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CERTAIN SUBCLASS OF ALPHA-CONVEX BI-UNIVALENT FUNCTIONS DEFINED USING q-DERIVATIVE OPERATOR

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Abstract. The present investigation deals with a new subclass of alpha-convex bi-univalent functions in the unit disc $E = \{z\colon \mid z \mid< 1\}$ defined with q-derivative operator. Bounds for the first two coefficients and Fekete-Szegö inequality are established for this class. Many known results follow as consequences of the results derived here.

1. Introduction

Let us consider the analytic functions f which have the expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k\,, \tag{1}$$

in the unit disc $E = \{z\colon |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The class of these functions is denoted by $\mathcal{A}$. Further, the class of functions $f \in \mathcal{A}$ and which are univalent in E, is denoted by $\mathcal{S}$. The functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc E and satisfy the conditions $u(0) = 0$ and $|u(z)| < 1$, are called Schwarz functions and the class of these functions is denoted by $\mathcal{U}$.

The classes $\mathcal{S}^*$ of starlike functions and $\mathcal{K}$ of convex functions are defined as follows:

$$\mathcal{S}^* = \Big\{f\colon f \in \mathcal{A}, Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, z \in E\Big\}$$

and

$$\mathcal{K} = \Big\{f\colon f \in \mathcal{A}, Re\Big(\frac{(zf'(z))'}{f'(z)}\Big) > 0, z \in E\Big\}\,.$$

For $0 \le \alpha \le 1$, Mocanu [17] introduced the class $\mathcal{M}(\alpha)$, which is a unification of the classes $\mathcal{S}^*$ and $\mathcal{K}$ and is defined as

$$\mathcal{M}(\alpha) = \Big\{f\colon f \in \mathcal{A}, Re\Big((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\Big) > 0, z \in E\Big\}\,.$$

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The functions in the class $\mathcal{M}(\alpha)$ are known as alpha-convex functions. In particular, $\mathcal{M}(0) \equiv \mathcal{S}^*$ and $\mathcal{M}(1) \equiv \mathcal{K}$.

If f and g are two analytic functions in E, then f is said to be subordinate to g (denoted as $f \prec g$) if there exists a Schwarz function $u \in \mathcal{U}$ such that $f(z) = g(u(z))$. By making use of a subordination theorem for analytic functions, many authors derived several subordination relationships between certain subclasses of analytic functions, for example, see [8, 16, 28]. Further, if g is univalent in E, then $f \prec g$ implies $f(0) = g(0)$ and $f(E) \subset g(E)$. For $-1 \leq B < A \leq 1$ and $0 \leq \eta < 1$, Polatoglu et al. [19] defined the class $\mathcal{P}(A, B; \eta)$ which consists of the functions $p(z)$ such that $p(z) \prec \dfrac{1 + [B + (A - B)(1 - \eta)]z}{1 + Bz}$. For $\eta = 0$, the class $\mathcal{P}(A, B; \eta)$ reduces to $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [12].

Quantum calculus is ordinary classical calculus which introduces q-calculus, where q stands for quantum. Nowadays, q-calculus has attracted many researchers as it is widely useful in various branches of Mathematics and Physics. The application of q-calculus was initiated by Jackson [10, 11] and he developed q-integral and q-derivative in a systematic way. For $0 < q < 1$, Jackson [10] defined the q-derivative of a function $f \in \mathcal{A}$ as

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0\,, \\ f'(0) & \text{for } z = 0\,, \end{cases} \tag{2}$$

where $D_q^2 f(z) = D_q(D_q f(z))$.
From (2), it is obvious that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}\,,$$

where $[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}$. If $q \to 1^-$, then $[k]_q \to k$. Further $D_q z^k = [k]_q z^{k-1}$ and $\lim_{q \to 1^-} D_q f(z) = f'(z)$.

Using q-derivative operator, Seoudy and Aouf [21] defined the subclasses of q-starlike and q-convex functions of order $\alpha (0 \leq \alpha < 1)$ as follows:

$$\mathcal{S}_q^*(\alpha) = \Big\{ f \colon f \in \mathcal{A}, Re\Big(\frac{z D_q f(z)}{f(z)} \Big) > \alpha, z \in E \Big\}$$

and

$$\mathcal{K}_q(\alpha) = \Big\{ f \colon f \in \mathcal{A}, Re\Big(\frac{D_q(z D_q f(z))}{D_q f(z)} \Big) > \alpha, z \in E \Big\}\,.$$

It is obvious that $f \in \mathcal{K}_q(\alpha)$ if and only if $f \in \mathcal{S}_q^*(\alpha)$. For $q \to 1^-$ and $\alpha = 0$, the classes $\mathcal{S}_q^*(\alpha)$ and $\mathcal{K}_q(\alpha)$ reduces to the classes $\mathcal{S}^*$ and $\mathcal{K}$, respectively.

