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# WEBSTER PSEUDO-TORSION FORMULAS OF CR MANIFOLDS

Ho Chor Yin

Abstract. In this article, we obtain a formula for Webster pseudo-torsion for the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$.

## 1. Introduction

The complete local invariants in the pseudoconformal geometry of a nondegenerate $CR$ manifold $M$ are defined on an $SU(p+1,q+1)$-bundle $Y$ over $M$, which generalizes the bundle of $Q$-frame as a real hyperquadric [1]. To reduce the structure group, Webster singles out a real nowhere vanishing one form $\theta$ on $M$ which annihilates the $CR$ structure of $M$. A $CR$ manifold $M$ with such a choice $\theta$ is called a pseudohermitian manifold [6]. The contact form $\theta$ is called a pseudohermitian structure. The structure group of the Chern bundle $Y$ is reduced to $U(p,q)$. In [6], Webster showed there is a natural connection in the bundle $T^{1,0}M$ adapted to $\theta$. This connection can be extended to a connection to $\mathbb{C}TM$. To solve the equivalence problem of pseudohermitian manifold, Webster derived the structure equations for $M$, from which the Webster Ricci curvature and Webster torsion tensor are defined. In [3], the author derived a formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$. In this article, we derive a formula for Webster pseudo-torsion for the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$ [3]. The main idea of the alternative proof is to describe the $CR$ structure using all Euclidean coordinates $z^1, z^2, \dots, z^{n+1}$ (see (39)). This new description of $CR$ structure using all Euclidean coordinates is originated in [4]. In other words, we dispense with distinguishing one coordinate, say $z^{n+1}$, such that $\frac{\partial r}{\partial z^{n+1}} \neq 0$, as is required in Chern-Moser and subsequent works. The organization of this article is as follows. In Section 2, we review pseudohermitian geometry following Webster and Tanaka. In Section 3, we derive a key identity for Webster pseudo-torsion computation in subsequent

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sections. In Section 4, we present the alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$. In Section 5, we obtain an explicit formula for Webster pseudo-torsion for the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$. To the best knowledge of the author, this formula obtained in Section 5 is a new result.

## 2. Pseudohermitian structures

In this section, we collect the basic facts on pseudohermitian geometry. Let $M$ be a $CR$ manifold with structure bundle $T^{1,0}M$ satisfying $T^{1,0}M \cap \overline{T^{1,0}} = \{0\}$ and $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Let $T^{0,1}M := \overline{T^{1,0}}$. Set $HM = \mathrm{Re}\,(T^{1,0}M \oplus T^{0,1}M)$. $HM$ is a $2n$ dimensional subbundle of $TM$ which carries a complex structure $J\colon HM \to HM$ given by $J(X + \overline{X}) = i(X - \overline{X})$ for $X \in T^{1,0}M$. Let $E \subset TM^*$ denote the real line subbundle which annihilates $HM$. Assuming $M$ is orientable, $E$ has a global nowhere vanishing section $\theta$. A choice of such a 1-form $\theta$ is called a *pseudohermitian structure* on $M$. The *Levi form* of $\theta$ is the Hermitian form $L_\theta$ on $TM^{1,0}$ defined by

$$L_\theta(V, \overline{W}) = L_\theta(\overline{W}, V) = -2\,\mathrm{i} d\,\theta(V \wedge \overline{W})\,.$$

For a nondegenerate (resp. strongly pseudoconvex) $CR$ manifold, $L_\theta$ is a nondegenerate (resp. positive definite) Hermitian form for any choice of $\theta$. The choice of $\theta$ determines a unique real vector field $\xi$ transverse to $HM$ such that $\theta(\xi) = 1$, $\xi \rfloor d\theta = 0$. An *admissible coframe* on an open subset of $M$ is a set of complex $(1,0)$-forms $\{\theta^1, \ldots, \theta^n\}$ form basis for $TM*^{1,0}$ and satisfies $\theta^\alpha(\xi) = 0$. Then we have $d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$ for some hermitian matrix of functions $h_{\alpha\bar{\beta}}$. In [6], Webster showed there is a natural connection in the bundle $T^{1,0}M$ adapted to $\theta$. This connection can be extended to a connection to $\mathbb{C}TM$. Webster showed that there are uniquely determined 1-forms $\omega_\alpha^\beta$, $\tau^\beta$ on $M$ satisfying

