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A DETERMINANT FORMULA FROM RANDOM WALKS

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Abstract. One usually studies the random walk model of a cat moving from one room to another in an apartment. Imagine now that the cat also has the possibility to go from one apartment to another by crossing some corridors, or even from one building to another. That yields a new probabilistic model for which each corridor connects the entrance rooms of several apartments. This article computes the determinant of the stochastic matrix associated to such random walks. That new model naturally allows to compute the determinant of a large class of matrices. Two examples involving digraphs and hyperplane arrangements are provided.

1. Introduction

The article uses the combinatorial notation $[n] := \{1, 2, \dots, n\}$ for a positive integer n. Recall that a random walk is a stochastic model describing the probability of random steps on some mathematical space. To describe our model, we consider a connected digraph $\mathrm{G} := (\mathrm{V}, \mathrm{E})$, where V and E are respectively the sets of vertices and of edges. For every pair (A, B) of vertices, there is a vertex sequence $(A = A_1, A_2, \dots, A_k = B)$ from A to B such that $(A_i, A_{i+1}) \in \mathrm{E}$ for $i \in [k-1]$. Denote the set formed by the vertex sequences from A to B by $\mathscr{S}(A, B)$. Define by $\mathrm{l}(A, B) := \min\big\{k \in \mathbb{N} \;\big|\; (A = A_1, A_2, \dots, A_k = B) \in \mathscr{S}(A, B)\big\}$ the length between A and B. Besides, denote the set formed by the minimal sequences from A to B by

$$\mathscr{M}(A, B) := \big\{(A = A_1, A_2, \dots, A_k = B) \in \mathscr{S}(A, B) \;\big|\; k = \mathrm{l}(A, B)\big\}.$$

This article treats a specific model of a cat in motion by located in a certain room at each step. The animal goes from room A to room B with the probability $\mathtt{p}(A, B)$. Our probabilistic graph for that model is the connected digraph $\mathrm{G} := (\mathrm{V}, \mathrm{E}, \mathtt{p})$ formed by the room set V, the set $\mathrm{E} \subseteq \mathrm{V}^2$ of 2-adjacent rooms containing also $\big\{(A_i, A_i)\big\}_{A_i \in \mathrm{V}}$, and the probability $\mathtt{p} : \mathrm{V}^2 \to [0, 1]$ labeling each pair $(A_i, A_j) \in \mathrm{E}$ by $\mathtt{p}(A_i, A_j)$. Moreover, the probability $\mathtt{p}$ must specifically have the following properties: Let $A, B \in \mathrm{V}$ such that $A \neq B$. Then,

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- $\sum_{A' \in \mathrm{V}} \mathtt{p}(A, A') = 1$,
- if $(A_1, A_2, \dots, A_k), (A'_1, A'_2, \dots, A'_k) \in \mathscr{M}(A, B)$, then as multisets
$$\big\{\mathtt{p}(A_i, A_{i+1})\big\}_{i \in [k-1]} = \big\{\mathtt{p}(A'_i, A'_{i+1})\big\}_{i \in [k-1]},$$
- if $(A_1, A_2, \dots, A_k) \in \mathscr{M}(A, B)$, then $\mathrm{p}(A, B) = \prod_{i \in [k-1]} \mathrm{p}(A_i, A_{i+1})$.

Let us call a digraph having such specifications "A Probabilistic Graph of a Walking Cat".

Definition 1.1. Let $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$ be a probabilistic graph of a walking cat. We say that a nonempty set $\mathrm{U} \subseteq \mathrm{V}$ is connected by a corridor if V can be partitioned into $\#\mathrm{U}$ sets $\mathrm{V}_1, \dots, \mathrm{V}_{\#\mathrm{U}}$ such that, for $i, j \in [\#\mathrm{U}]$,

- $\mathrm{V}_i \cap \mathrm{U}$ contains exactly one element which we denote C_i,
- if $i \neq j$, then $(C_i, C_j) \in \mathrm{E}$,
- if $A, B \in \mathrm{V}_i$, $A \neq B$, and $(A = A_1, A_2, \dots, A_k = B) \in \mathscr{M}(A, B)$, then $A_1, \dots, A_k \in \mathrm{V}_i$,
- if $i \neq j$, $(A, B) \in \mathrm{V}_i \times \mathrm{V}_j$, $(A_1, A_2, \dots, A_k) \in \mathscr{M}(A, C_i)$, $(B_1, B_2, \dots, B_l) \in \mathscr{M}(C_j, B)$, then $(A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_l) \in \mathscr{M}(A, B)$.

Example 1. In the probabilistic graph of Figure 1, the set $\{2, 3, 4\}$ is connected by a corridor, and the room set $\{1, 2, 3, 4, 5, 6\}$ can be partitioned into the subsets $\{1, 2\}$, $\{3, 5\}$, $\{4, 6\}$.

