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LINEARIZATION TECHNIQUE FOR OSCILLATION OF PERTURBED HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. It is shown that oscillation of perturbed second order half-linear differential equations can be derived from oscillation of second order linear differential equations associated with modified Riccati equations. In the main result of the present paper, some of technical assumptions in the known results of this type are removed.

1. Introduction

In this paper we consider the second order half-linear ordinary differential equation

$$\big(p(t)\Phi_\alpha(x')\big)' + q(t)\Phi_\alpha(x) = 0\,, \quad t \ge t_0\,, \tag{1.1}$$

where $\Phi_\alpha(x) = |x|^\alpha \operatorname{sgn} x$ with $\alpha > 0$, $p(t)$ and $q(t)$ are real-valued continuous functions on $[t_0,\infty)$, and $p(t) > 0$ for $t \ge t_0$. If $\alpha = 1$, then (1.1) reduces to the linear equation

$$\big(p(t)x'\big)' + q(t)x = 0\,, \quad t \ge t_0\,. \tag{1.2}$$

The half-linear equation (1.1) can be seen as a natural generalization of the linear equation (1.2).

It is well-known that all solutions of (1.1) exist on the entire interval $[t_0,\infty)$ and that if (1.1) has a nontrivial oscillatory [or nonoscillatory] solution, then any other nontrivial solution is also oscillatory [or nonoscillatory]. Equation (1.1) is said to be oscillatory [or nonoscillatory] if (1.1) has a nontrivial oscillatory [or nonoscillatory] solution. Clearly, if $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore we can suppose without loss of generality that a nonoscillatory solution of (1.1) is eventually positive.

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of half-linear differential equations. It is known that basic results for the second order linear equations can be generalized to the second order half-linear equations. The important works are summarized in the book of Došlý and Řehák [8]. For the recent results to half-linear equations

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we refer the reader to, for example, [1, 2, 3, 5, 6, 7, 12, 13, 14, 16, 17, 18, 19]. The present paper is strongly motivated by oscillatory and nonoscilaltory results in [5, 7, 11, 16].

Together with (1.1), we consider the equation of the same type

$$\big(p(t)\Phi_\alpha(x')\big)' + q_0(t)\Phi_\alpha(x) = 0\,, \quad t \ge t_0\,, \tag{1.3}$$

where $q_0(t)$ is a real-valued continuous function on $[t_0, \infty)$. Equation (1.1) is regarded as a perturbation of (1.3).

An important oscillatory result is the following theorem.

Theorem 1.1 (Došlý and Lomtatidze [7, Theorem 1])**.** *Suppose that equation* (1.3) *is nonoscillatory and let $x = x_0(t)$ be the principal solution of* (1.3) *satisfying $x_0(t) > 0$ for $t \ge T$. If*

$$\int_T^\infty x_0(t)^{\alpha+1}\,[q(t) - q_0(t)]\,dt = \infty\,, \tag{1.4}$$

then equation (1.1) *is oscillatory.*

For the concept of the principal solution, see Došlý and Řehák [8, Section 4.2]. In general, it is difficult to know whether a nonoscillatory solution $x_0(t)$ of (1.3) is principal or not.

In what follows, it will be assumed that (1.3) has a nonoscillatory solution $x = x_0(t)$ such that

$$x_0(t) > 0\,, \quad x_0'(t) \ne 0 \quad \text{for} \quad t \ge T\,, \tag{1.5}$$

$$\int_T^\infty \frac{1}{p(t)x_0(t)^2|x_0'(t)|^{\alpha-1}}\,dt = \infty\,, \tag{1.6}$$

$$\liminf_{t\to\infty} p(t)x_0(t)|x_0'(t)|^\alpha > 0\,, \tag{1.7}$$

and

$$\int_T^\infty x_0(t)^{\alpha+1}\,[q(t) - q_0(t)]\,dt \quad \text{is convergent.} \tag{1.8}$$

Condition (1.6) is closely related to an integral characterization of the principal solution of (1.3) (see Došlý and Elbert [4] and Došlá and Došlý [1, Proposition 2]). Condition (1.8) is a typical counterpart of (1.4).

For brevity, we set

$$P(t) = p(t)x_0(t)^2|x_0'(t)|^{\alpha-1} \quad \text{and} \quad Q(t) = x_0(t)^{\alpha+1}\,[q(t) - q_0(t)]\,. \tag{1.9}$$

Conditions (1.6) and (1.8) are rewritten as

$$\int_T^\infty \frac{1}{P(t)}\,dt = \infty$$

and

$$\int_T^\infty Q(t)\,dt \quad \text{is convergent,}$$

respectively.

The next theorem is known.

Theorem 1.2 (Došlý and Fišnarová [5, Theorem 3]). *Suppose that equation* (1.3) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.5)–(1.8). *If there exists* $\varepsilon > 0$ *such that the linear equation*

$$\big(P(t)x'\big)' + (1+\varepsilon)\frac{\alpha+1}{2\alpha}Q(t)x = 0 \tag{1.10}$$

is nonoscillatory, then equation (1.1) *is nonoscillatory.*

Corollary 1.3 (Došlý and Fišnarová [5, Corollary 1 (i)]). *Suppose that* (1.3) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.5)–(1.8). *If*

$$\begin{aligned}-\frac{3\alpha}{2(\alpha+1)} &< \liminf_{t\to\infty}\Big(\int_T^t \frac{1}{P(s)}\,ds\Big)\Big(\int_t^\infty Q(s)\,ds\Big)\\ &\le \limsup_{t\to\infty}\Big(\int_T^t \frac{1}{P(s)}\,ds\Big)\Big(\int_t^\infty Q(s)\,ds\Big) < \frac{\alpha}{2(\alpha+1)}\,,\end{aligned}$$

then (1.1) *is nonoscillatory.*

Corollary 1.3 is obtained by applying the classical Hille–Nehari nonoscillation criterion to the linear equation (1.10).

In this paper we will prove the following theorem.