$$d\theta = i\theta^\gamma \wedge \theta^{\overline{\gamma}}\,, \tag{1}$$

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta^{\ \alpha} + \theta \wedge \tau^\alpha\,, \tag{2}$$

$$\omega_{\alpha.}^{\ \beta} + \omega_{\overline{\beta}.}^{\ \overline{\alpha}} = 0\,, \quad \text{where } \omega_{\overline{\beta}.}^{\ \overline{\alpha}} = \overline{\omega_{\alpha.}^{\ \beta}}\,, \tag{3}$$

$$\tau^{\overline{\alpha}} = A_{\alpha\gamma}\theta^\gamma, \quad \text{where } \tau^{\overline{\alpha}} = \overline{\tau^\alpha}, \tag{4}$$

with

$$A_{\alpha\gamma} = A_{\gamma\alpha}\,, \tag{5}$$

and

$$h_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}\,. \tag{6}$$

This connection is called Webster connection. The curvature of the Webster connection, expressed in terms of the coframe is,

$$\begin{aligned} \Omega_{\beta.}^{\ \alpha} : &= d\omega_{\beta.}^{\ \alpha} - \omega_{\beta.}^{\ \gamma} \wedge \omega_{\gamma.}^{\ \alpha} - i\theta^{\bar{\beta}} \wedge \tau^\alpha + i\tau^{\bar{\beta}} \wedge \theta^\alpha, \\ &= R_{\beta\overline{\alpha}\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} + W_{\beta\overline{\alpha}\rho}\theta^\rho \wedge \theta - W_{\overline{\alpha}\beta\overline{\sigma}}\theta^{\overline{\sigma}} \wedge \theta \end{aligned} \tag{7}$$

where

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = \overline{R}_{\alpha\bar{\beta}\sigma\overline{\rho}} = R_{\overline{\alpha}\beta\overline{\sigma}\rho}\,, \tag{8}$$

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = R_{\rho\overline{\alpha}\beta\overline{\sigma}}\,, \tag{9}$$

$$W_{\overline{\alpha}\rho\overline{\sigma}} = W_{\overline{\sigma}\rho\overline{\alpha}}\,, \tag{10}$$

since by (6), $\Omega_{\beta.}^{\ \alpha} = \Omega_{\beta\overline{\alpha}}$. By (4), (7), we have

$$\begin{aligned} d\omega_{\beta.}^{\ \alpha} - \omega_{\beta.}^{\ \gamma} \wedge \omega_{\gamma.}^{\ \alpha} = -iA_{\beta\gamma}\theta^{\gamma} \wedge \theta^{\alpha} + R_{\beta\overline{\alpha}\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} + i\overline{A_{\alpha\gamma}}\theta^{\bar{\beta}} \wedge \theta^{\overline{\gamma}} \\ + W_{\beta\overline{\alpha}\rho}\theta^{\rho} \wedge \theta - W_{\overline{\alpha}\beta\overline{\sigma}}\theta^{\overline{\sigma}} \wedge \theta\,. \end{aligned} \tag{11}$$

We also put

$$\begin{aligned} \Omega^{\alpha} :&= d\tau^{\alpha} - \tau^{\beta} \wedge \omega_{\beta}^{\ \alpha}\,, \\ &= W_{\overline{\alpha}\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} - A_{\overline{\alpha\gamma}}\tau^{\overline{\gamma}} \wedge \theta + B_{\overline{\alpha\sigma}}\theta^{\overline{\sigma}} \wedge \theta\,, \end{aligned} \tag{12}$$

where

$$B_{\overline{\alpha\sigma}} = B_{\overline{\sigma\alpha}}\,. \tag{13}$$

Let $(\xi, X_{\alpha}, X_{\overline{\alpha}})$ be the dual frame to $(\theta, \theta^{\alpha}, \theta^{\overline{\alpha}})$. Define an operator $D$ locally by

$$DX_{\alpha} = \omega_{\alpha}^{\ \beta}X_{\beta}\,, \quad D\colon \Gamma\big(H(M)\big) \to \Gamma\big((T^*(M) \otimes H(M)\big)\,. \tag{14}$$