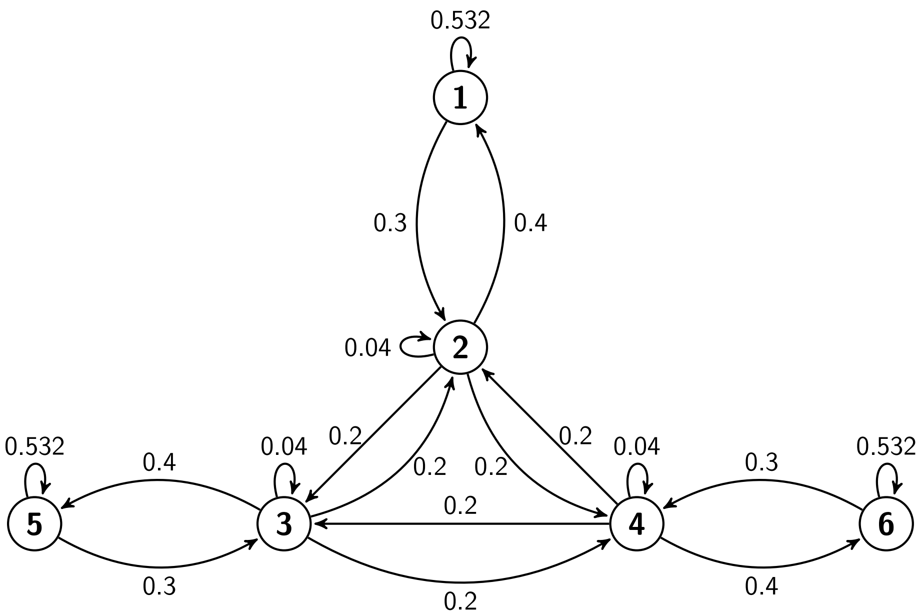

Fig. 1: A Probabilistic Graph of a Walking Cat

Before presenting the results, we need the following lemma that will be proven in Section 3.

Lemma 1.2. *Let* $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$ *be a probabilistic graph of a walking cat, and* $\mathrm{U}_1, \dots, \mathrm{U}_r$ *nonempty pairwise disjoint subsets of* V*. Suppose in addition that the sets* $\mathrm{U}_1, \dots, \mathrm{U}_r$ *are connected by a corridor. Then,* V *can be partitioned into* $s = \sum_{i \in [r]} \#\mathrm{U}_i - r + 1$ *sets* $\mathrm{V}_1, \dots, \mathrm{V}_s$ *such that, for* $l \in [s]$*,*

- $\exists i \in [r] : \mathrm{U}_i \cap \mathrm{V}_l \neq \emptyset$,
- $\forall i \in [r] : \#\mathrm{U}_i \cap \mathrm{V}_l \leq 1$,
- *if* $A, B \in \mathrm{V}_l$, $A \neq B$, *and* $(A = A_1, A_2, \dots, A_k = B) \in \mathscr{M}(A, B)$, *then* $A_1, \dots, A_k \in \mathrm{V}_l$.

The left stochastic matrix associated to a probabilistic graph of a walking cat $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$ is

$$\mathsf{S}_{\mathrm{G}} := \big(\mathtt{p}(B, A)\big)_{A, B \in \mathrm{V}} .$$

Define $\mathrm{E}_i := \big\{(A, B) \in E \bigm| A, B \in \mathrm{V}_i\big\}$ as well for each V_i mentioned in Lemma 1.2. It is clear that the induced subgraph $\mathrm{G}_i = (\mathrm{V}_i, \mathrm{E}_i)$ of G is connected. The matrix associated to that subgraph is $\mathsf{S}_{\mathrm{G}_i} := \big(\mathtt{p}(B, A)\big)_{A, B \in \mathrm{V}_i}$. We can now state the results.

Theorem 1.3. *Consider a probabilistic graph of a walking cat* $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$*. Let* $\mathrm{U}_1, \dots, \mathrm{U}_r$ *be nonempty pairwise disjoint subsets of* V *that are connected by a corridor, and partition* V *into* s *subsets* $\mathrm{V}_1, \dots, \mathrm{V}_s$ *like in Lemma 1.2. Assume additionally that, for every* $i \in [r]$*, there exists a real number* $c_i \in [0, 1]$ *such that, if* $A, B \in \mathrm{U}_i$ *and* $(A, B) \in E$*, then* $\mathtt{p}(A, B) = c_i$*. If* $\mathrm{G}_1 = (\mathrm{V}_1, \mathrm{E}_1)$, $\dots$, $\mathrm{G}_s = (\mathrm{V}_s, \mathrm{E}_s)$ *are the induced subgraphs, then*

$$\det \mathsf{S}_{\mathrm{G}} = \prod_{i \in [r]} \Big(1 + \sum_{\substack{\mathrm{K} \subseteq \mathrm{U}_i \\ \#\mathrm{K} \geq 2}} (-1)^{\#\mathrm{K} - 1} (\#\mathrm{K} - 1) \prod_{A \in \mathrm{K}} c_i \, \mathtt{p}(A, A)\Big) \prod_{k \in [s]} \det \mathsf{S}_{\mathrm{G}_k} .$$

Let $x_1, \dots, x_n$ be variables, and $\mathbb{M}_n$ the set formed by the monomials of $\mathbb{R}[x_1, \dots, x_n]$. We call the function $\mathtt{d} : \mathrm{V}^2 \to \mathbb{M}_n$ an exponential distance on a probabilistic graph of a walking cat $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$ if, for $A, B \in \mathrm{V}$ with $A \neq B$,

- $\mathtt{d}(A, A) = 1$,
- if $(A_1, A_2, \dots, A_k), (A'_1, A'_2, \dots, A'_k) \in \mathscr{M}(A, B)$, then as multisets
$$\big\{\mathtt{d}(A_i, A_{i+1})\big\}_{i \in [k-1]} = \big\{\mathtt{d}(A'_i, A'_{i+1})\big\}_{i \in [k-1]} ,$$
- if $(A_1, A_2, \dots, A_k) \in \mathscr{M}(A, B)$, then $\mathrm{d}(A, B) = \prod_{i \in [k-1]} \mathrm{d}(A_i, A_{i+1})$.