Theorem 1.4. *Suppose that* (1.3) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.5)–(1.8). *If there exists a number* ε *with* $0 < \varepsilon < 1$ *such that the linear equation*

$$\big(P(t)x'\big)' + (1-\varepsilon)\frac{\alpha+1}{2\alpha}Q(t)x = 0 \tag{1.11}$$

is oscillatory, then equation (1.1) *is oscillatory.*

Theorem 1.4 has been proved under various additional conditions. For example, see Theorems 4 and 5 in [5] and Theorem 1.6 in [16]. Applying the classical Hille–Nehari oscillation criterion to the linear equation (1.11), we get the next result.

Corollary 1.5. *Suppose that* (1.3) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.5)–(1.8). *If*

$$\liminf_{t\to\infty}\Big(\int_T^t \frac{1}{P(s)}\,ds\Big)\Big(\int_t^\infty Q(s)\,ds\Big) > \frac{\alpha}{2(\alpha+1)}\,,$$

then (1.1) *is oscillatory.*

Combining Theorem 1.4 with the classical Zlámal's oscillation criterion (for a half-linear extension, see Lemma 2.4 below), we obtain the next corollary.

Corollary 1.6. *Suppose that* (1.3) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.5)–(1.8). *If there is a number* λ *such that* $\lambda < 1$ *and*

$$\int_T^\infty \Big(\int_T^t \frac{1}{P(s)}\,ds\Big)^\lambda Q(t)\,dt = \infty\,,$$

then (1.1) *is oscillatory.*

In Corollary 1.6 we cannot take $\lambda = 0$ because of (1.8).

In the next section we state several basic (non)oscillatory results for the half-linear differential equation (1.1). Except for the last Lemma 2.4, the proofs are contained in the book of Došlý and Řehák [8]. Lemma 2.4 is due to Dosoudilová, Lomtatidze and Šremr [10]. In Section 3, we present a (non)oscillatory result for a specific half-linear equation with $\alpha > 1$. The result in Section 3 is used for the proof of Theorem 1.4. For the proof of Theorem 1.4 we also need some estimates for the function $F(u, v)$ which appears in the modified Riccati equation associated with (1.1). The estimates for $F(u, v)$ are stated in Section 4. The proof of Theorem 1.4 is given in Section 5. We present a few examples illustrating Corollaries 1.5 and 1.6 in Section 6.

2. Basic results

For the convenience of the reader we summarize basic (non)oscillatory results for the half-linear differential equation (1.1).

Lemma 2.1. *Equation* (1.1) *is nonoscillatory if and only if there is a function* $y \in C^1[T, \infty)$*,* $T \geq t_0$*, such that*

$$y' + q(t) + \alpha p(t)^{-1/\alpha} |y|^{(\alpha+1)/\alpha} \leq 0\,, \quad t \geq T\,.$$

Let us consider equation (1.1) under the conditions

$$\int_{t_0}^{\infty} p(t)^{-1/\alpha} dt = \infty \tag{2.1}$$

and

$$\int_{t_0}^{\infty} q(t)\, dt \quad \text{is convergent.} \tag{2.2}$$

Lemma 2.2. *Suppose that* (2.1) *and* (2.2) *hold. Equation* (1.1) *is nonoscillatory if and only if there is a function* $y \in C[T, \infty)$*,* $T \geq t_0$*, such that*

$$\int_T^{\infty} p(t)^{-1/\alpha} |y(t)|^{(\alpha+1)/\alpha}\, dt < \infty$$

and

$$y(t) = \int_t^{\infty} q(s)\, ds + \alpha \int_t^{\infty} p(s)^{-1/\alpha} |y(s)|^{(\alpha+1)/\alpha}\, ds\,, \quad t \geq T\,.$$

Lemma 2.3. *Consider equation* (1.1) *under conditions* (2.1) *and* (2.2)*.*

(I) *If*

$$\begin{aligned} -\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} &< \liminf_{t\to\infty} \Big(\int_{t_0}^{t} p(s)^{-1/\alpha}\, ds\Big)^{\alpha} \Big(\int_t^{\infty} q(s)\, ds\Big) \\ &\leq \limsup_{t\to\infty} \Big(\int_{t_0}^{t} p(s)^{-1/\alpha}\, ds\Big)^{\alpha} \Big(\int_t^{\infty} q(s)\, ds\Big) \\ &< \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}, \end{aligned}$$

then (1.1) *is nonoscillatory.*

(II) *If*

$$\liminf_{t\to\infty}\Big(\int_{t_0}^{t} p(s)^{-1/\alpha}\,ds\Big)^{\alpha}\Big(\int_{t}^{\infty} q(s)\,ds\Big) > \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha},$$

then (1.1) *is oscillatory.*

The results mentioned above are half-linear extensions of the classical results for the linear equation (1.2). For the proofs, see [8].

Lemma 2.4. *Consider equation* (1.1) *under condition* (2.1)*. If there is a number* λ *such that* $\lambda < \alpha$ *and*

$$\int_{t_0}^{\infty}\Big(\int_{t_0}^{t} p(s)^{-1/\alpha}\,ds\Big)^{\lambda} q(t)\,dt = \infty\,,$$

then (1.1) *is oscillatory.*

For the case $\alpha = 1$, Lemma 2.4 can be derived from Theorem 1 in the paper of Zlámal [20]. For the general case, Lemma 2.4 is a direct consequence of Corollary 2.7 in [10] (see also Corollary 3.1 in [15]).

3. An oscillatory result

In this section we suppose that $\alpha > 1$ and $p(t)$ satisfies

$$\int_{t_0}^{\infty} p(s)^{-1}\,ds = \infty \tag{3.1}$$

and $q(t)$ satisfies (2.2).