$D$ defines a connection on $H(M)$, see [6, p. 32]. We can define an hermitian metric $(\ \ ,\overline{\ \ })$ in the fibres of $H(M)$ by

$$(X_{\alpha}, \overline{X}_{\beta}) = \delta_{\alpha}^{\ \beta}\,. \tag{15}$$

Next, we turn to a formulation of the Webster connection by N. Tanaka [5]. We have $T^{1,0}M = \{X - iJX \mid X \in HM\}$ and using the decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}\xi$, we extend $J$ to $\mathbb{C}TM$ with $J\xi = 0$. Then we have

$$J^2X = -X + \theta(X)\xi\,, \quad X \in TM_x\,. \tag{16}$$

For, let $\mathrm{pr}\colon \mathbb{C}TM \to \mathbb{C}HM$ be the natural projection. Any $Y \in \mathbb{C}TM$ can be written as $Y = \mathrm{pr}(Y) + \theta(Y)\xi$ Then $J^2Y = -\mathrm{pr}(Y) = -Y + \theta(Y)\xi$. We put

$$\Omega = -d\theta\,. \tag{17}$$

We define a tensor field on $M$ by

$$g(X,Y) = \Omega(JX,Y)\,. \tag{18}$$

Then $g(X,Y) = g(Y,X)$, $g(JX,JY) = g(X,Y)$ and $g$ is positive definite on $HM$. Recall $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

**Theorem 2.1** (N. Tanaka [5, p. 29])**.** *There exists a unique affine connection*

$$\nabla\colon \Gamma(TM) \to \Gamma(TM \otimes TM^*)$$

*on $M$ such that*

(1) *The contact structure $HM$ is parallel, i.e.,*

$$\nabla_X \Gamma(HM) \subset \Gamma(HM) \quad \textit{for any} \quad X \in \Gamma(TM)\,. \tag{19}$$

(2) *The tensor field* $\xi, J, \Omega$ *are all parallel, i.e.,* $\nabla\xi = \nabla J = \nabla\Omega = 0$. *(It follows that* $\nabla\theta = \nabla g = 0$.*)*

(3) *The torsion* $T$ *of* $\nabla$ *satisfies:*

$$\begin{aligned} T(X,Y) &= -\Omega(X,Y)\xi\,,\\ T(\xi,JY) &= -JT(\xi,Y)\,, \quad X,\,Y \in HM_x\,. \end{aligned}$$

Let $X,\,Y \in \Gamma(\mathbb{C}HM)$. Denote by $[X,Y]_{HM}$ the $\mathbb{C}HM$-component of $[X,Y]$ in the decomposition:

$$\mathbb{C}TM = \mathbb{C}HM \oplus \mathbb{C} \otimes (TM/HM)\,.$$

Also denote by $[X,Y]_{1,0}$ (resp. by $[X,Y]_{0,1}$) the $TM^{1,0}$ component (resp. the $\overline{TM^{1,0}}$-component) of $[X,Y]_{HM}$ in the decomposition $\mathbb{C}HM = TM^{1,0} \oplus TM^{0,1}$. $\nabla$ can be extended to a differential operator of $\Gamma(\mathbb{C}TM)$ to $\Gamma(\mathbb{C}TM) \otimes \mathbb{C}TM^*$ in a natural manner. By (19), $\nabla J = 0$ and $T^{1,0}M = \{X - iJX \mid X \in HM\}$, we have

$$\begin{aligned} &\nabla_X \Gamma(TM^{1,0}) \subset \Gamma(TM^{1,0})\,,\\ &\nabla_X \Gamma(TM^{0,1}) \subset \Gamma(TM^{0,1})\,, \quad X \in \Gamma(\mathbb{C}TM)\,. \end{aligned}$$

Then we have

**Proposition 2.2** ([5, p. 31])**.** *The extension* $\nabla\colon \Gamma(\mathbb{C}TM) \to \Gamma(\mathbb{C}TM \otimes \mathbb{C}TM^*)$ *is given as follows. For* $X,\,Y \in \Gamma(TM^{1,0})$,