We say that the digraph $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathtt{d})$ is dual to the probability graph of a walking cat $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathtt{p})$ if

$$\forall A, B \in \mathrm{V} : \mathtt{p}(A, B) = \frac{\mathtt{d}(A, B)}{\sum_{C \in \mathrm{V}} \mathtt{d}(A, C)} .$$

Let us call such the digraph dual to a probability graph of a walking cat "An Exponential Distance Graph of a Walking Cat". The distance matrix associated to an exponential distance graph of a walking cat $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathtt{d})$ is

$$\mathsf{D}_{\bar{\mathrm{G}}} := \big(\mathtt{d}(B, A)\big)_{A,B \in \mathrm{V}}.$$

Besides, for each V_i defined in Lemma 1.2, the matrix associated to the induced subgraph $\bar{\mathrm{G}}_i = (\mathrm{V}_i, \mathrm{E}_i)$ of $\bar{\mathrm{G}}$ is $\mathsf{D}_{\bar{\mathrm{G}}_i} := \big(\mathtt{d}(B, A)\big)_{A,B \in \mathrm{V}_i}$.

Theorem 1.4. *Consider an exponential distance graph of a walking cat $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathtt{d})$. Let $\mathrm{U}_1, \dots, \mathrm{U}_r$ be nonempty pairwise disjoint subsets of V that are connected by a corridor, and partition V into s subsets $\mathrm{V}_1, \dots, \mathrm{V}_s$ like in Lemma 1.2. Besides, assume that, for every $i \in [r]$, there exists a real variable q_i such that, if $A, B \in \mathrm{U}_i$ and $(A, B) \in E$, then $\mathtt{d}(A, B) = q_i$. If $\bar{\mathrm{G}}_1 = (\mathrm{V}_1, \mathrm{E}_1)$, ..., $\bar{\mathrm{G}}_k = (\mathrm{V}_s, \mathrm{E}_s)$ are the induced subgraphs, then*

$$\det \mathsf{D}_{\bar{\mathrm{G}}} = \prod_{i \in [r]} \big(1 + (\#\mathrm{U}_i - 1)q_i\big)(1 - q_i)^{\#\mathrm{U}_i - 1} \prod_{k \in [s]} \det \mathsf{D}_{\bar{\mathrm{G}}_k}.$$

This article is structured as follows: We first compute a determinant constructed from a set of square matrices in Section 2. That determinant will be used to prove Theorem 1.3 and Theorem 1.4 in Section 3. We finish with the computation of the determinants of exponential distance matrices constructed from digraphs and from hyperplane arrangements in Section 4. The reader wishing to see a list of determinants inspired from algebraic and combinatorial problems may for example have a look at the articles of [3, 4].

2. A determinant based on a set of matrices

We compute a determinant defined from a set of square matrices. The author would like to thank Marcelo Aguiar for having led him to that computing.

Definition 2.1. Let $\mathsf{A}_1, \dots, \mathsf{A}_r$ be square matrices, and q a variable. Assume that, for each $k \in [r]$, A_k is indexed by $I_k = \{i_1^k, \dots, i_{n_k}^k\}$ and $\mathsf{A}_k = (a_{i,j})_{i,j \in I_k}$. Define the square matrix $M_q(\mathsf{A}_1, \dots, \mathsf{A}_r) = (m_{i,j})_{i,j \in I}$ indexed by $I = \bigsqcup_{k \in [r]} I_k$ as follows: if $i \in I_h$ and $j \in I_k$, then

$$m_{i,j} := \begin{cases} a_{i,j} & \text{if} \quad h = k, \\ q \cdot a_{i,i_1^h} \cdot a_{i_1^k,j} & \text{otherwise.} \end{cases}$$

Example 2. If $\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$M_q(\mathsf{A}, \mathsf{B}) = \begin{pmatrix} a_{11} & a_{12} & qa_{11}b_{11} & qa_{11}b_{12} & qa_{11}b_{13} \\ a_{21} & a_{22} & qa_{21}b_{11} & qa_{21}b_{12} & qa_{21}b_{13} \\ qa_{11}b_{11} & qa_{12}b_{11} & b_{11} & b_{12} & b_{13} \\ qa_{11}b_{21} & qa_{12}b_{21} & b_{21} & b_{22} & b_{23} \\ qa_{11}b_{31} & qa_{12}b_{31} & b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Denote by $\mathfrak{D}_n$ the set formed by the derangements of order n.

Lemma 2.2. *Take an integer $n \geq 2$, and n variables $a_1, \ldots, a_n$. Then,*

$$\det \begin{pmatrix} a_1 & 1 & \cdots & 1 \\ 1 & a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a_n \end{pmatrix} = \prod_{i\in[n]} a_i + \sum_{\substack{I\subseteq[n] \\ \#I\leq n-2}} (-1)^{n-\#I-1}(n-\#I-1)\prod_{i\in I} a_i\,.$$

Proof. We use the following notation from analytic combinatorics only in this proof: If $I \subseteq [n]$, then $\big[\prod_{i\in I} a_i\big]\Delta$ denotes the coefficient of the monomial $\big[\prod_{i\in I} a_i\big]$ in the polynomial Δ. Denoting by Δ the aimed determinant, it is clear that $\big[\prod_{i\in[n]} a_i\big]\Delta = 1$, and, for $I \subseteq [n]$ such that $\#I = n-1$, $\big[\prod_{i\in I} a_i\big]\Delta = 0$. Now if $\#I \leq n-2$, from Theorem 3.2 in the article of [6], we obtain

$$\big[\prod_{i\in I} a_i\big]\Delta = \prod_{\sigma\in\mathfrak{D}_{n-\#I}} \operatorname{sgn}\sigma = (-1)^{n-\#I-1}(n-\#I-1)\,. \quad \square$$

Denote by I_n the identity matrix of order n.

