) - f(z_{n+1})|}{|\psi(z_i,z_j)|} \\ &\leq c\,\frac{|\psi(z_{n+1},z_i)|^{n+1} + |\psi(z_{n+1},z_j)|^{n+1}}{|\psi(z_i,z_j)|} \\ &\leq c\,(|\psi(z_{n+1},z_i)|^n + |\psi(z_{n+1},z_j)|^n) \leq c\,G(\mathcal{Z})^n\,. \end{aligned}$$

Thus, by (10),

$$\begin{aligned} E[Q_n] &\leq c\,[1 + (1+2) + \cdots + (1+2+\cdots+2^{n-1})]G(\mathcal{Z})^{n+1} \\ &= c\,[2^{n+1} - (n+2)]G(\mathcal{Z})^{n+1} < c\,(2G(\mathcal{Z}))^{n+1}. \end{aligned}$$

Since $2G(\mathcal{Z}) < 1$, there exists N such that $E[Q_n] < \epsilon$ for all $n \geq N$.

(i) Condition $G(\mathcal{Z}) < \frac{1}{2}$ implies that the sequence $\mathcal{Z}$ is relatively compact in $\mathbb{D}$. So, let $(z_{n_i})_i$ be a convergent subsequence. Assume $m \geq N$ and $n_i > m$. Using the fact that Q_m is bounded independently of m, together with (5), and taking $m \geq n_j$ with sufficiently large n_j, we obtain

$$\begin{aligned} |Q_m(z_{n_i}) - f(z_{n_i})| &\leq |Q_m(z_{n_i}) - Q_m(z_{n_j})| + |f(z_{n_j}) - f(z_{n_i})| \\ &\leq c\,|\psi(z_{n_i},z_{n_j})| < \epsilon\,. \end{aligned}$$

(ii) Since $(Q_m)_{m\geq N}$ is uniformly bounded, by Montel's theorem, it has a subsequence $(Q_{m_k})_k$ that converges uniformly to a holomorphic function Q on the compact set $K = (z_{n_i})_i \cup \{\lim_i z_{n_i}\}$. Given $z_{n_i} \in K$, choosing m_k such that $m_k \geq n_i$, we have

$$|Q(z_{n_i}) - f(z_{n_i})| = |Q(z_{n_i}) - Q_{m_k}(z_{n_i})|\,,$$

and $|Q(z_{n_i}) - Q_{m_k}(z_{n_i})| < \epsilon$ if m_k is taken sufficiently large. So, $Q = f$ on K and (ii) holds. □

Proof of Proposition 1.2.

a) The proof is quite similar to that of the Proposition 1.1. Let $\epsilon > 0$. We define

$$\alpha_n(z) = \sum_{N=1}^{n} f^{(N)}(z)\frac{(z_{n+1}-z)^N}{N!}\,, \quad z \in \mathbb{D}\,, \tag{11}$$

where $n \geq 2$ will be determined later. Suppose $f \in \Lambda^s$, where $[s] = n$. We take $[z_i] := f(z_i) - \alpha(z_i)$, $[z_i, \dots, z_j]$ as in (4), and the polynomial p_n of degree n defined by

$$p_n(z) = [z_0] + \sum_{k=1}^{n}[z_0, \dots, z_k]\prod_{j=0}^{k-1}(z - z_j)\,, \quad z \in \mathbb{D}\,.$$

Note that $|\alpha_n(z)| = O(|z - z_{n+1}|)$, because $|f^{(n)}(z)| = O(1)$. We have that $P_n = p_n + \alpha_n$ is a polynomial of degree n interpolating f on $\mathcal{Z}_n$ and

$$E[P_n] = E[p_n] < |f(z_0) - \alpha_n(z_0) - f(z_{n+1})| + |z_{n+1} - z_0|\sum_{k=1}^{n}[z_0, \dots, z_k]\,. \tag{12}$$

By (2), the first term in (12) is $O(|z_{n+1} - z_0|^s)$. By (2) and (7),

$$\begin{aligned}|[z_i, z_j]| \leq& \frac{|f(z_i) - \alpha_n(z_i) - f(z_{n+1})| + |f(z_j) - \alpha_n(z_j) - f(z_{n+1})|}{|z_i - z_j|} \\ \leq& c\,\frac{|z_{n+1} - z_i|^s + |z_{n+1} - z_j|^s}{|z_i - z_j|} \leq c\,D(\mathcal{Z})^{s-1}\,.\end{aligned}$$

Thus, by (12) and continuing as in the proof of Proposition 1.1,

$$E[P_n] < c\,(2D(\mathcal{Z}))^{n+1}\,.$$

Since $2D(\mathcal{Z}) < 1$, choosing n sufficiently large, it follows $E[P_n] < \epsilon$.

To prove b) and c), take $\alpha_n(z)$ as in (11), $p - 1 = n$ and $q = n$, respectively, and proceed as in the proof of a). □

Remark 3.1. Regarding this topic, we believe that it could be interesting to identify more S-placed sequences and S-placed sequences in the weak sense.

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