(20) $\nabla_{\overline{X}} Y = [\overline{X}, Y]_{1,0}\,,$

(21) $\nabla_X Y$ *is given by* $\Omega(\nabla_X Y, \overline{Z}) = X\Omega(Y,\overline{Z}) - \Omega(Y, \overline{\nabla_{\overline{x}} Z})\ \forall Z \in \Gamma(TM^{1,0})\,,$

(22) $\nabla_\xi Y = [\xi, Y] - \dfrac{1}{2} J([\xi, JY] - J[\xi, Y]) = [\xi, Y]_{1,0}\,.$

$\nabla_X \overline{Y}$, $\nabla_{\overline{X}} \overline{Y}$, $\nabla_\xi \overline{Y}$ *are given by conjugations, and* $\nabla_X \xi$, $\nabla_{\overline{X}} \xi$, $\nabla_\xi \xi$ *are all zero.*

In the following, we shall identify $\nabla$ with Webster's $D$. We have

$$\begin{aligned} D_{\overline{X}_\beta} X_\alpha &= \omega_\alpha^{\ \gamma}(\overline{X}_\beta) X_\gamma \overset{(2)}{=} d\theta^\gamma(X_\alpha, \overline{X}_\beta) X_\gamma\\ &= -\theta^\gamma([X_\alpha, \overline{X}_\beta]) X_\gamma = [\overline{X}_\beta, X_\alpha]_{1,0} = \nabla_{\overline{X}_\beta} X_\alpha\,. \end{aligned}$$

And we check that

$$\begin{aligned} -d\theta(D_{X_\beta} X_\alpha, \overline{X}_\gamma) &= -i\theta^\rho \wedge \theta^{\overline{\rho}}\big(\omega_\alpha^{\ \sigma}(X_\beta) X_\sigma, \overline{X}_\gamma\big) = -i\omega_\alpha^{\ \gamma}(X_\beta) = i\overline{\omega_\gamma^{\ \alpha}}(X_\beta)\\ &= X_\beta\big(-i\theta^\rho \wedge \theta^{\overline{\rho}}(X_\alpha, \overline{X}_\gamma)\big) + i\theta^\rho \wedge \theta^{\overline{\rho}}\big(X_\alpha, \overline{\omega_\gamma^{\ \sigma}}(X_\beta)\overline{X}_\sigma\big)\\ &= X_\beta\big(-d\theta(X_\alpha, \overline{X}_\gamma)\big) - (-d\theta)(X_\alpha, \overline{\nabla_{\overline{X}_\beta} X_\gamma}) \quad \text{for all } X_\gamma\,. \end{aligned}$$

Hence, $D_{X_\beta} X_\alpha = \nabla_{X_\beta} X_\alpha$. We also have

$$D_\xi X_\alpha = \omega_\alpha^{\ \gamma}(\xi) X_\gamma \overset{(2)}{=} -d\theta^\gamma(\xi, X_\alpha) X_\gamma = \theta^\gamma([\xi, X_\alpha]) X_\gamma = [\xi, X_\alpha]_{1,0} = \nabla_\xi X_\alpha\,.$$

Then we identify the torsion terms. We have

$$\begin{aligned} T(X_\alpha, \overline{X}_\beta) &= \nabla_{X_\alpha}\overline{X}_\beta - \nabla_{\overline{X}_\beta}X_\alpha - [X_\alpha, \overline{X}_\beta] \\ &= [X_\alpha, \overline{X}_\beta]_{0,1} + [X_\alpha, \overline{X}_\beta]_{1,0} - [X_\alpha, \overline{X}_\beta] \\ &= -\theta([X_\alpha, \overline{X}_\beta])\xi \\ &= d\theta(X_\alpha, \overline{X}_\beta)\xi \\ &= i\delta_\alpha^{\ \beta}\xi = -\Omega(X_\alpha, \overline{X}_\beta)\xi, \end{aligned}$$

and

$$\begin{aligned} T(X_\alpha, X_\beta) &= (\omega_\beta^{\ \gamma}(X_\alpha) - \omega_\alpha^{\ \gamma}(X_\beta) - \theta^\gamma([X_\alpha, X_\beta]))X_\gamma \\ &= (\omega_\beta^{\ \gamma}(X_\alpha) - \omega_\alpha^{\ \gamma}(X_\beta) + d\theta^\gamma(X_\alpha, X_\beta))X_\gamma = 0\,, \end{aligned}$$

