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# THE CANONICAL CONSTRUCTIONS OF CONNECTIONS ON TOTAL SPACES OF FIBRED MANIFOLDS

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Abstract. We classify classical linear connections $A(\Gamma, \Lambda, \Theta)$ on the total space $Y$ of a fibred manifold $Y \to M$ induced in a natural way by the following three objects: a general connection $\Gamma$ in $Y \to M$, a classical linear connection $\Lambda$ on $M$ and a linear connection $\Theta$ in the vertical bundle $VY \to Y$. The main result says that if $\dim(M) \geq 3$ and $\dim(Y) - \dim(M) \geq 3$ then the natural operators $A$ under consideration form the 17 dimensional affine space.

## 1. Introduction

All manifolds considered in the paper are assumed to be finite dimensional and smooth (of class $C^\infty$). Maps between manifolds are assumed to be of class $C^\infty$.

Let $Y \to M$ be a fibred manifold and $E \to M$ be a vector bundle. Put $m = \dim(M)$ and $n = \dim(Y) - \dim(M)$.

A general connection in $Y \to M$ is a section $\Gamma\colon Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$ of $Y$, see [4]. It can be equivalently defined by its corresponding lifting map $\Gamma\colon Y \times_M TM \to TY$ (or $\Gamma\colon Y \to T^*M \otimes TY$) given by $\Gamma(y, v) := T_x\sigma(v)$, $j^1_x(\sigma) = \Gamma(y)$, $y \in Y_x$, $v \in T_xM$, $x \in M$. It can be also equivalently defined by its corresponding decomposition $TY = VY \oplus H^\Gamma Y$ of the tangent bundle $TY$ of $Y$, where $VY$ is the vertical bundle of $Y \to M$ and $H^\Gamma Y$ is the so-called $\Gamma$-horizontal sub-bundle being the image of $\Gamma\colon Y \times_M TM \to TY$.

A general connection $D\colon E \to J^1E$ in $E \to M$ is called a linear connection in $E \to M$ if $D$ is a vector bundle map. It can be equivalently defined by its covariant differentiation $D$ given by $(D_X\sigma)(x) :=$ the vertical part of $T_x\sigma(X_x)$ (modulo the identification $V_{\sigma(x)}E = E_x$) for all vector fields $X$ on $M$ and all sections $\sigma$ of $E \to M$ and all $x \in M$.

A linear connection $\Theta\colon VY \to J^1(VY)$ in the vertical bundle $VY \to Y$ of $Y \to M$ is called a vertical classical linear connection on $Y \to M$.

A linear connection $\Lambda\colon TM \to J^1(TM)$ in the tangent bundle $TM \to M$ of $M$ is called a classical linear connection on $M$. (Then the covariant differentiation of $\Lambda$ is a linear connection on $M$ in the sense of [2].)

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In [5], we described all classical linear connections $A(\Gamma,\Lambda,\Theta)\colon TY \to J^1(TY)$ on $Y$ induced in natural way by the following three objects: a general connection $\Gamma$ in $Y \to M$, a torsion free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$. We proved that if $m \geq 3$ and $n \geq 3$ then the induced connections $A(\Gamma,\Lambda,\Theta)$ form 12-parameter family.

In the present note, we describe all classical linear connections $A(\Gamma,\Lambda,\Theta)\colon TY \to J^1(TY)$ on $Y$ induced in natural way by the following three objects: a general connection $\Gamma$ in $Y \to M$, a (not necessarily torsion free) classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$. The main result (Theorem 8.3) is that if $m \geq 3$ and $n \geq 3$ then the induced connections $A(\Gamma,\Lambda,\Theta)$ form the 17-dimensional affine space. In particular, the basis of the affine space is presented.

An important example of $A(\Gamma,\Lambda,\Theta)$ is $\Psi(\Gamma,\Lambda,\Theta)$, see Example 2.1. In Proposition 2.3, we deduce that every $\Gamma$-lift of a geodesic of $\Lambda$ is a geodesic of $\Psi(\Gamma,\Lambda,\Theta)$.

In the case of a vector bundle $E \to M$, there exists a classical linear connection $G(D,\Lambda)\colon TE \to J^1(TE)$ on $E$ induced in natural way by the following two objects: a linear connection $D$ in $E \to M$ and a (not necessarily torsion free) classical linear connection $\Lambda$ on $M$, see Example 3.1. This connection was discovered by J. Gancarzewicz, [1]. Quite similarly, there is a vertical classical linear connection $\theta(D) : VE \to J^1(VE)$ on $E \to M$ induced in natural way be a linear connection $D$ in $E \to M$, see Example 3.2. In Proposition 3.3, we observe that $\Psi(D,\Lambda,\Theta)$ generalizes $G(D,\Lambda)$. Namely, we prove $G(D,\Lambda) = \Psi(D,\Lambda,\theta(D))$. In Remark 3.4, we generalize this fact (we reobtain some construction by I. Kolář [3]).

General connections globalize first order differential systems. Indeed, if $x^1,\dots,x^m$, $y^1,\dots,y^n$ are fibre manifold local coordinates on $Y \to M$, then a general connection $\Gamma\colon Y \to T^*M \otimes TY$ has the coordinate expression $dx^i \otimes \frac{\partial}{\partial x^i} + \Gamma^p_i(x,y)dx^i \otimes \frac{\partial}{\partial y^p}$ or (dually) $dy^p = \Gamma^p_i(x,y)dx^i$, where we use the Einstein convention of summation. Classical linear connections play fundamental role in the differential geometry. They globalize parallel transport and involve in the equations of geodesics, and therefore they have applications in many physical theories. Canonical constructions on connections have been studied in many papers, e.g. [1, 3, 4, 5].

From now on, let $x^1,\dots,x^m,y^1,\dots,y^n$ be fibre manifold local coordinates on a fibred manifold $Y \to M$, $\eta^1,\dots,\eta^n$ be the additional coordinates in the vertical bundle $VY$ and $\xi^1,\dots,\xi^m$ be the additional coordinates in the tangent bundle $TM$. Let the same letters $\xi^1,\dots,\xi^m$ and $\eta^1,\dots,\eta^n$ denote the additional coordinates on $TY$. If $Y = E \to M$ is a vector bundle, we assume that $x^1,\dots,x^m,y^1,\dots,y^n$ are vector bundle local coordinates. If $Y = \mathbf{R}^{m,n} = \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ is the obvious trivial bundle, we assume that $x^1,\dots,x^m,y^1,\dots,y^n$ are the usual coordinates.