By Koebe one-quarter theorem [7], every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z(z \in E)$$

and

$$f(f^{-1}(w)) = w\Big(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\Big)$$

where

(3) $$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ for which both f and f^{-1} are univalent in E, is called a bi-univalent function. The class of functions of the form (1) and which are bi-univalent in E, is denoted by Σ. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, are some of the examples of the functions in the class Σ. The well known Koebe function $f(z) = \frac{z}{(1-z)^2}$ is not a member of the class Σ.

Lewin [13] was the first, who investigated the class Σ and proved that $|a_2| < 1.51$. Subsequently, bounds for the initial coefficients of numerous sub-classes of bi-univalent functions were studied by various authors in [3, 5, 9, 18, 22, 23, 24, 25, 26, 27]. Further several subclasses of bi-univalent functions defined with q-derivative operator were studied by various authors including [1, 2, 6, 14, 15, 20, 29, 30].

Using the notion of q-derivative, now we define a subclass of alpha-convex bi-univalent functions and establish the bounds of $|a_2|$, $|a_3|$ and Fekete-Szegö inequality for this class.

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$ if the following conditions are satisfied:

$$(1-\lambda)\frac{zD_qf(z)}{f(z)} + \lambda\frac{D_q(zD_qf(z))}{D_qf(z)} \prec \left(\frac{1+[B+(A-B)(1-\eta)]z}{1+Bz}\right)^{\alpha}$$

and

$$(1-\lambda)\frac{wD_qg(w)}{g(w)} + \lambda\frac{D_q(wD_qg(w))}{D_qg(w)} \prec \left(\frac{1+[B+(A-B)(1-\eta)]w}{1+Bw}\right)^{\alpha},$$

where $g(w) = f^{-1}(w)$ as given in (3), $-1 \leq B < A \leq 1$, $0 \leq \lambda \leq 1$, $0 < \alpha \leq 1$ and $0 \leq \eta < 1$.

The following observations are obvious:

(i) $\mathcal{M}\Sigma_q(1-2\beta, -1; 0; 1; \lambda) \equiv \mathcal{B}\Sigma_q(\beta, \lambda)$.

(ii) $\mathcal{M}\Sigma_q(1, -1; 0; \alpha; \lambda) \equiv \mathcal{M}\Sigma_q(\alpha, \lambda)$.

(iii) For $q \to 1^-$, $\mathcal{M}\Sigma_q(1-2\beta, -1; 0; 1; \lambda) \equiv \mathcal{B}_\Sigma(\beta, \lambda)$, the class studied by Li and Wang [14].

(iv) For $q \to 1^-$, $\mathcal{M}\Sigma_q(1, -1; 0; \alpha; \lambda) \equiv \mathcal{M}_\Sigma(\alpha, \lambda)$, the class introduced by Li and Wang [14].

(v) $\mathcal{M}\Sigma_q(1,-1;0;\alpha;\lambda) \equiv \mathcal{S}^*_{\Sigma q}(\alpha,\lambda)$.

(vi) $\mathcal{M}\Sigma_q(1-2\beta,-1;0;1;\lambda) \equiv \mathcal{K}_{\Sigma q}(\beta,\lambda)$.

Throughout this paper, we make the assumptions that $0 < \alpha \le 1$, $0 \le \lambda \le 1$, $0 \le \beta < 1$, $0 \le \eta < 1$, $-1 \le B < A \le 1$, $z \in E$, $w \in E$ and $g(w) = f^{-1}(w)$ as given in (3).

For deriving the main results, we use the following lemma:

Lemma 1.1 ([4]). *If* $p(z) = \dfrac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, $u(z) \in \mathcal{U}$, *then*

$$|p_n| \le (A-B)(1-\eta)\,, \quad n \ge 1\,.$$

2. The class $\mathcal{M}\Sigma_q(A,B;\eta;\alpha;\lambda)$

Theorem 2.1. *If* $f \in \mathcal{M}\Sigma_q(A,B;\eta;\alpha;\lambda)$, *then*

$$|a_2| \le \frac{\sqrt{2\alpha^2(A-B)(1-\eta)}}{\sqrt{2\alpha[([3]_q-[2]_q)+\lambda(-2[3]_q+[3]_q^2+[2]_q+[2]_q^2-[2]_q^3)]+(1-\alpha)[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}} \tag{4}$$

and

$$|a_3| \le \frac{\alpha(A-B)(1-\eta)}{([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)} + \frac{\alpha^2(A-B)^2(1-\eta)^2}{[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,. \tag{5}$$