and

$$\begin{aligned} T(\xi, X_\alpha) &= \nabla_\xi X_\alpha - \nabla_{X_\alpha}\xi - [\xi, X_\alpha] \\ &= [\xi, X_\alpha]_{1,0} - [\xi, X_\alpha] \\ &= -\theta^{\bar\beta}([\xi, X_\alpha])\overline{X}_\beta - \theta([\xi, X_\alpha])\xi \\ &= (\theta^{\overline{\gamma}} \wedge \overline{\omega_\gamma^{\ \beta}} + \theta \wedge \tau^{\bar\beta})(\xi, X_\alpha)\overline{X}_\beta \\ &= \tau^{\bar\beta}(X_\alpha)\overline{X}_\beta \\ &= A_{\alpha\beta}\overline{X}_\beta\,. \end{aligned}$$

Finally, we identify the curvatures terms. We have

$$\begin{aligned} R(Y,Z)X_\beta &= \nabla_Y\nabla_Z X_\beta - \nabla_Z\nabla_Y X_\beta - \nabla_{[Y,Z]}X_\beta \\ &= ((Y\omega_\beta^{\ \alpha}(Z) + \omega_\beta^{\ \gamma}(Z)\omega_\gamma^{\ \alpha}(Y)) - \big(Z\omega_\beta^{\ \alpha}(Y) + \omega_\beta^{\ \gamma}(Y)\omega_\gamma^{\ \alpha}(Z)\big) \\ &\quad - \omega_\beta^{\ \alpha}([Y,Z]))X_\alpha \overset{(11)}{=} \big((d\omega_\beta^{\ \alpha} - \omega_\beta^{\ \gamma} \wedge \omega_\gamma^{\ \alpha})(Y,Z)\big)X_\alpha\,, \end{aligned}$$

and

$$\begin{aligned} R(X_\rho, X_\sigma)X_\beta &= (-iA_{\beta\gamma}\theta^\gamma \wedge \theta^\alpha)(X_\rho, X_\sigma)X_\alpha \\ &= -iA_{\beta\gamma}(\delta_\rho^\gamma\delta_\sigma^\alpha - \delta_\sigma^\gamma\delta_\rho^\alpha)X_\alpha \\ &= -i(A_{\beta\rho}X_\sigma - A_{\beta\sigma}X_\rho)\,, \\ R(X_\rho, \overline{X}_\sigma)X_\beta &= R_{\beta\overline{\alpha}\rho\overline{\sigma}}X_\alpha\,, \\ R(\overline{X}_\rho, \overline{X}_\sigma)X_\beta &= (i\overline{A_{\alpha\gamma}}\theta^{\bar\beta} \wedge \theta^{\overline{\gamma}})(\overline{X}_\rho, \overline{X}_\sigma)X_\alpha \\ &= i\overline{A_{\alpha\gamma}}(\delta_\beta^\rho\delta_\gamma^\sigma - \delta_\beta^\sigma\delta_\gamma^\rho)X_\alpha \\ &= i(\delta_\beta^\rho\overline{A_{\alpha\sigma}} - \delta_\beta^\sigma\overline{A_{\alpha\rho}})X_\alpha\,, \\ R(X_\rho, \xi)X_\beta &= W_{\beta\overline{\alpha}\rho}X_\alpha\,, \\ R(\overline{X}_\sigma, \xi)X_\beta &= -W_{\overline{\alpha}\beta\overline{\sigma}}X_\alpha\,. \end{aligned}$$

## 3. A key identity for Webster pseudo-torsion computation

In this section, we obtain a key identity (53) for Webster pseudo-torsion computation in Section 5.