Theorem 2.3. *Let $\mathsf{A}_1, \ldots, \mathsf{A}_r$ be square matrices, and q a variable. Assume that, for each $k \in [r]$, A_k is indexed by $I_k = \{i_1^k, \ldots, i_{n_k}^k\}$ and $\mathsf{A}_k = (a_{i,j})_{i,j\in I_k}$. Then,*

$$\det M_q(\mathsf{A}_1, \ldots, \mathsf{A}_r) = \Big(1 + \sum_{\substack{K\subseteq[r] \\ \#K\geq 2}} (-1)^{\#K-1}(\#K-1)\prod_{k\in K} q a_{i_1^k, i_1^k}\Big)\prod_{k\in[r]} \det \mathsf{A}_k\,.$$

Proof. Remark first that $M_q(\mathsf{A}_1, \ldots, \mathsf{A}_r)$ is equal to the product of the square matrix $\overrightarrow{\bigoplus_{k\in[r]}} \mathsf{A}_k$ with the square matrix $\mathsf{F} = \begin{pmatrix} \mathsf{F}_{11} & \mathsf{F}_{12} & \cdots & \mathsf{F}_{1r} \\ \mathsf{F}_{21} & \mathsf{F}_{22} & \cdots & \mathsf{F}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{F}_{r1} & \mathsf{F}_{r2} & \cdots & \mathsf{F}_{rr} \end{pmatrix}$, where F_{hk} is the $n_k \times n_h$ matrix such that

$$\mathsf{F}_{hk} = \begin{cases} \mathsf{I}_{n_h} & \text{if} \quad h = k\,, \\ \begin{pmatrix} q a_{i_1^k, i_1^k} & q a_{i_1^k, i_2^k} & q a_{i_1^k, i_3^k} & \cdots & q a_{i_1^k, i_{n_k}^k} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} & \text{otherwise}\,. \end{cases}$$

In the case of Example 2 for instance, we have

$$M_q(\mathsf{A}, \mathsf{B}) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & q b_{11} & q b_{12} & q b_{13} \\ 0 & 1 & 0 & 0 & 0 \\ q a_{11} & q a_{12} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $I = \bigsqcup_{k \in [r]} I_k$ and $J = \bigsqcup_{k \in [r]} \{i_1^k\}$. Using the determinantal formula that one can find in the book of [2], we obtain

$$\det \mathsf{F} = \det \mathsf{F}[J] \ \det\big(\mathsf{F}[I \setminus J] - \mathsf{F}[I \setminus J, J]\mathsf{F}[J]^{-1}\mathsf{F}[J, I \setminus J]\big)$$

where $\mathsf{F}[J]$ is the $r \times r$ circulant matrix $\begin{pmatrix} 1 & qa_{i_1^2, i_1^2} & \cdots & qa_{i_1^r, i_1^r} \\ qa_{i_1^1, i_1^1} & 1 & \cdots & qa_{i_1^r, i_1^r} \\ \vdots & \ddots & \ddots & \vdots \\ qa_{i_1^1, i_1^1} & \cdots & qa_{i_1^{r-1}, i_1^{r-1}} & 1 \end{pmatrix}$,
$\mathsf{F}[I \setminus J] = \mathsf{I}_{\#I \setminus J}$, and $\mathsf{F}[I \setminus J, J]$ is the $\#I \setminus J \times r$ null matrix. Using Lemma 2.2, we obtain

$$\begin{aligned} \det \mathsf{F}[J] &= \prod_{k \in [r]} qa_{i_1^k, i_1^k} \times \begin{vmatrix} \frac{1}{qa_{i_1^1, i_1^1}} & 1 & \cdots & 1 \\ 1 & \frac{1}{qa_{i_1^2, i_1^2}} & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & \frac{1}{qa_{i_1^r, i_1^r}} \end{vmatrix} \\ &= 1 + \sum_{\substack{K \subseteq [r] \\ \#K \geq 2}} (-1)^{\#K-1}(\#K - 1) \prod_{k \in K} qa_{i_1^k, i_1^k}. \end{aligned}$$

Finally, with $\det \overrightarrow{\bigoplus_{k \in [r]}} \mathsf{A}_k = \prod_{k \in [r]} \det \mathsf{A}_k$, we get the result. $\square$

3. Proof of Lemma 1.2, Theorem 1.3, and Theorem 1.4

We begin by proving Lemma 1.2, then Theorem 1.3, and finally Theorem 1.4.