Theorem 3.1. *Let* $\alpha > 1$*. Suppose that* (2.2) *and* (3.1) *hold. If there exists a function* $y \in C^1[T,\infty)$*,* $T \geq t_0$*, such that*

$$y'(t) + q(t) + \alpha p(t)^{-1}|y(t)|^{(\alpha+1)/\alpha} \leq 0\,, \quad t \geq T\,, \tag{3.2}$$

then, for any constant $M > 0$*, there exists a function* $w \in C^1[T_1,\infty)$*,* $T_1 \geq T$*, such that*

$$w'(t) + Mq(t) + p(t)^{-1}w(t)^2 \leq 0\,, \quad t \geq T_1\,. \tag{3.3}$$

Proof. Suppose that there is a function $y \in C^1[T,\infty)$, $T \geq t_0$, satisfying (3.2). By Lemma 2.1, the half-linear equation

$$\big(p(t)^{\alpha}\Phi_{\alpha}(x')\big)' + q(t)\Phi_{\alpha}(x) = 0$$

is nonoscillatory. Note that (3.1) implies

$$\int_{t_0}^{\infty}\big(p(t)^{\alpha}\big)^{-1/\alpha}\,dt = \int_{t_0}^{\infty} p(t)^{-1}\,dt = \infty\,.$$

Since (2.2) is assumed to hold, Lemma 2.2 implies that there is a function $v \in C^1[T_0,\infty)$, $T_0 \geq t_0$, such that

$$\int_{T_0}^{\infty} p(t)^{-1}|v(t)|^{(\alpha+1)/\alpha}\,dt < \infty$$

and

$$v(t)=\int_t^\infty q(s)\,ds+\alpha\int_t^\infty p(s)^{-1}|v(s)|^{(\alpha+1)/\alpha}\,ds, \quad t\ge T_0\,.$$

Then we have $\lim_{t\to\infty} v(t)=0$ and

$$v'(t)+q(t)+\alpha p(t)^{-1}|v(t)|^{(\alpha+1)/\alpha}=0\,, \quad t\ge T_0\,.$$

Let $M>0$ be an arbitrary constant. Since $\alpha>1$ and $\lim_{t\to\infty} v(t)=0$, we have

$$\alpha|v(t)|^{(\alpha+1)/\alpha}\ge Mv(t)^2$$

for sufficiently large t, say $t\ge T_1$. Then it is easy to see that the function

$$w(t)=Mv(t)\,, \quad t\ge T_1\,,$$

satisfies (3.3). The proof is complete. □

The next corollary is a restatement of Theorem 3.1.

Corollary 3.2. *Let $\alpha>1$. Suppose that* (2.2) *and* (3.1) *hold. If the half-linear equation*

$$\big(p(t)^\alpha\Phi_\alpha(x')\big)'+q(t)\Phi_\alpha(x)=0 \tag{3.4}$$

is nonoscillatory, then the linear equation

$$\big(p(t)x'\big)'+Mq(t)x=0 \tag{3.5}$$

is nonoscillatory for any $M>0$. Equivalently, if the linear equation (3.5) *is oscillatory for some $M>0$, then the half-linear equation* (3.4) *is oscillatory.*

4. Lemmas

It is known that the function

$$F(u,v)=|u+v|^{(\alpha+1)/\alpha}-|v|^{(\alpha+1)/\alpha}-\frac{\alpha+1}{\alpha}\Phi_{1/\alpha}(v)u\,, \quad u,v\in\mathbb{R}\,, \tag{4.1}$$

plays a crucial role in the study of the oscillation and nonoscillation of (1.1).

Lemma 4.1 (see, e.g., Došlý and Fišnarová [5, Lemma 4])**.** *Let $x=x(t)$ and $x=x_0(t)$ be nonoscillatory solutions of* (1.1) *and* (1.3)*, respectively. Suppose that $x(t)>0$ and $x_0(t)>0$ for $t\ge T$ $(\ge t_0)$. Then the function*

$$u(t)=p(t)x_0(t)^{\alpha+1}\left[\Phi_\alpha\Big(\frac{x'(t)}{x(t)}\Big)-\Phi_\alpha\Big(\frac{x_0'(t)}{x_0(t)}\Big)\right]\,, \quad t\ge T\,, \tag{4.2}$$

is a solution of the modified Riccati differential equation

$$\begin{aligned}&u'(t)+x_0(t)^{\alpha+1}\big[q(t)-q_0(t)\big]\\&\qquad+\alpha p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F\big(u(t),p(t)x_0(t)\Phi_\alpha\big(x_0'(t)\big)\big)=0\,, \quad t\ge T\,,\end{aligned} \tag{4.3}$$

where $F(u,v)$ is defined by (4.1).

Lemma 4.2. *Let $F(u,v)$ be the function which is defined by* (4.1).

(i) *$F(u,v)\ge 0$ for all $u,v\in\mathbb{R}$; $F(u,v)=0$ if and only if $u=0$.*

(ii) *Let k_1 and k_2 be constants satisfying $0 < k_1 < k_2$. Then there is a constant $L(k_1, k_2) > 0$ such that $F(u, v)$ can be expressed in the following form*

$$F(u,v) = \frac{\alpha+1}{2\alpha^2}|v|^{-(\alpha-1)/\alpha}u^2\bigl(1 + R(u,v)\bigr)$$

with

$$\bigl|R(u,v)\bigr| \le \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u|$$

for $|u| \le k_1 < k_2 \le |v|$.

Proof. For the proofs of the part (i) and the part (ii) of the case $v > 0$, see Lemma 3.2 (i) and (iii) in [16]. In general, we have

$$F(-u,-v) = F(u,v)\,, \quad u, v \in \mathbb{R}\,.$$

Therefore the part (ii) of the case $v < 0$ can be derived from the case $v > 0$. The proof is complete. □

The function $F(u, v)$ is closely related to the function

$$P(u,v) = \frac{1}{\alpha+1}|u|^{\alpha+1} - uv + \frac{\alpha}{\alpha+1}|v|^{(\alpha+1)/\alpha}\,, \quad u, v \in \mathbb{R}\,.$$

In fact, it is shown (Došlý and Fišnarová [5, the proof of Lemma 6]) that

$$F(u,v) = \frac{\alpha+1}{\alpha}P(\Phi_{1/\alpha}(v), u+v)\,, \quad u, v \in \mathbb{R}\,. \tag{4.4}$$

The following lemma is well-known (see Došlý and Elbert [4, Lemma 2.4]).