Let $M$ be the boundary of a strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Let $r$ be a smooth real-valued defining function of $M$ i.e. $M = \{r = 0\}$ and $dr \neq 0$. Throughout this section, the range of indices are: $0 \leq i, j, k \cdots \leq n+1$, $0 \leq \alpha, \beta, \gamma \cdots \leq n$. Coordinates for $\mathbb{C}^{n+1}$ will be given by $(z_1, z_2, \ldots, z_{n+1})$. We will use the conventions: $r_j = \frac{\partial r}{\partial z^j}$, $r_{j\overline{k}} = \frac{\partial^2 r}{\partial z^j \partial \overline{z}^k}$. The $CR$ structure is on $M$ is given by

$$T^{1,0}M = \{X = x_j \frac{\partial}{\partial z^j} : dr(X) = x^j r_j = 0\}\,. \tag{23}$$

We define a $2n$ dimensional subbundle of $TM$ by

$$\mathbb{C}HM = T^{1,0}M \oplus T^{0,1}M \quad \text{where} \quad T^{0,1}M := \overline{T^{1,0}M}\,, \tag{24}$$

and $HM := \mathrm{Re}\,(T^{1,0}M \oplus T^{0,1}M)$. $HM$ carries a complex structure map

$$J \colon HM \to HM, \quad J^2 = -Id\,, \tag{25}$$

and we denote its extension to $\mathbb{C}TM$ by $J$,

$$J \colon \mathbb{C}HM \to \mathbb{C}HM,\ J^2 = -Id \text{ and } J|_{T^{1,0}M} = \text{ multiplication by } i = \sqrt{-1}\,. \tag{26}$$

Define a one form $\theta$ on $\mathbb{C}^{n+1}$ by

$$\theta = -i\partial r = -i r_j dz^j\,. \tag{27}$$

On $CTM$, $\theta$ is a real one form annihilating $T^{1,0}M \oplus T^{0,1}M$,

$$\theta = i\partial r = i\overline{\partial} r = \frac{i}{2}(\overline{\partial} r - \partial r)\,. \tag{28}$$

For $X$, $Y \in T^{1,0}M$,

$$\begin{aligned} &\theta([X,Y]) = 0\,, \quad \theta([\overline{X},\overline{Y}]) = 0\,, \\ \text{and } \ &\theta([X,\overline{Y}]) = -d\theta(X,\overline{Y}) = -i\partial\overline{\partial} r(X,\overline{Y})\,. \end{aligned} \tag{29}$$

For $X$, $Y \in T^{1,0}M$, the Levi form is given by

$$L_\theta(X,\overline{Y}) = \theta([JX,\overline{Y}]) = -d\theta(JX,\overline{Y}) = \partial\overline{\partial} r(X,\overline{Y})\,. \tag{30}$$

$M$ is said to be *strongly pseudoconvex* if $L_\theta(X,\overline{Y})$ is positive definite as a Hermitian form on $T^{1,0}M$. In other words,

$$\forall\ w^j \frac{\partial}{\partial z^j} \neq 0\,, \quad w^j r_j = 0 \Rightarrow r_{j\overline{k}} w^j w^{\overline{k}} > 0\,. \tag{31}$$

Note that the matrix $r_{j\overline{k}}$ is not necessary invertible though (31) is satisfied.

**Example 3.1.** The real hyperquadric in $\mathbb{C}^2$ given by

$$M := \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0,\ r = z_1\overline{z}_1 - \frac{z_2 - \overline{z}_2}{2i}\} \quad \text{which is s.p.c.}$$

$T^{1,0}M$ is spanned by $\frac{\partial}{\partial z_1} + 2i\overline{z}_1 \frac{\partial}{\partial z_2}$. We see that

$$\begin{pmatrix} r_{1\overline{1}} & r_{1\overline{2}} \\ r_{2\overline{1}} & r_{2\overline{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{while} \quad \begin{pmatrix} 1 & 2i\overline{z}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2iz_1 \end{pmatrix} = 1\,.$$

Neither does s.p.c., (31) imply the positive definiteness of $r_{j\bar{k}}$, as we see from

**Example 3.2.** $M := \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0,\ r = 1 + z_1\overline{z}_1 - z_2\overline{z}_2\}$ which is s.p.c.
$T^{1,0}M$ is spanned by $\overline{z}_2 \frac{\partial}{\partial z_1} + \overline{z}_1 \frac{\partial}{\partial z_2}$. We see that

$$\begin{pmatrix} r_{1\bar{1}} & r_{1\bar{2}} \\ r_{2\bar{1}} & r_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{while} \quad \begin{pmatrix} \overline{z}_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = 1\,.$$