Proof. Consider first U_1 partitioning V into $\mathrm{V}_1^{(1)}, \dots, \mathrm{V}_{\#\mathrm{U}_1}^{(1)}$. For $i \in [\#\mathrm{U}_1]$, set $\mathrm{U}_1 \cap \mathrm{V}_i^{(1)} = \{C_i^{(1)}\}$. If $i, j \in [\#\mathrm{U}_1]$ with $i \neq j$, as $\big\{(A, B) \mid A \in \mathrm{V}_i^{(1)}, B \in \mathrm{V}_j^{(1)}\big\} = \big\{(C_i^{(1)}, C_j^{(1)})\big\}$, U_2 is then included in some $\mathrm{V}_i^{(1)}$ that we assume to be $\mathrm{V}_{\#\mathrm{U}_1}^{(1)}$. From its definition, U_2 also partitions $\mathrm{V}_{\#\mathrm{U}_1}^{(1)}$ into $\mathrm{V}_1^{(2)}, \dots, \mathrm{V}_{\#\mathrm{U}_2}^{(2)}$, and the partition $\mathrm{V}_1^{(1)}, \dots, \mathrm{V}_{\#\mathrm{U}_1 - 1}^{(1)}, \mathrm{V}_1^{(2)}, \dots, \mathrm{V}_{\#\mathrm{U}_2}^{(2)}$ has the property of Lemma 1.2 for U_1 and U_2. By induction, we obtain the partition of $\sum_{i \in [r-1]} \#\mathrm{U}_i - r + 2$ sets $\mathrm{V}_1^{(1)}, \dots, \mathrm{V}_{\#\mathrm{U}_1 - 1}^{(1)}, \mathrm{V}_1^{(2)}, \dots, \mathrm{V}_{\#\mathrm{U}_2 - 1}^{(2)}, \dots, \mathrm{V}_1^{(n-1)}, \dots, \mathrm{V}_{\#\mathrm{U}_{n-1}}^{(n-1)}$ having the property of Lemma 1.2 after the $(n-1)^{\text{th}}$ step. Taking any two different sets $\mathrm{V}_i^{(l)}, \mathrm{V}_j^{(k)}$ of those latter, either $\big\{(A, B) \mid A \in \mathrm{V}_i^{(l)}, B \in \mathrm{V}_j^{(k)}\big\}$ is equal to some $\big\{(C_i^{(l)}, C_m^{(l)})\big\}$ or is empty. Hence, U_n is included in exactly one of these $\sum_{i \in [r-1]} \#\mathrm{U}_i - r + 2$ sets that we assume to be $\mathrm{V}_{\#\mathrm{U}_{n-1}}^{(n-1)}$. After its partitioning by U_n, we finally obtain the desired

$\sum_{i\in[r]}\#\mathrm{U}_i - r + 1$ sets $\mathrm{V}_1^{(1)},\ldots,\mathrm{V}_{\#\mathrm{U}_1-1}^{(1)},\mathrm{V}_1^{(2)},\ldots,\mathrm{V}_{\#\mathrm{U}_2-1}^{(2)},\ldots,\mathrm{V}_1^{(n-1)},\ldots,\mathrm{V}_{\#\mathrm{U}_{n-1}-1}^{(n-1)}$, $\mathrm{V}_1^{(n)},\ldots,\mathrm{V}_{\#\mathrm{U}_n}^{(n)}$. □

Proof. Considering the sets $\mathrm{V}_1^{(1)},\ldots,\mathrm{V}_{\#\mathrm{U}_1}^{(1)},\ldots,\mathrm{V}_1^{(n)},\ldots,\mathrm{V}_{\#\mathrm{U}_n}^{(n)}$ in the proof of Lemma 1.2, let $\mathsf{S}_{\mathrm{G}_i^{(k)}} := \big(\mathtt{p}(B,A)\big)_{A,B\in\mathrm{V}_i^{(k)}}$. Using Theorem 2.3, we successively get

$$\begin{aligned}
\det\mathsf{S}_{\mathrm{G}} &= \det M_{c_1}(\mathrm{V}_1^{(1)},\ldots,\mathrm{V}_{\#\mathrm{U}_1}^{(1)})\\
&= \Big(1+\sum_{\substack{\mathrm{K}\subseteq\mathrm{U}_1\\ \#\mathrm{K}\geq 2}}(-1)^{\#\mathrm{K}-1}(\#\mathrm{K}-1)\prod_{A\in\mathrm{K}}c_1\,\mathtt{p}(A,A)\Big)\prod_{k\in[\#\mathrm{U}_1-1]}\det\mathsf{S}_{\mathrm{G}_k^{(1)}}\\
&\quad\times\det M_{c_2}(\mathrm{V}_1^{(2)},\ldots,\mathrm{V}_{\#\mathrm{U}_2}^{(2)})\\
&= \prod_{i\in[r]}\Big(1+\sum_{\substack{\mathrm{K}\subseteq\mathrm{U}_i\\ \#\mathrm{K}\geq 2}}(-1)^{\#\mathrm{K}-1}(\#\mathrm{K}-1)\prod_{A\in\mathrm{K}}c_i\,\mathtt{p}(A,A)\Big)\\
&\quad\times\prod_{l\in[n]}\prod_{k\in[\#\mathrm{U}_l-1]}\det\mathsf{S}_{\mathrm{G}_k^{(l)}}\times\det\mathsf{S}_{\mathrm{G}_{\#\mathrm{U}_n}^{(n)}}.
\end{aligned}$$