Lemma 4.3. *We have*

$$P(u,v) \ge \frac{1}{2}|u|^{-\alpha+1}\bigl(v - \Phi_\alpha(u)\bigr)^2 \quad (0 < \alpha \le 1)\,,$$
$$P(u,v) \le \frac{1}{2}|u|^{-\alpha+1}\bigl(v - \Phi_\alpha(u)\bigr)^2 \quad (\alpha > 1)$$

for all $u, v \in \mathbb{R}$, $u \neq 0$.

The following lemma was proved by Došlý and Řezníčková [9, Lemma 4].

Lemma 4.4. *Let $M > 0$ be an arbitrary number. Then there exists a constant $K = K(M) > 0$ such that*

$$P(u,v) \le K|u|^{-\alpha+1}\bigl(v - \Phi_\alpha(u)\bigr)^2 \quad (0 < \alpha \le 1)\,,$$

$$P(u,v) \ge K|u|^{-\alpha+1}\bigl(v - \Phi_\alpha(u)\bigr)^2 \quad (\alpha > 1)$$

for all $u, v \in \mathbb{R}$ satisfying

$$u \neq 0,\ \left|\frac{v}{\Phi_\alpha(u)}\right| \le M\,. \tag{4.5}$$

Combining Lemma 4.3 with Lemma 4.4, we have the next corollary.

Corollary 4.5. *Let $M > 0$ be an arbitrary number. Then there exist constants $K_1 = K_1(M) > 0$ and $K_2 = K_2(M) > 0$ such that*

$$K_1|u|^{-\alpha+1}(v - \Phi_\alpha(u))^2 \le P(u,v) \le K_2|u|^{-\alpha+1}(v - \Phi_\alpha(u))^2$$

for all $u, v \in \mathbb{R}$ satisfying (4.5).

Lemma 4.6. *Let $F(u,v)$ be the function defined by* (4.1)*. Then,*

$$F(u,v) \ge \frac{\alpha+1}{2\alpha}|v|^{-(\alpha-1)/\alpha}u^2 \quad (0 < \alpha \le 1), \tag{4.6}$$

$$F(u,v) \le \frac{\alpha+1}{2\alpha}|v|^{-(\alpha-1)/\alpha}u^2 \quad (\alpha > 1) \tag{4.7}$$

for all $u, v \in \mathbb{R}$ with $v \ne 0$.

Moreover, for every $M > 0$, there are constants $L_1 = L_1(M) > 0$ and $L_2 = L_2(M) > 0$ such that

$$L_1|v|^{-(\alpha-1)/\alpha}u^2 \le F(u,v) \le L_2|v|^{-(\alpha-1)/\alpha}u^2 \quad \text{for } v \ne 0,\ \left|\frac{u}{v}\right| \le M\,. \tag{4.8}$$

Proof. The idea of the proof is due to Došlý and Fišnarová [5, Lemma 6]. We have the formula (4.4). Therefore, (4.6) and (4.7) follow from Lemma 4.3, and (4.8) follows from Corollary 4.5. □

Lemma 4.7. *Let $\alpha > 1$, and let $F(u,v)$ be the function defined by* (4.1)*. Let $c > 0$ be an arbitrary number. Then there is a constant $L = L(c) > 0$ such that, for all $u, v \in \mathbb{R}$ satisfying $u \le -c$ and $|v| \ge c$,*

$$F(u,v) \ge L|v|^{-(\alpha-1)/\alpha}|u|^{(\alpha+1)/\alpha}\,. \tag{4.9}$$

Proof. The idea of the proof is due to Došlý and Fišnarová [6, Lemma 3.4]. Fix $v \ne 0$, and put

$$\varphi(u) = \frac{F(u,v)}{|u|^{(\alpha+1)/\alpha}} \quad \text{for } u < 0\,.$$

Then,

$$\varphi'(u) = \frac{\alpha+1}{\alpha}\,\frac{1}{u^2\Phi_{1/\alpha}(u)}\psi(u)\,, \tag{4.10}$$

where

$$\psi(u) = -v\Phi_{1/\alpha}(u+v) + \frac{1}{\alpha}\Phi_{1/\alpha}(v)u + |v|^{(\alpha+1)/\alpha}\,.$$

If $u + v \ne 0$, then

$$\psi'(u) = -\frac{v}{\alpha}\left[|u+v|^{-(\alpha-1)/\alpha} - |v|^{-(\alpha-1)/\alpha}\right]\,.$$

In what follows we distinguish the cases $v \le -c$ and $v \ge c$. First consider the case $v \le -c$. Since $u < 0$, we have $u + v < 0$ and $|u+v| > |v|$. Then, by the condition $\alpha > 1$, we see that $\psi'(u) < 0$ for $u < 0$. Since $\lim_{u\to 0^-}\psi(u) = 0$, this shows that $\psi(u) > 0$ for $u < 0$. Hence, by (4.10), we have $\varphi'(u) < 0$ for $u < 0$. Therefore, if $u \le -c$, then $\varphi(u) \ge \varphi(-c)$, that is,

$$\frac{F(u,v)}{|u|^{(\alpha+1)/\alpha}} \ge \frac{F(-c,v)}{c^{(\alpha+1)/\alpha}}\,, \quad u \le -c\,. \tag{4.11}$$

Since $v \le -c$, we have $|-c/v| \le 1$. An application of Lemma 4.6 to the case $M = 1$ implies that there is a positive constant L_1 such that

$$F(-c,v) \ge L_1 |v|^{-(\alpha-1)/\alpha} c^2\,.$$

Thus, by (4.11), we see that

$$F(u,v) \ge L_1 c^{(\alpha-1)/\alpha} |v|^{-(\alpha-1)/\alpha} |u|^{(\alpha+1)/\alpha} \quad \text{for } u \le -c \text{ and } v \le -c\,. \tag{4.12}$$