Proof. From Definition 1.1, using the concept of subordination, we have

$$\begin{aligned}(1-\lambda)\frac{zD_qf(z)}{f(z)} &+ \lambda\frac{D_q(zD_qf(z))}{D_qf(z)} \\ &= \Big(\frac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)}\Big)^{\alpha} = [p(z)]^{\alpha}, u \in \mathcal{U}\end{aligned} \tag{6}$$

and

$$\begin{aligned}(1-\lambda)\frac{wD_qg(w)}{g(w)} &+ \lambda\frac{D_q(wD_qg(w))}{D_qg(w)} \\ &= \Big(\frac{1+[B+(A-B)(1-\eta)]v(w)}{1+Bv(w)}\Big)^{\alpha} = [q(w)]^{\alpha}, v \in \mathcal{U}\,,\end{aligned} \tag{7}$$

where $p(z) = 1 + p_1z + p_2z^2 + \ldots$ and $q(w) = 1 + q_1w + q_2w^2 + \ldots$.
Expanding and equating the coefficients of z and z^2 in (6) and of w and w^2 in (7), we obtain

$$\big[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)\big]a_2 = \alpha p_1\,, \tag{8}$$

$$\begin{aligned}&\big[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)\big]a_3\\ &\quad+\big[(1-[2]_q)+\lambda(-1+[2]_q+[2]_q^2-[2]_q^3)\big]a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)p_1^2}{2}\end{aligned}\tag{9}$$

and

$$-\big[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)\big]a_2=\alpha q_1\,,\tag{10}$$

$$\begin{aligned}&\big[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)\big](2a_2^2-a_3)\\ &\quad+\big[(1-[2]_q)+\lambda(-1+[2]_q+[2]_q^2-[2]_q^3)\big]a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)q_1^2}{2}\,.\end{aligned}\tag{11}$$

(8) and (10) together gives

$$p_1=-q_1\tag{12}$$

and

$$2\big[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)\big]^2a_2^2=\alpha^2(p_1^2+q_1^2)\,.\tag{13}$$

Adding (9) and (11) and using (13), it yields

$$\begin{aligned}&2\alpha\big[([3]_q-[2]_q)+\lambda(-2[3]_q+[3]_q^2+[2]_q+[2]_q^2-[2]_q^3)\big]a_2^2\\ &\quad=\alpha^2(p_2+q_2)+(\alpha-1)\big[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)\big]^2a_2^2\,,\end{aligned}\tag{14}$$

which gives

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{2\alpha[([3]_q-[2]_q)+\lambda(-2[3]_q+[3]_q^2+[2]_q+[2]_q^2-[2]_q^3)]+(1-\alpha)[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,.\tag{15}$$

Taking modulus, applying triangle inequality and using Lemma 1.1 in (15), we can easily obtain (4).
Now subtracting (11) from (9), we get

$$2\big[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)\big](a_3-a_2^2)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)\,.\tag{16}$$

Using (12), (13) and (16), it gives

$$a_3=\frac{\alpha(p_2-q_2)}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}+\frac{\alpha^2(p_1^2+q_1^2)}{2[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,.\tag{17}$$

Using (12) and (13) in (17), and applying triangle inequality, we obtain

$$|a_3|\le\frac{\alpha(|p_2|+|q_2|)}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}+\frac{\alpha^2(2|p_1|^2)}{2[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,.\tag{18}$$

Using Lemma 1.1, the result (5) can be easily obtained from (18). □

On putting $A=1-2\beta$, $B=-1$, $\eta=0$, $\alpha=1$, Theorem 2.1 gives the following result.

Corollary 2.1. *If $f \in \mathcal{B}\Sigma_q(\beta;\lambda)$, then*

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{2\alpha[([3]_q-[2]_q)+\lambda(-2[3]_q+[3]_q^2+[2]_q+[2]_q^2-[2]_q^3)]}}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)} + \frac{4(1-\beta)^2}{[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,.$$

For $A=1$, $B=-1$, $\eta=0$, Theorem 2.1 yields the following result.