## 2. Connections $\Psi(\Gamma,\Lambda,\Theta)$ and their ($\Gamma$-horizontal) geodesics

Let $Y \to M$ be a fibred manifold. Let $\Gamma\colon Y \to J^1Y$ be a general connection in $Y \to M$ and $\Lambda\colon TM \to J^1(TM)$ be a classical linear connection on $M$ and $\Theta\colon VY \to J^1(VY)$ be a vertical classical linear connection on $Y \to M$.

**Example 2.1** ([5])**.** Let $Z \in T_yY$, $y \in Y_x$, $x \in M$. We decompose $Z \in T_yY$ into the horizontal part $h(Z) = \Gamma(y, Z_o)$, $Z_o \in T_xM$, and the vertical part $vZ$. We take a vector field $X$ on $M$ such that $j^1_x X = \Lambda(Z_o)$ and construct its $\Gamma$-lift $\Gamma X\colon Y \to TY$, and we take a vertical vector field $\theta Z\colon Y \to VY$ such that $j^1_y(\theta Z) = \Theta(vZ)$. For every $Z \in T_yY$ we put $\Psi(\Gamma,\Lambda,\Theta)(Z) := j^1_y(\Gamma X + \theta Z)$ . Thus $\Psi(\Gamma,\Lambda,\Theta)\colon TY \to J^1(TY)$ is a classical linear connection on $Y$.

**Lemma 2.2** ([5])**.** *Let $dy^p = \Gamma^p_i(x,y)\,dx^i$ be the coordinate expression of $\Gamma$,*

$$d\xi^i = \Lambda^i_{jk}(x)\xi^j\,dx^k \tag{1}$$

*be the coordinate expression of $\Lambda$, and $d\eta^p = \Theta^p_{qi}(x,y)\eta^q\,dx^i + \Theta^p_{qs}(x,y)\eta^q\,dy^s$ be the coordinate expression of $\Theta$. The coordinate expression of $\Psi = \Psi(\Gamma,\Lambda,\Theta)$ is* (1) *and*

$$\begin{aligned} d\eta^p = {} & (\tfrac{\partial\Gamma^p_i}{\partial x^j} + \Gamma^p_k\Lambda^k_{ij} - \Theta^p_{qj}\Gamma^q_i)\xi^i\,dx^j + \Theta^p_{qj}\eta^q\,dx^j + \\ & + (\tfrac{\partial\Gamma^p_i}{\partial y^s} - \Theta^p_{qs}\Gamma^q_i)\xi^i\,dy^s + \Theta^p_{qs}\eta^q\,dy^s\,. \end{aligned} \tag{2}$$

**Proof.** ([5]) Let $\xi^i = X^i(x)$ and $\eta^p = (\theta Z)^p(x,y)$ be the coordinate expression of $X$ or $\theta Z$, respectively. Hence

$$\frac{\partial X^i}{\partial x^j} = \Lambda^i_{kj}X^k\,, \quad \frac{\partial(\theta Z)^p}{\partial x^j} = \Theta^p_{qj}(\theta Z)^q\,, \quad \frac{\partial(\theta Z)^p}{\partial y^s} = \Theta^p_{qs}(\theta Z)^q\,.$$

Then the coordinate expression of $\Gamma X + \theta Z$ is $\xi^i = X^i(x)$ and

$$\eta^p = \Gamma^p_i(x,y)X^i(x) + (\theta Z)^p(x,y)\,.$$

Differentiating this relation we obtain (2). □

**Proposition 2.3.** *Every $\Gamma$-lift $(x^i(t), y^p(t))$ of a geodesic $x^i(t)$ of $\Lambda$ is a geodesic of $\Psi = \Psi(\Gamma,\Lambda,\Theta)$ for arbitrary $\Theta$.*

**Proof.** The $\Gamma$-lift satisfies $\frac{dy^p}{dt} = \Gamma^p_i(x(t),y(t))\frac{dx^i}{dt}$. Differentiating this and using $x^i(t)$ is a geodesic of $\Lambda$ we obtain

$$\frac{d^2x^i}{dt^2} = \Lambda^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \quad \text{and} \quad \frac{d^2y^p}{dt^2} = (\frac{\partial\Gamma^p_i}{\partial x^j} + \Gamma^p_k\Lambda^k_{ij})\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial\Gamma^p_i}{\partial y^q}\frac{dx^i}{dt}\frac{dy^q}{dt}\,.$$

Since $\frac{dy^p}{dt} = \Gamma^p_i\frac{dx^i}{dt}$, then we obtain $\frac{d^2x^i}{dt^2} = \Lambda^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$ and

$$\begin{aligned} \tfrac{d^2y^p}{dt^2} = {} & (\tfrac{\partial\Gamma^p_i}{\partial x^j} + \Gamma^p_k\Lambda^k_{ij} - \Theta^p_{qj}\Gamma^q_i)\tfrac{dx^i}{dt}\tfrac{dx^j}{dt} + \Theta^p_{qj}\tfrac{dy^q}{dt}\tfrac{dx^j}{dt} + \\ & + (\tfrac{\partial\Gamma^p_i}{\partial y^s} - \Theta^p_{qs}\Gamma^q_i)\tfrac{dx^i}{dt}\tfrac{dy^s}{dt} + \Theta^p_{qs}\tfrac{dy^q}{dt}\tfrac{dy^s}{dt} \end{aligned}$$

for any $\Theta$. But these equations are the ones of geodesics of $\Psi = \Psi(\Gamma,\Lambda,\Theta)$ (with the coordinate expression (1) and (2)). □

## 3. Connections $G(D,\Lambda)$ and their relations with $\Psi(D,\Lambda,\Theta)$

Let $E\to M$ be a vector bundle and $D$ be a linear connection in $E\to M$ and $\Lambda$ be a classical linear connection on $M$. Thus, if $X$ is a vector field on $M$ and $s$ is a section of $E$, then $D_Xs$ is a section of $E$. Further, let $X^D$ denote the horizontal lift of vector field $X$ with respect to $D$. Moreover, using the translation in the individual fibres of $E$, we derive from every section $s\colon M\to E$ a vertical vector field $s^V$ on $E$ called the vertical lift of $s$.