Next consider the case $v \ge c$. Then, $-2v < -c$. We examine the function $\varphi(u)$ on the intervals $[-2v,-c]$ and $(-\infty,-2v)$, separately. If $u \in [-2v,-c]$, then $|u/v| \le 2$. Applying Lemma 4.6 to the case $M = 2$, we deduce that there is a positive constant $\widehat{L}_1$ such that

$$F(u,v) \ge \widehat{L}_1 |v|^{-(\alpha-1)/\alpha} u^2\,.$$

Therefore, by the condition $\alpha > 1$,

$$F(u,v) \ge \widehat{L}_1 c^{(\alpha-1)/\alpha} |v|^{-(\alpha-1)/\alpha} |u|^{(\alpha+1)/\alpha} \quad (-2v \le u \le -c,\ v \ge c)\,. \tag{4.13}$$

If $u \in (-\infty,-2v)$, then $|u+v| > |v|$. Hence, because of $\alpha > 1$, we see that $\psi'(u) > 0$ for $u < -2v$. This implies that $\psi(u)$ is increasing on $(-\infty,-2v)$. We have

$$\psi(-2v) = 2\Big(1 - \frac{1}{\alpha}\Big)|v|^{(\alpha+1)/\alpha} > 0$$

and

$$\lim_{u\to-\infty} \frac{\psi(u)}{u} = \frac{1}{\alpha}\Phi_{1/\alpha}(v) > 0\,.$$

Therefore, there is a number $u_1 < -2v$ such that $\psi(u) < 0$ for $u < u_1$, $\psi(u) = 0$ for $u = u_1$, and $\psi(u) > 0$ for $u_1 < u < -2v$. This implies that $\varphi'(u) > 0$ for $u < u_1$, $\varphi'(u) = 0$ for $u = u_1$, and $\varphi'(u) < 0$ for $u_1 < u < -2v$. The function $\varphi(u)$ has the local maximum at $u = u_1$. We have

$$\lim_{u\to-\infty} \varphi(u) = 1$$

and, since $\alpha > 1$,

$$\varphi(-2v) = \frac{\alpha+1}{\alpha}\frac{1}{2^{1/\alpha}} > 1\,.$$

Therefore it is found that

$$\varphi(u) > 1, \text{ i.e., } F(u,v) > |u|^{(\alpha+1)/\alpha} \quad \text{for } u < -2v\,.$$

Hence the condition $v \ge c$ yields

$$F(u,v) \ge c^{(\alpha-1)/\alpha} |v|^{-(\alpha-1)/\alpha} |u|^{(\alpha+1)/\alpha} \quad (u < -2v,\ v \ge c)\,. \tag{4.14}$$

By (4.13) and (4.14), we see that

$$F(u,v) \ge \min\{\widehat{L}_1, 1\} c^{(\alpha-1)/\alpha} |v|^{-(\alpha-1)/\alpha} |u|^{(\alpha+1)/\alpha} \quad \text{for } u \le -c \text{ and } v \ge c\,. \tag{4.15}$$

From (4.12) and (4.15) it is concluded that there is a constant $L>0$ such that (4.9) holds for $u\le -c$ and $|v|\ge c$. The proof of Lemma 4.7 is complete. □

5. Proof of Theorem 1.4

Proof of Theorem 1.4. Let $x=x_0(t)$ be a nonoscillatory solution of (1.3) which satisfies (1.5)–(1.8). By (1.5) and (1.7), there is a constant $c>0$ such that

$$p(t)x_0(t)|x_0'(t)|^\alpha \ge c\,,\quad t\ge T\,. \tag{5.1}$$

Suppose that there is $\varepsilon\in(0,1)$ such that (1.11) is oscillatory. Assume that equation (1.1) has a nonoscillatory solution $x(t)$. We may suppose that $x(t)>0$ for $t\ge T$. Then, by Lemma 4.1, the function $u(t)$ defined by (4.2) satisfies (4.3). Integrating (4.3) from T to t, we obtain

$$\begin{aligned}&u(t)-u(T)+\int_T^t x_0(s)^{\alpha+1}[q(s)-q_0(s)]\,ds\\&\qquad+\alpha\int_T^t p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F\big(u(s),p(s)x_0(s)\Phi_\alpha\big(x_0'(s)\big)\big)\,ds=0\end{aligned} \tag{5.2}$$

for $t\ge T$. Since the integrand of the last integral in the left-hand side of (5.2) is nonnegative for $t\ge T$ (see Lemma 4.2 (i)), we have either

$$\int_T^\infty p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F\big(u(s),p(s)x_0(s)\Phi_\alpha\big(x_0'(s)\big)\big)\,ds=\infty \tag{5.3}$$

or

$$\int_T^\infty p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F\big(u(s),p(s)x_0(s)\Phi_\alpha\big(x_0'(s)\big)\big)\,ds<\infty\,. \tag{5.4}$$

Suppose first that (5.3) holds. Since (1.8) is assumed to hold, it follows from (5.2) that

$$\lim_{t\to\infty}u(t)=-\infty\,. \tag{5.5}$$

Let $\alpha>1$. By (5.5) we may suppose that $u(t)\le -c$ for $t\ge T$. Here, the number c is a constant satisfying (5.1). Then, applying Lemma 4.7 to the case $u=u(t)$ and $v=p(t)x_0(t)\Phi_\alpha(x_0'(t))$, we find that there is a constant $L>0$ such that

$$\begin{aligned}&F\big(u(t),p(t)x_0(t)\Phi_\alpha\big(x_0'(t)\big)\big)\\&\qquad\ge Lp(t)^{-(\alpha-1)/\alpha}x_0(t)^{-(\alpha-1)/\alpha}|x_0'(t)|^{-\alpha+1}|u(t)|^{(\alpha+1)/\alpha}\,,\quad t\ge T\,.\end{aligned}$$

Therefore, (4.3) yields

$$u'(t)+Q(t)+\alpha LP(t)^{-1}|u(t)|^{(\alpha+1)/\alpha}\le 0\,,\quad t\ge T\,,$$

where $P(t)$ and $Q(t)$ are given by (1.9). Then the function $y(t)=L^\alpha u(t)$ $(t\ge T)$ satisfies

$$y'(t)+L^\alpha Q(t)+\alpha P(t)^{-1}|y(t)|^{(\alpha+1)/\alpha}\le 0\,,\quad t\ge T\,.$$

Let $M > 0$ be an arbitrary number. By Theorem 3.1, there is a function $w \in C^1[T_1,\infty)$, $T_1 \geq T$, such that

$$w'(t) + MQ(t) + P(t)^{-1}w(t)^2 \leq 0\,, \quad t \geq T_1\,. \tag{5.6}$$

Hence, by the linear version of Lemma 2.1, the equation

$$\big(P(t)x'\big)' + MQ(t)x = 0 \quad (M > 0) \tag{5.7}$$

is nonoscillatory. This is a contradiction to the condition that (1.11) is oscillatory.