□

Proof. With an argument similar to the proof of Theorem 1.3, we obtain

$$\begin{aligned}
\det\mathsf{D}_{\bar{\mathrm{G}}} &= \prod_{i\in[r]}\Big(1+\sum_{\substack{\mathrm{K}\subseteq\mathrm{U}_i\\ \#\mathrm{K}\geq 2}}(-1)^{\#\mathrm{K}-1}(\#\mathrm{K}-1)\prod_{A\in\mathrm{K}}q_i\,\mathtt{d}(A,A)\Big)\\
&\quad\times\prod_{l\in[n]}\prod_{k\in[\#\mathrm{U}_l-1]}\det\mathsf{D}_{\bar{\mathrm{G}}_k^{(l)}}\times\det\mathsf{D}_{\bar{\mathrm{G}}_{\#\mathrm{U}_n}^{(n)}}.
\end{aligned}$$

Then, $\mathtt{d}(A,A)=1$ on one side, and on the other side

$$\begin{aligned}
\big(1+(\#\mathrm{U}_i-1)q_i\big)(1-q_i)^{\#\mathrm{U}_i-1} &= \big(1+(\#\mathrm{U}_i-1)q_i\big)\sum_{k=0}^{\#\mathrm{U}_i-1}(-1)^k\binom{\#\mathrm{U}_i-1}{k}q_i^k\\
&= 1+(-1)^{\#\mathrm{U}_i-1}(\#\mathrm{U}_i-1)q_i^{\#\mathrm{U}_i}\\
&\quad+\sum_{k=0}^{\#\mathrm{U}_i-2}\Big((-1)^k(\#\mathrm{U}_i-1)\binom{\#\mathrm{U}_i-1}{k}+(-1)^{k+1}\binom{\#\mathrm{U}_i-1}{k+1}\Big)q_i^{k+1}\\
&= 1+(-1)^{\#\mathrm{U}_i-1}(r-1)q_i^{\#\mathrm{U}_i}+\sum_{k=0}^{\#\mathrm{U}_i-2}(-1)^k k\binom{\#\mathrm{U}_i}{k+1}q_i^{k+1}\\
&= 1+\sum_{k=2}^{\#\mathrm{U}_i}(-1)^{k-1}(k-1)\binom{\#\mathrm{U}_i}{k}q_i^k.
\end{aligned}$$

□

4. Examples of random walks

We compute the determinant of matrices associated to two exponential distance graphs.
Indirectly acyclic digraph. Transform a digraph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ to an undirected graph $\mathrm{u(G)} = (\mathrm{V}, \mathrm{u(E)})$ by defining $\mathrm{u(E)} := \big\{\{A, B\} \in \binom{\mathrm{V}}{2} \mid (A, B) \in \mathrm{E}\big\}$. We say that the digraph G is indirectly acyclic if the undirected graph u(G) is acyclic.

Lemma 4.1. *Let $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathsf{d})$ be an indirectly acyclic exponential distance graph of a walking cat. Then,*

$$\det \mathsf{D}_{\bar{\mathrm{G}}} = \prod_{\{A,B\} \in \mathrm{u(E)}} \big(1 - \mathsf{d}(A, B)\, \mathsf{d}(B, A)\big).$$

Proof. We proceed by induction on the number of rooms. Assume $\mathrm{V} = \{A_1, \dots, A_n\}$, and Lemma 4.1 for $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathsf{d})$. Then, consider the extension $\bar{\mathrm{G}}' = (\mathrm{V}', \mathrm{E}', \mathsf{d}')$ of $\bar{\mathrm{G}}$ such that $\mathrm{V}' = \mathrm{V} \sqcup \{B\}$, $\mathrm{E}' = \mathrm{E} \sqcup \big\{(A_n, B), (B, A_n)$, and $\mathsf{d}'(A_i, A_j) = \mathsf{d}(A_i, A_j)$ for $i, j \in [n]$. Hence,

$$\det \mathsf{D}_{\bar{\mathrm{G}}'} = \begin{vmatrix} 1 & \mathsf{d}(A_2, A_1) & \cdots & \mathsf{d}(A_n, A_1) & \mathsf{d}'(B, A_n)\, \mathsf{d}(A_n, A_1) \\ \mathsf{d}(A_1, A_2) & 1 & \cdots & \mathsf{d}(A_n, A_2) & \mathsf{d}'(B, A_n)\, \mathsf{d}(A_n, A_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathsf{d}(A_1, A_n) & \mathsf{d}(A_2, A_n) & \cdots & 1 & \mathsf{d}'(B, A_n) \\ \mathsf{d}'(A_1, B) & \mathsf{d}'(A_2, B) & \cdots & \mathsf{d}'(A_n, B) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \mathsf{d}(A_2, A_1) & \cdots & \mathsf{d}(A_n, A_1) & 0 \\ \mathsf{d}(A_1, A_2) & 1 & \cdots & \mathsf{d}(A_n, A_2) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathsf{d}(A_1, A_n) & \mathsf{d}(A_2, A_n) & \cdots & 1 & 0 \\ \mathsf{d}'(A_1, B) & \mathsf{d}'(A_2, B) & \cdots & \mathsf{d}'(A_n, B) & 1 - \mathsf{d}'(B, A_n)\, \mathsf{d}'(A_n, B) \end{vmatrix}$$

$$= \det \mathsf{D}_{\bar{\mathrm{G}}} \times \big(1 - \mathsf{d}'(B, A_n)\, \mathsf{d}'(A_n, B)\big).$$