**Example 3.1.** In [1], J. Gancarzewicz proved that there exists a unique classical linear connection $G=G(D,\Lambda)$ on the total space $E$ such that

$$G_{X^D}Z^D=(\Lambda_XZ)^D\,,\quad G_{\sigma^V}Z^D=0\,,\quad G_{X^D}s^V=(D_X\sigma)^V\,,\quad G_{s^V}\sigma^V=0$$

for all vector fields $X$ and $Z$ on $M$ and all sections $s$ and $\sigma$ of $E\to M$. It generalizes the (usual) horizontal lift $\nabla^H$ of a classical linear connection $\nabla$ on $M$ to $TM$. (Namely, $\nabla^H=G(\nabla,\nabla)$.)

**Example 3.2.** Quite similarly, there is a unique vertical classical linear connection $\theta=\theta(D)$ on $E\to M$ such that

$$\theta_{X^D}s^V=(D_X\sigma)^V,\ \theta_{s^V}\sigma^V=0$$

for all vector fields $X$ on $M$ and all sections $s,\sigma$ of $E$.

**Proposition 3.3.** *We have $G(D,\Lambda)=\Psi(D,\Lambda,\theta(D))$ , where $\Psi(\Gamma,\Lambda,\Theta)$ is as in Example 2.1.*

**Proof.** Let $dy^p=D^p_{qi}(x)y^qdx^i$ be the coordinate expression of $D$ and (1) be the coordinate expression of $\Lambda$ and let

$$d\xi^i=(G^i_{jk}\xi^j+G^i_{pk}\eta^p)\,dx^k+(G^i_{jq}\xi^j+G^i_{pq}\eta^p)\,dy^q\,,$$
$$d\eta^p=(G^p_{ij}\xi^i+G^p_{qj}\eta^q)\,dx^j+(G^p_{ir}\xi^i+G^p_{qr}\eta^q)\,dy^r$$

be the coordinate expression of $G=G(D,\Lambda)$. Then, by Section 54.2 in [4], we have

$$G^i_{jk}=\Lambda^i_{jk}\,,\qquad G^p_{ij}=(\frac{\partial D^p_{qi}}{\partial x^j}-D^p_{rj}D^r_{qi}+D^p_{qk}\Lambda^k_{ij})y^q\,,$$
$$G^j_{ip}=G^j_{pi}=0\,,\qquad G^p_{iq}=G^p_{qi}=D^p_{qi}\,,\qquad G^i_{pq}=0\,,\qquad G^p_{qr}=0\,.$$

Then the coordinate expression of $G(D,\Lambda)$ is (1) and

(3) $$d\eta^p=(\tfrac{\partial D^p_{qi}}{\partial x^j}y^q\xi^i+D^p_{qk}\Lambda^k_{ij}y^q\xi^i+D^p_{qj}(\eta^q-D^q_{si}y^s\xi^i))dx^j+D^p_{si}\xi^idy^s\,.$$

Further let

$$d\eta^p=\theta^p_{qj}\eta^q\,dx^j+\theta^p_{qr}\eta^q\,dy^r$$

be the coordinate expression of $\theta=\theta(D)$. Then, by quite similar (almost the same) proof as in Section 54.2 in [4], we have

$$\theta^p_{qj}=D^p_{qj}\,,\qquad \theta^p_{qs}=0\,.$$

Then, by Lemma 2.2 with $\Gamma^p_i(x,y)$ replaced by $D^p_{qi}(x)y^q$ and with $\Theta$ replaced by $\theta(D)$, the coordinate expression of $\Psi=\Psi(D,\Lambda,\theta(D))$ is (1) and (3), as well. $\square$

**Remark 3.4.** The result of this section can be generalized as follows. Let $p\colon Y \to M$ be a fibred manifold and $E \to M$ be a vector bundle with the same base $M$. Let $\Gamma$ be a general connection in $Y \to M$, $\Lambda$ be a classical linear connection on $M$, $\Delta$ be a linear connection in $E \to M$ and $\Phi\colon Y \times_M E \to VY$ be a vertical parallelism on $Y \to M$, i.e. a system of parallelism $\Phi_x\colon Y_x \times E_x \to TY_x$, $x \in M$ such that the resulting map $\Phi\colon Y \times_M E \to VY$ is a vector bundle isomorphism covering the identity map of $Y$. The system $(\Delta, \Lambda, \Phi)$ determines a vertical classical linear connection $\Theta = \Theta(\Lambda, \Phi, \Delta)\colon VY \to J^1(VY \to Y)$ on $Y \to M$. Indeed, for any point $v = \Phi(y, Z_1) \in V_yY$, $y \in Y_x$, $Z_1 \in E_x$, $x = p(y)$, we take a section $\sigma : M \to E$ such that $j^1_x\sigma = \Delta(Z_1)$ and we define $\Theta(v) = j^1_y(\varphi(\sigma))$ , where $\varphi(\sigma) : Y \to VY$ is a vertical vector field given by $\varphi(\sigma)(y) = \Phi(y, \sigma(p(y)))$. Now, using our general construction from Example 2.1, one can define a classical linear connection $(\Gamma, \Lambda, \Phi, \Delta) := \Psi(\Gamma, \Lambda, \Theta(\Lambda, \Phi, \Delta))$ on $Y$. Clearly, it is the same as in [3]. If $Y = E \to M$ is a vector bundle and $\Gamma = \Delta = D$ and $\Phi = \Phi^o$ is the canonical vertical parallelism on $E \to M$, then $(D, \Lambda, \Phi^o, D)$ coincides with the (mentioned in Example 3.1) classical linear connection $G(D, \Lambda)$ on $E$.