Let $0 < \alpha \leq 1$. Then, using (4.6) in Lemma 4.6, we see that

$$\begin{aligned} &F\big(u(t), p(t)x_0(t)\Phi_\alpha\big(x_0'(t)\big)\big) \\ &\quad\geq \frac{\alpha+1}{2\alpha} p(t)^{-(\alpha-1)/\alpha} x_0(t)^{-(\alpha-1)/\alpha} |x_0'(t)|^{-\alpha+1} u(t)^2\,, \quad t \geq T\,. \end{aligned}$$

It follows from (4.3) that

$$u'(t) + Q(t) + \frac{\alpha+1}{2} P(t)^{-1} u(t)^2 \leq 0\,, \quad t \geq T\,,$$

where $P(t)$ and $Q(t)$ are given by (1.9). Let β be a fixed number satisfying $\beta > 1$. By (5.5) we see that

$$\frac{\alpha+1}{2} u(t)^2 \geq \beta |u(t)|^{(\beta+1)/\beta}$$

for sufficiently large t. We may suppose that this inequality is satisfied for $t \geq T$. Then we have

$$u'(t) + Q(t) + \beta P(t)^{-1} |u(t)|^{(\beta+1)/\beta} \leq 0\,, \quad t \geq T\,.$$

The rest of the proof is similar to the case $\alpha > 1$. Let $M > 0$ be an arbitrary number. By Theorem 3.1 with α replaced by β, there is a function $w \in C^1[T_1,\infty)$, $T_1 \geq T$, such that (5.6) holds. By the linear version of Lemma 2.1, equation (5.7) is nonoscillatory. This is a contradiction to the condition that (1.11) is oscillatory. Thus we find that (5.3) does not hold.

Suppose next that (5.4) holds. Using (1.8), (5.2) and (5.4), we see that $\lim_{t\to\infty} u(t)$ exists and is finite. Put $\lim_{t\to\infty} u(t) = \ell$ $(\in \mathbb{R})$. Integrating the equality (4.3) from t to τ $(T \leq t \leq \tau)$ and letting $\tau \to \infty$, we obtain

$$\begin{aligned} u(t) = \ell &+ \int_t^\infty x_0(s)^{\alpha+1}[q(s) - q_0(s)]\,ds \\ &+ \alpha \int_t^\infty p(s)^{-1/\alpha} x_0(s)^{-(\alpha+1)/\alpha} F\big(u(s), p(s)x_0(s)\Phi_\alpha\big(x_0'(s)\big)\big)\,ds \end{aligned}$$

for $t \geq T$. Since $u(t)$ has a finite limit as $t \to \infty$, it is bounded on $[T,\infty)$. Let $C = \sup_{t\geq T} |u(t)|$. By (5.1), we have

$$\left|\frac{u(t)}{p(t)x_0(t)\Phi_\alpha(x_0'(t))}\right| \leq \frac{C}{c}\,, \quad t \geq T\,.$$

Applying the latter part of Lemma 4.6 to the case

$$u = u(t)\,, \quad v = p(t)x_0(t)\Phi_\alpha\big(x_0'(t)\big) \quad \text{and} \quad M = C/c\,,$$

we see that there is a constant $L_1 > 0$ such that

$$L_1 \big| p(t) x_0(t) \Phi_\alpha\big(x_0'(t)\big)\big|^{-(\alpha-1)/\alpha} u(t)^2 \le F\big(u(t), p(t) x_0(t) \Phi_\alpha\big(x_0'(t)\big)\big)$$

for $t \ge T$. Thus we get

$$L_1 P(t)^{-1} u(t)^2 \le p(t)^{-1/\alpha} x_0(t)^{-(\alpha+1)/\alpha} F\big(u(t), p(t) x_0(t) \Phi_\alpha\big(x_0'(t)\big)\big)$$

for $t \ge T$, and so (5.4) gives

$$\int_T^\infty P(t)^{-1} u(t)^2 \, dt < \infty \, .$$

If $\lim_{t\to\infty} u(t) = \ell \ne 0$, then the above fact contradicts (1.6). Therefore we deduce that $\ell = 0$.

Since

$$\lim_{t\to\infty} u(t) = \ell = 0 \, , \tag{5.8}$$

we find that

$$|u(t)| \le \frac{c}{2} < c \le p(t) x_0(t) |x_0'(t)|^\alpha \tag{5.9}$$

for sufficiently large t. Here, c is a positive constant satisfying (5.1). We may suppose that (5.9) holds for $t \ge T$. Then, applying Lemma 4.2 (ii) to the case $k_1 = c/2$, $k_2 = c$, $u = u(t)$ and $v = p(t)x_0(t)\Phi_\alpha(x_0'(t))$, we deduce that $F(u(t), p(t)x_0(t)\Phi_\alpha(x_0'(t)))$ is expressed as

$$\begin{aligned} &F\big(u(t), p(t) x_0(t) \Phi_\alpha\big(x_0'(t)\big)\big) \\ &\quad = \frac{\alpha+1}{2\alpha^2} |p(t) x_0(t) \Phi_\alpha(x_0'(t))|^{-(\alpha-1)/\alpha} u(t)^2 \big(1 + R(t)\big) \end{aligned}$$

with

$$|R(t)| \le \frac{|\alpha - 1|}{3\alpha} L\big(c/2, c\big) \big|u(t)\big| \tag{5.10}$$

for $t \ge T$. Here, $L\big(c/2, c\big)$ is a positive constant. Therefore we get

$$\begin{aligned} &p(t)^{-1/\alpha} x_0(t)^{-(\alpha+1)/\alpha} F\big(u(t), p(t) x_0(t) \Phi_\alpha\big(x_0'(t)\big)\big) \\ &\quad = \frac{\alpha+1}{2\alpha^2} P(t)^{-1} u(t)^2 \big(1 + R(t)\big) \, , \quad t \ge T \, . \end{aligned} \tag{5.11}$$