□

Example 3. The determinant of the matrix associated to the exponential distance graph of a walking cat represented in Figure 2 is

$$\begin{vmatrix} 1 & a^- & a^-b^+ & a^-c^+ & a^-d^+ & a^-d^+e^+ \\ a^+ & 1 & b^+ & c^+ & d^+ & d^+e^+ \\ a^+b^- & b^- & 1 & b^-c^+ & b^-d^+ & b^-d^+e^+ \\ a^+c^- & c^- & b^+c^- & 1 & c^-d^+ & c^-d^+e^+ \\ a^+d^- & d^- & b^+d^- & c^+d^- & 1 & e^+ \\ a^+d^-e^- & d^-e^- & b^+d^-e^- & c^+d^-e^- & e^- & 1 \end{vmatrix}$$

$$= \begin{matrix} (1 - a^+a^-)\,(1 - b^+b^-)\,(1 - c^+c^-) \\ (1 - d^+d^-)\,(1 - e^+e^-). \end{matrix}$$

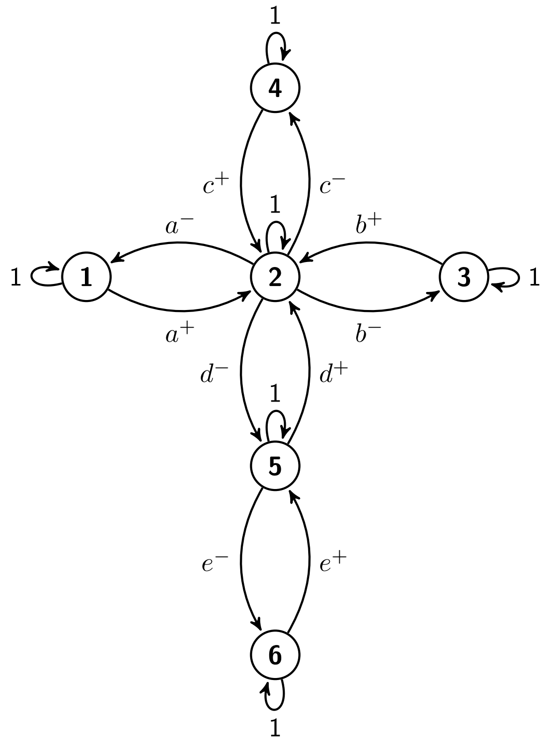

FIG. 2: An Exponential Distance Graph

Proposition 4.2. *Consider an exponential distance graph of a walking cat* $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathtt{d})$. *Let* $\mathrm{U}_1, \ldots, \mathrm{U}_r$ *be nonempty pairwise disjoint subsets of* V *that are connected by a corridor, and partition* V *into* s *subsets* $\mathrm{V}_1, \ldots, \mathrm{V}_s$ *like in Lemma 1.2. Besides, assume that*

- *for every* $i \in [r]$, *there is a real variable* q_i *such that, if* $A, B \in \mathrm{U}_i$, *then* $\mathtt{d}(A, B) = q_i$,
- *for every* $k \in [s]$, *the induced subgraph* $\bar{\mathrm{G}}_k = (\mathrm{V}_k, \mathrm{E}_k, \mathtt{d})$ *is an indirectly acyclic digraph.*

We obtain,

$$\det \mathsf{D}_{\bar{\mathrm{G}}} = \prod_{i \in [r]} \big(1 + (\#\mathrm{U}_i - 1) q_i\big)(1 - q_i)^{\#\mathrm{U}_i - 1} \prod_{k \in [s]} \prod_{\{A,B\} \in \mathrm{u}(\mathrm{E}_k)} \big(1 - \mathtt{d}(A, B)\, \mathtt{d}(B, A)\big).$$

Proof. Use Theorem 1.4 and Lemma 4.1. $\square$

Hyperplane arrangement. To every hyperplane H in $\mathbb{R}^n$ can be associated two connected open half-spaces H^+ and H^- such that $H^+ \sqcup H^0 \sqcup H^- = \mathbb{R}^n$ and $\overline{H^+} \cap \overline{H^-} = H^0$, letting $H^0 := H$. A face of a hyperplane arrangement $\mathcal{A}$ is a nonempty subset of $\mathbb{R}^n$ having the form $F := \bigcap_{H \in \mathcal{A}} H^{\epsilon_H(F)}$ with $\epsilon_H(F) \in \{+, 0, -\}$.

Denote the set formed by the faces of $\mathcal{A}$ by $F_{\mathcal{A}}$. A chamber of $\mathcal{A}$ is a face $F \in F_{\mathcal{A}}$ such that $\epsilon_H(F) \neq 0$ for every $H \in \mathcal{A}$. Denote the set formed by the chambers of $\mathcal{A}$ by $C_{\mathcal{A}}$. For $A, B \in C_{\mathcal{A}}$, the set of half-spaces containing A but not B is $\mathscr{H}(A,B) := \big\{H^{\epsilon_H(A)} \;\big|\; H \in \mathcal{A},\, \epsilon_H(A) = -\epsilon_H(B)\big\}$. Assign a variable h_H^{ε} to every half-space H^{ε}, and define the polynomial ring $R_{\mathcal{A}} := \mathbb{R}\big[h_H^{\varepsilon} \;\big|\; \varepsilon \in \{+,-\},\, H \in \mathcal{A}\big]$. The exponential distance $\mathrm{v} : C_{\mathcal{A}} \times C_{\mathcal{A}} \to R_{\mathcal{A}}$, defined by [1] in their monograph, is

$$\mathrm{v}(A,A) = 1 \quad \text{and} \quad \mathrm{v}(A,B) = \prod_{H^{\varepsilon} \in \mathscr{H}(A,B)} h_H^{\varepsilon} \;\text{ if }\; A \neq B\,.$$

The centralization to a face $F \in F_{\mathcal{A}} \setminus C_{\mathcal{A}}$ is defined by $\mathcal{A}_F := \{H \in \mathcal{A} \mid F \subseteq H\}$, its weight $\mathrm{b}_F := \prod_{H \in \mathcal{A}_F} h_H^{+}\, h_H^{-}$, and its multiplicity $\beta_F := \dfrac{\#\{C \in C_{\mathcal{A}} \mid \overline{C} \cap H = F\}}{2}$ which is independent of the chosen $H \in \mathcal{A}_F$ as can be seen in Theorem 5.7 of the article of [5].