## 4. Examples of tensor fields of type $(1, 2)$ on $Y$ induced by $(\Gamma, \Lambda, \Theta)$

Let $Y \to M$ be a fibred manifold. Let $\Gamma\colon Y \to J^1Y$ be a general connection in $Y \to M$ and $\Lambda\colon TM \to J^1(TM)$ be a classical linear connection on $M$ and $\Theta\colon VY \to J^1(VY)$ be a vertical classical linear connection on $Y \to M$. Let $\Psi = \Psi(\Gamma, \Lambda, \Theta)$ be the classical linear connection on $Y$ as in Example 2.1 and let $\mathrm{Tor}(\Psi)$ be the torsion tensor (of type $(1, 2)$) of $\Psi$ on $Y$.

Using the usual $\Gamma$-decomposition $TY = VY \oplus_Y H^\Gamma Y$, we can write

$$T^*Y \otimes TY = (V^*Y \otimes VY) \oplus_Y (V^*Y \otimes H^\Gamma Y)$$
$$\oplus_Y ((H^\Gamma Y)^* \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes H^\Gamma Y)\,.$$

Let $\mathrm{id}_{HY}$ be the tensor field of type $(1, 1)$ on $Y$ being the $(H^\Gamma Y)^* \otimes H^\Gamma Y$-component of the identity tensor field $\mathrm{id}_{TY}$ on $Y$ (the other 3 components of $\mathrm{id}_{HY}$ are 0) and let $\mathrm{id}_{VY}$ be the tensor field of type $(1, 1)$ on $Y$ being the $V^*Y \otimes VY$-component of $\mathrm{id}_{TY}$. Similarly, we can write

$$T^*Y \otimes T^*Y \otimes TY = (V^*Y \otimes V^*Y \otimes VY) \oplus_Y (V^*Y \otimes V^*Y \otimes H^\Gamma Y)$$
$$\oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes VY) \oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y)$$
$$\oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes H^\Gamma Y)$$
$$\oplus_Y ((H^\Gamma Y)^* \otimes H^\Gamma Y)^* \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y)\,.$$

Let $\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$ and $\mathrm{Tor}^{H^*\otimes H^*\otimes V}(\Psi)$ and $\mathrm{Tor}^{V^*\otimes H^*\otimes V}(\Psi)$ and $\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$ and $\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)$ be the tensor fields of type $(1, 2)$ on $Y$ being the $H^\Gamma Y \otimes V^*Y \otimes VY$- and $(H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes VY$- and the $V^*Y \otimes (H^\Gamma Y)^* \otimes VY$- and the $V^*Y \otimes V^*Y \otimes VY$- and the $(H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y$-component of $\mathrm{Tor}(\Psi)$, respectively. Applying contraction $C^1_2$ to the above tensor fields of type $(1, 2)$, we get (in particular) 1-forms $C^1_2\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$ and $C^1_2\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$

and $C^1_2\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)$ on $Y$. We can tensor these 1-forms by the tensor fields $\mathrm{id}_{HY}$ and $\mathrm{id}_{VY}$ of type $(1,1)$ and obtain respective tensor fields of type $(1,2)$ on $Y$. Thus we have the following tensor fields $\tau_i = \tau_i(\Gamma,\Lambda,\Theta)$ of type $(1,2)$ on $Y$ canonically depending on $(\Gamma,\Lambda,\Theta)$.

**Example 4.1.** $\tau_1 := \mathrm{Tor}^{H^*\otimes H^*\otimes V}(\Psi)$.

**Example 4.2.** $\tau_2 := \mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.3.** $\tau_3 := \mathrm{id}_{HY}\otimes C^1_2\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.4.** $\tau_4 := C^1_2\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)\otimes \mathrm{id}_{HY}$.

**Example 4.5.** $\tau_5 := C^1_2\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)\otimes \mathrm{id}_{VY}$.

**Example 4.6.** $\tau_6 := \mathrm{id}_{VY}\otimes C^1_2\mathrm{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.7.** $\tau_7 := \mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.8.** $\tau_8 := \mathrm{id}_{HY}\otimes C^1_2\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.9.** $\tau_9 := C^1_2\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)\otimes \mathrm{id}_{HY}$.

**Example 4.10.** $\tau_{10} := \mathrm{id}_{VY}\otimes C^1_2\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

**Example 4.11.** $\tau_{11} := C^1_2\mathrm{Tor}^{V^*\otimes V^*\otimes V}(\Psi)\otimes \mathrm{id}_{VY}$.

**Example 4.12.** $\tau_{12} := \mathrm{Tor}^{V^*\otimes H^*\otimes V}(\Psi)$.

**Example 4.13.** $\tau_{13} := \mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)$.

**Example 4.14.** $\tau_{14} := \mathrm{id}_{VY}\otimes C^1_2\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)$.

**Example 4.15.** $\tau_{15} := C^1_2\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)\otimes \mathrm{id}_{VY}$.

**Example 4.16.** $\tau_{16} := \mathrm{id}_{HY}\otimes C^1_2\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)$.

**Example 4.17.** $\tau_{17} := C^1_2\mathrm{Tor}^{H^*\otimes H^*\otimes H}(\Psi)\otimes \mathrm{id}_{HY}$.

## 5. The natural operator approach

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with $m$-dimensional bases and $n$-dimensional fibres and their fibred (local) diffeomorphisms. The general concept of natural operators can be found in [4]. We need the following particular cases.