Remember that $\varepsilon \in (0,1)$ is the number such that (1.11) is oscillatory. By (5.8) and (5.10), we have $\lim_{t\to\infty} R(t) = 0$, and hence

$$R(t) \ge -\varepsilon \tag{5.12}$$

for sufficiently large t, say $t \ge T_1$. Then, by (4.3), (5.11) and (5.12), we find that

$$u'(t) + Q(t) + (1-\varepsilon) \frac{\alpha+1}{2\alpha} P(t)^{-1} u(t)^2 \le 0 \, , \quad t \ge T_1 \, .$$

Therefore the function

$$y(t) = (1-\varepsilon) \frac{\alpha+1}{2\alpha} u(t) \, , \quad t \ge T_1 \, ,$$

satisfies

$$y'(t) + (1-\varepsilon)\frac{\alpha+1}{2\alpha}Q(t) + P(t)^{-1}y(t)^2 \le 0\,, \quad t \ge T_1\,.$$

Hence, the linear version of Lemma 2.1 implies that equation (1.11) is nonoscillatory. This is a contradiction. Thus we find that (5.4) also does not hold. Consequently, equation (1.1) does not have nonoscillatory solutions. The proof of Theorem 1.4 is complete. □

6. Examples

In this section we provide a few examples illustrating our results.

Example 6.1. Let

$$\gamma_\alpha = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}, \tag{6.1}$$

and let $p(t)$ be a continuous function on $[0,\infty)$ such that $p(t) > 0$ for $t \ge 0$ and

$$\int_0^\infty p(t)^{-1/\alpha}\,dt = \infty\,.$$

For simplicity of notation, we put

$$I_p(t) = \int_0^t p(s)^{-1/\alpha}\,ds\,.$$

Then, let us consider the half-linear differential equation

$$\big(p(t)\Phi_\alpha(x')\big)' + \Big(\gamma_\alpha p(t)^{-1/\alpha} I_p(t)^{-\alpha-1} + c(t)\Big)\,\Phi_\alpha(x) = 0\,, \quad t \ge t_0\,, \tag{6.2}$$

where $c(t)$ is a continuous function on $[t_0,\infty)$, $t_0 > 0$. Equation (6.2) is regarded as a perturbation of the Euler type equation

$$\big(p(t)\Phi_\alpha(x')\big)' + \gamma_\alpha p(t)^{-1/\alpha} I_p(t)^{-\alpha-1}\Phi_\alpha(x) = 0\,, \quad t \ge t_0\,. \tag{6.3}$$

Equations (6.2) and (6.3) are the special cases of (1.1) and (1.3) with

$$q(t) = \gamma_\alpha p(t)^{-1/\alpha} I_p(t)^{-\alpha-1} + c(t)$$

and

$$q_0(t) = \gamma_\alpha p(t)^{-1/\alpha} I_p(t)^{-\alpha-1}\,,$$

respectively.

Equation (6.3) has the exact solution

$$x_0(t) = I_p(t)^{\alpha/(\alpha+1)}\,, \quad t \ge t_0\,.$$

It is clear that this solution $x_0(t)$ satisfies (1.5) with $T = t_0$. The functions $P(t)$ and $Q(t)$ defined by (1.9) are equal to

$$P(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha-1} p(t)^{1/\alpha} I_p(t) \quad \text{and} \quad Q(t) = I_p(t)^\alpha c(t)\,,$$

respectively. We have

$$\int_{t_0}^t \frac{1}{P(s)}\,ds \sim \Big(\frac{\alpha}{\alpha+1}\Big)^{-\alpha+1} \log I_p(t) \quad (t \to \infty)$$

and

$$p(t)x_0(t)\big|x_0'(t)\big|^{\alpha} = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}, \quad t \ge t_0 .$$

Therefore, all the conditions (1.5)–(1.7) are satisfied. By Corollary 1.5 we can conclude that if $c(t)$ satisfies

$$\int_{t_0}^{\infty} I_p(t)^{\alpha} c(t)\,dt \quad \text{is convergent}$$

and

$$\liminf_{t\to\infty}\,(\log I_p(t))\Big(\int_t^{\infty} I_p(s)^{\alpha} c(s)\,ds\Big) > \frac{1}{2}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha},$$

then (6.2) is oscillatory.

Example 6.2. Let γ_α be the constant given by (6.1), and let $p(t)$ be a continuous function on $[t_0,\infty)$ such that $p(t) > 0$ for $t \ge t_0$ and

$$\int_{t_0}^{\infty} p(t)^{-1/\alpha}\,dt < \infty .$$

For simplicity of notation, we put

$$J_p(t) = \int_t^{\infty} p(s)^{-1/\alpha}\,ds, \quad t \ge t_0 .$$

Consider the half-linear equations

$$\big(p(t)\Phi_\alpha(x')\big)' + \big(\gamma_\alpha p(t)^{-1/\alpha} J_p(t)^{-\alpha-1} + c(t)\big)\Phi_\alpha(x) = 0, \quad t \ge t_0, \tag{6.4}$$

and

$$\big(p(t)\Phi_\alpha(x')\big)' + \gamma_\alpha p(t)^{-1/\alpha} J_p(t)^{-\alpha-1}\Phi_\alpha(x) = 0, \quad t \ge t_0 . \tag{6.5}$$

Here, $c(t)$ is a continuous function on $[t_0,\infty)$. Equations (6.4) and (6.5) are the special cases of (1.1) and (1.3) with

$$q(t) = \gamma_\alpha p(t)^{-1/\alpha} J_p(t)^{-\alpha-1} + c(t)$$

and

$$q_0(t) = \gamma_\alpha p(t)^{-1/\alpha} J_p(t)^{-\alpha-1},$$

respectively.