Corollary 2.2. *If $f \in \mathcal{M}\Sigma_q(\alpha;\lambda)$, then*

$$|a_2| \leq$$

$$\frac{2\alpha}{\sqrt{2\alpha[([3]_q-[2]_q)+\lambda(-2[3]_q+[3]_q^2+[2]_q+[2]_q^2-[2]_q^3)]+(1-\alpha)[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}}$$

and

$$|a_3| \leq \frac{2\alpha}{([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)} + \frac{4\alpha^2}{[([2]_q-1)+\lambda(1-2[2]_q+[2]_q^2)]^2}\,.$$

For $q \to 1^-$ and on putting $A=1, B=-1, \eta=0$, Theorem 2.1 agrees with the following result due to Li and Wang [14].

Corollary 2.3. *If $f \in \mathcal{M}_\Sigma(\alpha;\lambda)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda-\alpha\lambda)}}$$

and

$$|a_3| \leq \frac{\alpha}{1+2\lambda} + \frac{4\alpha^2}{(1+\lambda)^2}\,.$$

For $q \to 1^-$ and on substituting $A = 1-2\beta, B=-1, \eta=0, \alpha=1$, Theorem 2.1 coincides with the following result due to Li and Wang [14].

Corollary 2.4. *If $f \in \mathcal{B}_\Sigma(\beta;\lambda)$, then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+\lambda}}$$

and

$$|a_3| \leq \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}\,.$$

On putting $A=1-2\beta$, $B=-1$, $\eta=0$, $\alpha=1$, $\lambda=1$, the following result can be easily obtained from Theorem 2.1.

Corollary 2.5. *If $f \in \mathcal{K}_{\Sigma q}(\beta)$, then*

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{2\alpha[-[3]_q + [3]_q^2 + [2]_q^2 - [2]_q^3]}}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{-[3]_q + [3]_q^2} + \frac{4(1-\beta)^2}{[-[2]_q + [2]_q^2]^2}\,.$$

On putting $A = 1$, $B = -1$, $\eta = 0$, $\lambda = 0$ in Theorem 2.1, the following result is obvious.

Corollary 2.6. *If $f \in \mathcal{S}^*_{\Sigma q}(\alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha([3]_q - [2]_q) + (1-\alpha)([2]_q - 1)^2}}$$

and

$$|a_3| \leq \frac{2\alpha}{[3]_q - 1} + \frac{4\alpha^2}{([2]_q - 1)^2}\,.$$

Theorem 2.2. *If $f \in \mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha(1-\eta)(A-B)}{([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)}, & \text{if } \; 0 \leq |l(\mu)| < \frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}, \\ 2\alpha(1-\eta)(A-B)|l(\mu)|, & \text{if } \; |l(\mu)| \geq \frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}, \end{cases}$$

where

$$l(\mu) = \frac{\alpha(1-\mu)}{2\alpha[([3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3)] + (1-\alpha)[([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}\,. \tag{19}$$

Proof. Using (13) in (17), we have

$$a_3 = \frac{\alpha(p_2 - q_2)}{2[([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} + a_2^2\,. \tag{20}$$

Making use of (20), it yields

$$a_3 - \mu a_2^2 = \frac{\alpha(p_2 - q_2)}{2[([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} + (1-\mu)\,. \tag{21}$$

Further (21) can be expressed as

$$\begin{aligned} a_3 - \mu a_2^2 = \alpha \Bigg[&\Big(l(\mu) + \frac{1}{2[([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} \Big) p_2 \\ &+ \Big(l(\mu) - \frac{1}{2[([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} \Big) q_2 \Bigg]\,, \end{aligned} \tag{22}$$

where $l(\mu)$ is defined in (19).
Taking modulus, applying triangle inequality and using Lemma 1.1, (22) yields

$$|a_3-\mu a_2^2| \leq \alpha(1-\eta)(A-B)\Big|\Big(l(\mu)+\frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}\Big)$$

$$+\Big(l(\mu)-\frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}\Big)\Big|. \tag{23}$$

For $0 \leq |l(\mu)| < \frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}$,

$$|a_3-\mu a_2^2| \leq \frac{\alpha(1-\eta)(A-B)}{([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)}. \tag{24}$$

For $|l(\mu)| \geq \frac{1}{2[([3]_q-1)+\lambda(1-2[3]_q+[3]_q^2)]}$,

$$|a_3-\mu a_2^2| \leq 2\alpha(1-\eta)(A-B)|l(\mu)|. \tag{25}$$

The proof of Theorem 2.2 is obvious from (24) and (25).

□

Conclusion

This paper is concerned with the study of a new and generalized class of alpha-convex bi-univalent functions using q-derivative operator. The class is defined using the concept of subordination. Some earlier known results follow as special cases of the results proved here. This paper will work as a motivation to the other researchers to study some more relevant subclasses of bi-univalent functions using subordination.

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