**Definition 5.1.** An $\mathcal{FM}_{m,n}$-natural operator $A$ sending a triple $(\Gamma,\Lambda,\Theta)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y\to M$ and a classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y\to M$ into a classical linear connection $A(\Gamma,\Lambda,\Theta)$ on $Y$ is an $\mathcal{FM}_{m,n}$-invariant system of regular operators

$$A\colon \mathrm{Con}(Y)\times \mathrm{Con}_{\mathrm{clas}}(M)\times \mathrm{Con}_{\mathrm{vert-clas}}(Y)\to \mathrm{Con}_{\mathrm{clas}}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y=(Y\to M)$, where $\mathrm{Con}(Y)$ is the set of general connections $\Gamma$ in $Y\to M$, $\mathrm{Con}_{\mathrm{clas}}(M)$ is the set of classical linear connections $\Lambda$ on $M$,

$\mathrm{Con}_{\mathrm{vert-clas}}(Y)$ is the set of vertical classical linear connections $\Theta$ on $Y \to M$ and $\mathrm{Con}_{\mathrm{clas}}(Y)$ is the set of classical linear connections on $Y$. The $\mathcal{FM}_{m,n}$-invariance of $A$ means that if $(\Gamma, \Lambda, \Theta) \in \mathrm{Con}(Y) \times \mathrm{Con}_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y)$ and $(\Gamma_1, \Lambda_1, \Theta_1) \in \mathrm{Con}(Y_1) \times \mathrm{Con}_{\mathrm{clas}}(M_1) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y_1)$ are $f$-related for an $\mathcal{FM}_{m,n}$-map $f\colon Y \to Y_1$, then so are $A(\Gamma, \Lambda, \Theta)$ and $A(\Gamma_1, \Lambda_1, \Theta_1)$. The regularity of $A$ means that $A$ transforms smoothly parametrized families into smoothly parametrized families.

For example, $A(\Gamma, \Lambda, \Theta) := \Psi(\Gamma, \Lambda, \Theta)$ defines a natural operator in the sense of Definition 5.1.

**Definition 5.2.** An $\mathcal{FM}_{m,n}$-natural operator $\Delta$ sending a triple $(\Gamma, \Lambda, \Theta)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$ into a tensor field $\Delta(\Gamma, \Lambda, \Theta)$ of type $(1, 2)$ on $Y$ is an $\mathcal{FM}_{m,n}$-invariant system of regular operators

$$\Delta : \mathrm{Con}(Y) \times \mathrm{Con}_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y) \to \mathrm{Ten}^{(1,2)}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $\mathrm{Ten}^{(1,2)}(Y)$ is the space of tensor fields of type $(1, 2)$ on $Y$ and the other spaces are as in Definition 5.1.

Clearly, any natural operator $A$ in the sense of Definition 5.1 is of the form

$$A(\Gamma, \Lambda, \Theta) = \Psi(\Gamma, \Lambda, \Theta) + \Delta(\Gamma, \Lambda, \Theta)$$

for a unique natural operator $\Delta$ in the sense of Definition 5.2. So, to describe all natural operators in the sense of Definition 5.1 it is sufficient to describe the ones in the sense of Definition 5.2.

**Definition 5.3.** An $\mathcal{FM}_{m,n}$-natural operator $\Delta$ sending a 4-tuple $(\Gamma, \Lambda, \Theta, S)$ consisting of a general connections $\Gamma$ in an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$ and a skew-symmetric tensor field $S$ of type $(1, 2)$ on $M$ into a tensor field $\Delta(\Gamma, \Lambda, \Theta, S)$ of type $(1, 2)$ on $Y$ is an $\mathcal{FM}_{m,n}$-invariant family of regular operators

$$\Delta\colon \mathrm{Con}(Y) \times \mathrm{Con}^o_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y) \times \mathrm{Ten}^{(1,2)}_{\mathrm{skew-sym}}(M) \to \mathrm{Ten}^{(1,2)}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $\mathrm{Con}^o_{\mathrm{clas}}(M)$ is the set of all torsion-free classical linear connections on $M$, $\mathrm{Ten}^{(1,2)}_{\mathrm{skew-sym}}(M)$ is the space of all skew-symmetric tensor fields on $M$ of type $(1, 2)$ and the other spaces are as in Definitions 5.1 and 5.2.

For any classical linear connection $\nabla$ on a manifold $M$, the classical linear connection $\nabla - \frac{1}{2}\mathrm{Tor}(\nabla)$ is torsion free and

$$\nabla = (\nabla - \frac{1}{2}\mathrm{Tor}(\nabla)) + \frac{1}{2}\mathrm{Tor}(\nabla)\,.$$

Hence, there is the obvious bijection between the natural operators $\Delta$ in the sense of Definition 5.2 and the ones $\Delta$ in the sense of Definition 5.3. So, to describe all natural operators in the sense of Definition 5.1 it is sufficient to describe the ones in the sense of Definition 5.3.

**Definition 5.4** ([5])**.** An $\mathcal{FM}_{m,n}$-natural operator $\Delta$ sending a 3-tuple $(\Gamma, \Lambda, \Theta)$ consisting of a general connections $\Gamma$ in an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to Y$ into a tensor field $\Delta(\Gamma, \Lambda, \Theta)$ of type $(1, 2)$ on $Y$ is an $\mathcal{FM}_{m,n}$-invariant family of regular operators

$$\Delta \colon \mathrm{Con}(Y) \times \mathrm{Con}^o_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y) \to \mathrm{Ten}^{(1,2)}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$.

**Proposition 5.5** ([5])**.** *Let $m \geq 2$ and $n \geq 3$. Let $\Delta$ be an $\mathcal{FM}_{m,n}$-natural operator in the sense of Definition 5.4. Then there are (uniquely determined) real numbers $\lambda^1, \dots, \lambda^{12}$ such that*

$$\Delta(\Gamma, \Lambda, \Theta) = \sum_{i=1}^{12} \lambda^i \tau_i(\Gamma, \Lambda, \Theta)$$

*for any triple $(\Gamma, \Lambda, \Theta)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$, where $\tau_i$ are as in Examples 4.1–4.12.*

Let $\Delta$ be a natural operator in the sense of Definition 5.3. Then we have an $\mathcal{FM}_{m,n}$-natural operator $\Delta^{(1)}$ in the sense of Definition 5.4 given by

$$\Delta^{(1)}(\Gamma, \Lambda, \Theta) := \Delta(\Gamma, \Lambda, \Theta, 0)$$

for any triple $(\Gamma, \Lambda, \Theta)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$. Clearly,

$$\Delta(\Gamma, \Lambda, \Theta, S) = \Delta^{(1)}(\Gamma, \Lambda, \Theta) + \Delta^{(2)}(\Gamma, \Lambda, \Theta, S)$$

for any 4-tuple $(\Gamma, \Lambda, \Theta, S)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$ and a skew-symmetric tensor field $S$ of type $(1, 2)$ on $M$, where $\Delta^{(2)}$ is the natural operator in the sense of Definition 5.3 satisfying $\Delta^{(2)}(-, -, -, 0) = 0$.