Proposition 4.3. *Consider an exponential distance graph of a walking cat* $\bar{\mathrm{G}} = (\mathrm{V}, \mathrm{E}, \mathtt{d})$. *Let* $\mathrm{U}_1, \dots, \mathrm{U}_r$ *be nonempty pairwise disjoint subsets of* V *that are connected by a corridor, and partition* V *into* s *subsets* $\mathrm{V}_1, \dots, \mathrm{V}_s$ *like in Lemma 1.2. Besides, assume that*

- *for every* $i \in [r]$, *there is a real variable* q_i *such that, if* $A, B \in \mathrm{U}_i$, *we have* $\mathtt{d}(A,B) = q_i$,
- *for every* $k \in [s]$, *there exists a hyperplane arrangement* $\mathcal{A}_k$ *such that, if* $\bar{\mathrm{G}}_k = (\mathrm{V}_k, \mathrm{E}_k, \mathtt{d})$ *is the subgraph induced by* V_k, *then* $\mathrm{V}_k = C_{\mathcal{A}_k}$, $\mathrm{E}_k = \big\{(A,B) \in C_{\mathcal{A}_k}^2 \;\big|\; \#\mathscr{H}(A,B) = 1\big\}$, *and* $\mathtt{d}(A,B) = \mathrm{v}(A,B)$ *for* $A, B \in \mathrm{V}_k$.

We obtain

$$\det \mathsf{D}_{\bar{\mathrm{G}}} = \prod_{i \in [r]} \big(1 + (\#\mathrm{U}_i - 1)q_i\big)(1 - q_i)^{\#\mathrm{U}_i - 1} \prod_{k \in [s]} \prod_{F \in F_{\mathcal{A}_k} \setminus C_{\mathcal{A}_k}} (1 - \mathrm{b}_F)^{\beta_F}\,.$$

Proof. Use Theorem 1.4 and Corollary 1.4 in the article of [5]. $\square$

Example 4. Consider the exponential distance graph with induced subgraphs and entrance rooms respectively represented by the hyperplane arrangements $\mathcal{A}_1, \mathcal{A}_2$ and the set $\{C_1, C_2\}$ in Figure 3. In order to have a determinant computable with a computer algebra system, we set $h_{H_i}^{+} = h_{H_i}^{-} = h_1$ for each $i \in [3]$, $h_H^{+} = h_H^{-} = h_2$, and $\mathtt{d}(C_1, C_2) = \mathtt{d}(C_2, C_1) = q$. The determinant of the matrix associated to that exponential distance graph is

$$\begin{vmatrix}
1 & h_1^2 & h_1 & h_1^2 & h_1 & h_1 & h_1^2 & q & qh_2 \\
h_1^2 & 1 & h_1 & h_1^2 & h_1 & h_1^3 & h_1^2 & h_1^2q & h_1^2qh_2 \\
h_1 & h_1 & 1 & h_1 & h_1^2 & h_1^2 & h_1^3 & h_1q & h_1qh_2 \\
h_1^2 & h_1^2 & h_1 & 1 & h_1^3 & h_1 & h_1^2 & h_1^2q & h_1^2qh_2 \\
h_1 & h_1 & h_1^2 & h_1^3 & 1 & h_1^2 & h_1 & h_1q & h_1qh_2 \\
h_1 & h_1^3 & h_1^2 & h_1 & h_1^2 & 1 & h_1 & h_1q & h_1qh_2 \\
h_1^2 & h_1^2 & h_1^3 & h_1^2 & h_1 & h_1 & 1 & h_1^2q & h_1^2qh_2 \\
q & h_1^2q & h_1q & h_1^2q & h_1q & h_1q & h_1^2q & 1 & h_2 \\
qh_2 & h_1^2qh_2 & h_1qh_2 & h_1^2qh_2 & h_1qh_2 & h_1qh_2 & h_1^2qh_2 & h_2 & 1
\end{vmatrix}$$

$$= (1-q^2)(1-h_1^2)^9(1-h_2^2)\,.$$

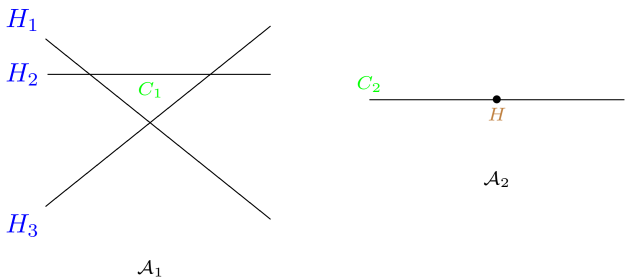

FIG. 3: Hyperplane arrangements and Entrance Rooms

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