Equation (6.5) has the exact solution

$$x_0(t) = J_p(t)^{\alpha/(\alpha+1)}, \quad t \ge t_0 .$$

It can be checked that all the conditions (1.5)–(1.7) are satisfied. By Corollary 1.5 we can conclude that if $c(t)$ satisfies

$$\int_{t_0}^{\infty} J_p(t)^{\alpha} c(t)\,dt \quad \text{is convergent}$$

and

$$\liminf_{t\to\infty}\,|\log J_p(t)|\Big(\int_t^{\infty} J_p(s)^{\alpha} c(s)\,ds\Big) > \frac{1}{2}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha},$$

then (6.4) is oscillatory. The details are left to the reader.

Example 6.3. Consider the half-linear differential equation

$$\big(\Phi_\alpha(x')\big)' + \big(\gamma_\alpha t^{-\alpha-1} + c(t)\big)\,\Phi_\alpha(x) = 0\,, \quad t \ge t_0\ (> 1)\,, \tag{6.6}$$

where γ_α is the positive constant given by (6.1) and the function $c(t)$ is given by

$$c(t) = -t^{-\alpha}\frac{d}{dt}\Big(\frac{1+\cos(\log t)}{(\log t)^{1/2}}\Big)\,, \quad t \ge t_0\,. \tag{6.7}$$

Equation (6.6) is regarded as a perturbation of the equation

$$\big(\Phi_\alpha(x')\big)' + \gamma_\alpha t^{-\alpha-1}\Phi_\alpha(x) = 0\,, \quad t \ge t_0\,. \tag{6.8}$$

Equation (6.8) has the exact solution

$$x_0(t) = t^{\alpha/(\alpha+1)}\,, \quad t \ge t_0\,.$$

It is clear that this solution $x_0(t)$ satisfies (1.5) with $T = t_0$. The functions $P(t)$ and $Q(t)$ defined by (1.9) are equal to

$$P(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha-1} t \quad \text{and} \quad Q(t) = t^\alpha c(t)\,,$$

respectively. Therefore we have

$$\int_{t_0}^{t} \frac{1}{P(s)}\,ds = \Big(\frac{\alpha}{\alpha+1}\Big)^{-\alpha+1}(\log t - \log t_0)$$

and

$$Q(t) = -\frac{d}{dt}\Big(\frac{1+\cos(\log t)}{(\log t)^{1/2}}\Big) = \frac{1}{2}\,\frac{1+\cos(\log t)}{t(\log t)^{3/2}} + \frac{\sin(\log t)}{t(\log t)^{1/2}}\,.$$

It is easy to see that (1.6)–(1.8) are satisfied. We apply Zlámal type oscillation criterion (Corollary 1.6, $\lambda = 1/2$). Since

$$\begin{aligned}\int_{t_0}^{t} (\log s)^{1/2} Q(s)\,ds &= \frac{1}{2}\int_{t_0}^{t}\frac{1}{s\log s}\,ds + \frac{1}{2}\int_{t_0}^{t}\frac{\cos(\log s)}{s\log s}\,ds \\ &\quad + \int_{t_0}^{t}\frac{\sin(\log s)}{s}\,ds \\ &= \frac{1}{2}[\log(\log t) - \log(\log t_0)] + \frac{1}{2}\int_{\log t_0}^{\log t}\frac{\cos\sigma}{\sigma}\,d\sigma \\ &\quad + \int_{\log t_0}^{\log t}\sin\sigma d\sigma\,,\end{aligned}$$

we get

$$\int_{t_0}^{t} (\log s)^{1/2} Q(s)\,ds \sim \frac{1}{2}\log(\log t) \quad (t \to \infty)\,. \tag{6.9}$$

Integration by parts yields

$$\begin{aligned}\int_{t_0}^{t} (\log s - \log t_0)^{1/2} Q(s)\, ds &= \int_{t_0}^{t} \Big(1 - \frac{\log t_0}{\log s}\Big)^{1/2} (\log s)^{1/2} Q(s)\, ds \\ &= \Big(1 - \frac{\log t_0}{\log t}\Big)^{1/2} \int_{t_0}^{t} (\log s)^{1/2} Q(s)\, ds \\ &\quad - \int_{t_0}^{t} \Big(\int_{t_0}^{s} (\log \sigma)^{1/2} Q(\sigma)\, d\sigma\Big) \frac{d}{ds}\Big(1 - \frac{\log t_0}{\log s}\Big)^{1/2} ds\,.\end{aligned}$$

Since

$$\frac{d}{ds}\Big(1 - \frac{\log t_0}{\log s}\Big)^{1/2} = O\Big(\frac{1}{s(\log s)^2}\Big) \quad (s \to \infty)\,,$$

it follows from (6.9) that

$$\int_{t_0}^{t} (\log s - \log t_0)^{1/2} Q(s)\, ds \sim \frac{1}{2} \log(\log t) \quad (t \to \infty)\,,$$

and so

$$\int_{t_0}^{\infty} \Big(\int_{t_0}^{t} \frac{1}{P(s)}\, ds\Big)^{1/2} Q(t)\, dt = \infty\,.$$

Thus, by Corollary 1.6 of the case $\lambda = 1/2$, equation (6.6) with (6.7) is oscillatory.

Note that Hille–Nehari type oscillation criterion (Corollary 1.5) cannot be applied to equation (6.6) with (6.7), because

$$\begin{aligned}&\liminf_{t\to\infty} \Big(\int_{t_0}^{t} \frac{1}{P(s)}\, ds\Big)\Big(\int_{t}^{\infty} Q(s)\, ds\Big) \\ &= \Big(\frac{\alpha}{\alpha+1}\Big)^{-\alpha+1} \liminf_{t\to\infty} (\log t - \log t_0) \frac{1 + \cos(\log t)}{(\log t)^{1/2}} = 0\,.\end{aligned}$$

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