So, to describe all natural operators $A$ in the sense of Definition 5.1 it is sufficient to classify all natural operators $\Delta$ in the sense of Definition 5.3 satisfying $\Delta(-, -, -, 0) = 0$.

## 6. The dimension of the vector space of all $\Delta$ with $\Delta(-, -, -, 0) = 0$

Let $Y = \mathbf{R}^{m,n}$ be the trivial bundle $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$. We denote the trivial general connection in $\mathbf{R}^{m,n}$ by $\Gamma^o$ (i.e. $\Gamma^o = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}$), the flat classical linear connection on $\mathbf{R}^m$ by $\Lambda^o$ (i.e. $\Lambda^o = (0)$) and the trivial vertical classical linear connection on $\mathbf{R}^{m,n}$ by $\Theta^o$ (i.e. $\Theta^o = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{p=1}^n dy^p \otimes \frac{\partial}{\partial y^p}$).

We study a natural operator $\Delta$ in the sense of Definition 5.3 satisfying the condition $\Delta(-, -, -, 0) = 0$. Such $\Delta$ is determined by the values

$$\Delta(\Gamma, \Lambda, \Theta, S)(y) \in T^*_y Y \otimes T^*_y Y \otimes T_y Y$$

for $\mathcal{FM}_{m,n}$-objects $Y \to M$, general connections $\Gamma$ in $Y \to M$, torsion free classical linear connections $\Lambda$ on $M$, vertical classical linear connections $\Theta$ on $Y \to M$, skew-symmetric tensor fields $S$ of type $(1,2)$ on $M$ and $y \in Y_x$, $x \in M$. Using the invariance of $\Delta$ with respect to (respective) fibred manifold charts, we can assume $Y = \mathbf{R}^{m,n}$, $y = (0,0)$. Further, similarly as in Section 4 in [5], using Corollary 19.8 in [4], we may assume

$$\Gamma = \Gamma^o + \sum \Gamma^p_{j;\alpha\beta} x^\alpha y^\beta \, dx^j \otimes \tfrac{\partial}{\partial y^p}\,, \tag{4}$$

where the sum is over all $m$-tuples $\alpha$ and all $n$-tuples $\beta$ of non-negative integers and $j = 1, \dots, m$ and $p = 1, \dots, n$ with $1 \le |\alpha| + |\beta| \le K$,

$$\Lambda = \big(\sum \Lambda^i_{jk;\gamma} x^\gamma\big)_{i,j,k=1,\dots,m}\,,\ \ \Lambda^i_{jk;\gamma} = \Lambda^i_{kj;\gamma}\,, \tag{5}$$

where the sums are over all $m$-tuples $\gamma$ of non-negative integers with $1 \le |\gamma| \le K$,

$$\Theta = \Theta^o + \sum \Theta^r_{ip;\delta\sigma} x^\delta y^\sigma \eta^p \, dx^i \otimes \tfrac{\partial}{\partial \eta^r} + \sum \Theta^r_{sp;\delta\sigma} x^\delta y^\sigma \eta^p \, dy^s \otimes \tfrac{\partial}{\partial \eta^r}\,, \tag{6}$$

where the first sum is over all $m$-tuples $\delta$ and all $n$-tuples $\sigma$ of non-negative integers and $i = 1, \dots, m$ and $r, p = 1, \dots, n$ with $0 \le |\delta| + |\sigma| \le K$ and the second sum is over all $m$-tuples $\delta$ and $n$-tuples $\sigma$ of non-negative integers and $r, s, p = 1, \dots, n$ with $0 \le |\delta| + |\sigma| \le K$,

$$S = \sum S^k_{ij;\xi} x^\xi (dx^i \wedge dx^j) \otimes \tfrac{\partial}{\partial x^k}\,, \tag{7}$$

where the sum is over all $m$-tuples $\xi$ and all integers $i, j, k = 1, \dots, m$ with $0 \le |\xi| \le K$, where $K$ is an arbitrary positive integer. Here (and from now on)

$$dx^i \wedge dx^j := \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i)\,.$$

So, $\Delta$ is determined by the collection of smooth maps $\Delta_K \colon \mathbf{R}^{n(K)} \to \mathbf{R}^q = T^*_{(0,0)}\mathbf{R}^{m,n} \otimes T^*_{(0,0)}\mathbf{R}^{m,n} \otimes T_{(0,0)}\mathbf{R}^{m,n}$ $(K = 1, 2, \dots)$ given by

$$\Delta_K\big((\Gamma^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (\Theta^r_{ip;\delta\sigma}), (\Theta^r_{sp;\delta\sigma}), (S^k_{ij;\xi})\big) := \Delta(\Gamma, \Lambda, \Theta, S)(0,0)\,, \tag{8}$$

where $\Gamma, \Lambda, \Theta, S$ are as in (4)–(7).

Using the invariance of $\Delta$ with respect to $\varphi_t \times \phi_t$, $\varphi_t = t\,\mathrm{id}_{\mathbf{R}^m}$, $\phi_t = t\,\mathrm{id}_{\mathbf{R}^n}$, $t > 0$, we get the homogeneous condition

$$\begin{aligned} t\Delta_K\big(&(\Gamma^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (\Theta^r_{ip;\delta\sigma}), (\Theta^r_{sp;\delta\sigma}), (S^k_{ij;\xi})\big) \\ &= \Delta_K\big((t^{|\alpha|+|\beta|}\Gamma^p_{j;\alpha\beta}), (t^{|\gamma|+1}\Lambda^i_{jk;\gamma}), (t^{|\delta|+|\sigma|+1}\Theta^r_{ip;\delta\sigma}), \\ &\quad (t^{|\delta|+|\sigma|+1}\Theta^r_{sp;\delta\sigma}), (t^{|\xi|+1}S^k_{ij;\xi})\big)\,. \end{aligned}$$

By the homogeneous function theorem, from this homogeneity condition we obtain that $\Delta_K$ is the linear combination with real coefficients of $\Theta^r_{ip;(0)(0)}$, $\Theta^r_{sp;(0)(0)}$, $S^k_{ij;(0)}$ and $\Gamma^p_{j;\alpha\beta}$ with $|\alpha| + |\beta| = 1$, $i, j, k = 1, \dots, m$, $p, r, s = 1, \dots, n$. So, since $\Delta$ satisfies the condition $\Delta(\Gamma, \Lambda, \Theta, 0) = 0$, we get

**Lemma 6.1.** *If $m \geq 2$ and $n \geq 1$, $\Delta_K$ is the linear combination of $S^k_{ij;(0)}$ with real coefficients.*

We prove the following lemma.

**Lemma 6.2.** *Let $m \geq 2$ and $n \geq 1$. The natural operator $\Delta$ is fully determined by the value*

$$\Delta^1 := \Delta(\Gamma^o, \Lambda^o, \Theta^o, (dx^1 \wedge dx^2) \otimes \tfrac{\partial}{\partial x^1})(0,0)\,. \tag{9}$$

**Proof.** We know that the collection of maps $\Delta_K$ for $K = 1, 2, \dots$ determines $\Delta$. Then, using Lemma 6.1, it remains to prove that $\Delta(\Gamma^o, \Lambda^o, \Theta^o, (dx^j \wedge dx^i) \otimes \frac{\partial}{\partial x^k})(0,0)$ is determined by $\Delta^1$ for $i, j, k = 1, \dots, m$. The case $m = 2$ is easy. Let $m \geq 3$. Using the invariance of $\Delta$ with respect to the $\mathcal{FM}_{m,n}$-map $\varphi := (x^1, x^2, x^3 + x^1, x^4, \dots, x^m, y^1, \dots, y^n)$, we see that $\tilde{\Delta} := \Delta(\Gamma^o, \Lambda^o, \Theta^o, (dx^1 \wedge dx^2) \otimes (\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}))(0,0)$ is determined by $\Delta^1$ (it is the image of $\Delta^1$ by $\varphi$), and then $\Delta(\Gamma^o, \Lambda^o, \Theta^o, dx^1 \wedge dx^2) \otimes \frac{\partial}{\partial x^3})(0,0)$ is determined by $\Delta^1$ (it is $\tilde{\Delta} - \Delta^1$). Now, using the invariance of $\Delta$ with respect to permutations of the coordinates $x^1, \dots, x^m$, we complete the proof. □

We else prove the following lemma.

**Lemma 6.3.** *Let $m \geq 3$ and $n \geq 1$. Let $\Delta^1$ be as in (9). There are $a_1, \dots, a_5 \in \mathbf{R}$ such that*

$$\begin{aligned}
\Delta^1 = a_1 &\sum_{i=1}^m d_{(0,0)}x^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\
&+ a_2 \sum_{i=1}^m d_{(0,0)}x^i \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\
&+ a_3 \sum_{p=1}^n d_{(0,0)}x^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \\
&+ a_4 \sum_{p=1}^n d_{(0,0)}y^p \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\
&+ a_5 \Big( d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} \\
&- d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} \Big)\,.
\end{aligned}$$

**Proof.** By the invariance of $\Delta$ with respect to $(t^1x^1, \dots, t^mx^m, \tau^1y^1, \dots, \tau^ny^n)$ for $t^1 > 0, \dots, t^m > 0$ and $\tau^1 > 0, \dots, \tau^n > 0$ we get easily $\Delta^1 = \sum_{p=1}^n b_p d_{(0,0)}y^p \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{p=1}^n c_p d_{(0,0)}x^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{i=1}^m d_i d_{(0,0)}x^i \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} + \sum_{i=1}^m e_i d_{(0,0)}x^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)}$. Then by the invariance of $\Delta$ with respect to respective permutation of coordinates, we deduce $b_1 = \dots = b_n$,

$c_1 = \cdots = c_n$, $d_3 = \cdots = d_m$, $e_3 = \cdots = e_m$. Then

$$\begin{aligned}\Delta^1 = a_1 &\sum_{i=1}^{m} d_{(0,0)}x^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\ &+ a_2 \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\ &+ a_3 \sum_{p=1}^{n} d_{(0,0)}x^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ a_4 \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ \lambda_1 d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} \\ &+ \lambda_2 d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} \\ &+ \lambda_3 d_{(0,0)}x^2 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^2}_{|(0,0)} .\end{aligned}$$

Then by the invariance of $\Delta$ with respect to $(x^1, x^2, x^3 + x^2, \dots, x^m, y^1, \dots, y^n)$ (here we apply $m \geq 3$), we get $\Delta^1 = \Delta^1 + \lambda^3 d_{(0,0)}x^2 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^3}_{|(0,0)}$. So, $\lambda_3 = 0$. Then by the invariance of $\Delta$ with respect to $(x^1 - x^2, x^2, \dots, x^m, y^1, \dots, y^n)$ we get $\Delta^1 = \Delta^1 + \lambda_1 d_{(0,0)}x^2 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} + \lambda_2 d_{(0,0)}x^2 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)}$. So, $\lambda_1 = -\lambda_2$. □

From Lemmas 6.2 and 6.3 it follows immediately the following proposition.

**Proposition 6.4.** *Let $m \geq 3$ and $n \geq 1$. The real vector space of all natural operators $\Delta$ in the sense of Definition 5.3 satisfying the condition $\Delta(-,-,-,0) = 0$ is of dimension $\leq 5$.*

## 7. Linear independence of some natural operators

Let $\tau_i$ $(i = 13, \dots, 17)$ be as in Examples 4.13–4.17. We define natural operators $\chi_j$ (for $j = 1, \dots, 5$) in the sense of Definition 5.3 by

$$\chi_j(\Gamma, \Lambda, \Theta, S) := \tau_{j+12}(\Gamma, \Lambda + S, \Theta) - \tau_{j+12}(\Gamma, \Lambda, \Theta) \tag{10}$$

for any 4-tuple $(\Gamma, \Lambda, \Theta, S)$ consisting of a general connection $\Gamma$ in a $\mathcal{FM}_{m,n}$-object $Y \to M$ and a torsion-free classical linear connection $\Lambda$ on $M$ and a vertical classical linear connection $\Theta$ on $Y \to M$ and a skew-symmetric tensor field $S$ of type $(1, 2)$ on $M$. Clearly, $\chi_j(-,-,-,0) = 0$.

**Proposition 7.1.** *Let $m \geq 3$ and $n \geq 1$. The collection of natural operators $\chi_1, \dots, \chi_5$ is linearly independent over $\mathbf{R}$.*

**Proof.** Denote $\Sigma := (\Gamma^o, \Lambda^o, \Theta^o, (dx^1 \wedge dx^2) \otimes \frac{\partial}{\partial x^1})$ . Then

$$\chi_j(\Sigma) = \tau_{12+j}(\Sigma^1) - \tau_{12+j}(\Sigma^o) ,$$

where $\Sigma^1 := (\Gamma^o, \Lambda, \Theta^o)$, $\Sigma^o := (\Gamma^o, \Lambda^o, \Theta^o)$ and $\Lambda := \Lambda^o + (dx^1 \wedge dx^2) \otimes \frac{\partial}{\partial x^1}$. Clearly,

$$(\Gamma^o)^p_i = 0\,, \quad (\Theta^o)^p_{sj} = 0\,, \quad (\Theta^o)^p_{qs} = 0\,, \quad \Lambda^1_{12} = -\Lambda^1_{21} = \frac{1}{2} \quad \text{and other} \quad \Lambda^k_{ij} = 0\,.$$

Then, by Lemma 2.2, $\Psi(\Sigma^1)$ has expression

$$d\eta^p = 0\,, \quad d\xi^1 = \frac{1}{2}(\xi^1 dx^2 - \xi^2 dx^1) \quad \text{and other} \quad d\xi^i = 0\,.$$

Then we have

$$\mathrm{Tor}(\Psi(\Sigma^1))(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} ,$$

where $\mathrm{Tor}(\Psi(\Sigma^1))$ is the torsion tensor of $\Psi(\Sigma^1)$. Similarly, $\mathrm{Tor}(\Psi(\Sigma^o)) = 0$. Then

$$\tilde\chi_1 = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^1}_{|(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^1}_{|(0,0)},$$

$$\tilde\chi_2 = -\sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)}, \ \tilde\chi_3 = -\sum_{p=1}^{n} d_{(0,0)}x^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)},$$

$$\tilde\chi_4 = -\sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)}, \ \tilde\chi_5 = -\sum_{i=1}^{m} d_{(0,0)}x^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

where $\tilde\chi_j := \chi_j(\Sigma)(0,0)$. Then $\chi_1, \dots, \chi_5$ are linearly independent because $\tilde\chi_1, \dots, \tilde\chi_5$ are (here we use $m \geq 3$ and $n \geq 1$). □

## 8. The main result

**Proposition 8.1.** *Let $m \geq 3$ and $n \geq 3$. Any natural operator $A$ in the sense of Definition 5.1 is of the form*

$$A(\Gamma, \Lambda, \Theta) = \Psi(\Gamma, \Lambda, \Theta) + \sum_{i=1}^{12} \lambda^i \tau_i(\Gamma, \Lambda - \frac{1}{2}\mathrm{Tor}(\Lambda), \Theta)$$
$$+ \sum_{j=1}^{5} \mu^j \chi_j\big(\Gamma, \Lambda - \frac{1}{2}\mathrm{Tor}(\Lambda), \Theta, \frac{1}{2}\mathrm{Tor}(\Lambda)\big)$$

*for (uniquely determined by $A$) real numbers $\lambda^i$, $\mu^j$, where $\tau_i$ are as in Examples 4.1–4.12, $\chi_j$ are defined by* (10), $\mathrm{Tor}(\Lambda)$ *is the torsion tensor of* $\Lambda$ *and* $\Psi(\Gamma, \Lambda, \Theta)$ *is as in Example 2.1. Consequently, the vector space of all $\mathcal{FM}_{m,n}$-natural operators in the sense of Definition 5.2 is of dimension* 17 *over* $\mathbf{R}$.

**Proof.** It follows immediately from Propositions 6.4 and 7.1 and Section 5 (in particular Proposition 5.5). □

**Proposition 8.2.** *Let $m \geq 3$ and $n \geq 3$. The collection of $\mathcal{FM}_{m,n}$-natural operators $\tau_1, \dots, \tau_{17}$ from Examples 4.1–4.17 is linearly independent over* $\mathbf{R}$.

**Proof.** By Section 7, we can write

$$\tilde{\chi}_j := \chi_j(\Sigma)(0,0) = \tau_{12+j}(\Sigma^1)(0,0) - \tau_{12+j}(\Sigma^o)(0,0) = \tau_{12+j}(\Sigma^1)(0,0)\,.$$

Then the collection $\tau_{13}(\Sigma^1)(0,0), \dots, \tau_{17}(\Sigma^1)(0,0)$ is linearly independent over $\mathbf{R}$, see Proof of Proposition 7.1. Further, $\tau_1(\Sigma^1)(0,0), \dots, \tau_{12}(\Sigma^1)(0,0)$ are 0. Moreover, the collection $\tau_1, \dots, \tau_{12}$ is linearly independent over $\mathbf{R}$ because of Proposition 5.5. So, the collection $\tau_1, \dots, \tau_{17}$ is linearly independent over $\mathbf{R}$. □

**Theorem 8.3** (The main result)**.** Let $m \geq 3$ and $n \geq 3$. Any $\mathcal{FM}_{m,n}$-natural operator $A$ in the sense of Definition 5.1 is of the form

$$A(\Gamma, \Lambda, \Theta) = \Psi(\Gamma, \Lambda, \Theta) + \sum_{i=1}^{17} \nu^i \tau_i(\Gamma, \Lambda, \Theta)$$

for (uniquely determined by $A$) real numbers $\nu^i$, where $\tau_1, \dots, \tau_{17}$ are as in Examples 4.1–4.17 and $\Psi$ is as in Example 2.1.

**Proof.** It follows immediately from Propositions 8.1 and 8.2. □

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