Let $\xi$ be the unique real vector field on $M$ such that

$$\theta(\xi) = 1\,, \tag{32}$$

$$\xi \rfloor d\theta = 0\,. \tag{33}$$

Let

$$\xi = \xi^j \frac{\partial}{\partial z^j} + \overline{\xi^j} \frac{\partial}{\partial \overline{z}^j}\,. \tag{34}$$

We have

$$\theta(\xi) = 1 \quad \text{means} \quad i r_{\bar{k}} \xi^{\bar{k}} = 1 \quad \text{or} \quad r_j \xi^j = i\,, \tag{35}$$

$$\xi \rfloor d\theta = 0 \quad \text{means} \quad x^j r_j = 0 \Rightarrow x^j r_{j\bar{k}} \xi^{\bar{k}} = 0\,. \tag{36}$$

Let $TM = HM \oplus \mathbb{R}\xi$, we extend (25),

$$J \colon TM \to TM \quad \text{by} \quad J\xi = 0\,. \tag{37}$$

Then, $J$ as a $\binom{1}{1}$ tensor satisfies

$$J^2 X = -X + \theta(X)\xi \tag{38}$$

for all $X \in TM$. With $J$ as a $\binom{1}{1}$ tensor, we regard $g(X, Y) := -d\theta(JX, Y) = L_\theta(X, Y)$ as $\binom{0}{2}$ tensor on $TM$. Note that , for $X, Y \in TM$, $\theta([JX, Y]) \neq -d\theta(JX, Y)$ since $\theta([X, Y])$ is not a tensor, for instance, we have $\theta([f\xi, \xi]) = \theta(\xi(f)\xi) = \xi(f)$. In the following, we write $\langle X, Y\rangle := g(X, \overline{Y})$. Choose $X_1, \ldots, X_n$ in $T^{1,0}_p M$ for some point $p$ in $M$. Let

$$X_\alpha = x^j_\alpha \frac{\partial}{\partial z^j} \tag{39}$$

satisfying

$$x^j_\alpha r_j = 0, \tag{40}$$

$$x^j_\alpha r_{j\bar{k}} \overline{x^k_\beta} = \delta^\beta_\alpha\,. \tag{41}$$

Note that we use all Euclidean coordinates $z^1, \ldots, z^{n+1}$ in the description of the $CR$ structure of $M$. In this way, we dispense with distinguishing one coordinate, say $z^{n+1}$, such that $\frac{\partial r}{\partial z^{n+1}} \neq 0$, as is required in Chern-Moser and subsequent works. Our computation is therefore symmetric in all $z^1, \ldots, z^{n+1}$. Write

$$J(u) := (-1)^{n+1} \det \begin{pmatrix} u & u_{\overline{1}} & \cdots & u_{\overline{n+1}} \\ u_1 & u_{1\overline{1}} & \cdots & u_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n+1} & u_{n+1\overline{1}} & \cdots & u_{n+1\overline{n+1}} \end{pmatrix}, \tag{42}$$

$$F := \begin{pmatrix} r & r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_1 & r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n+1} & r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix}, \tag{43}$$

and

$$\langle\langle \xi, \xi \rangle\rangle := \xi^j r_{j\overline{k}} \xi^{\overline{k}} . \tag{44}$$

Then, we have

$$-\langle\langle \xi, \xi \rangle\rangle r_j + i r_{j\overline{k}} \xi^{\overline{k}} = 0 . \tag{45}$$

**Proof of** (45)**.** $(r_j dz^j)(X_\alpha) = x^j_\alpha r_j \overset{(40)}{=} 0$ and $(r_{j\overline{k}} \xi^{\overline{k}} dz^j)(X_\alpha) = x^j_\alpha r_{j\overline{k}} \xi^{\overline{k}} = 0$ for all $\alpha$, implies that, since $dr \neq 0$, $r_{j\overline{k}} \xi^{\overline{k}} = b r_j$ for some $b$. By contraction with $\xi^j$, $\langle\langle \xi, \xi \rangle\rangle = bi$. Thus, we obtain (45). Write

$$a^{\overline{j}k} := \overline{x^j_\alpha} x^k_\alpha . \tag{46}$$

Then

$$r_{\overline{j}} a^{\overline{j}k} = 0 . \tag{47}$$

Write

$$X_{n+1} := \xi^j \frac{\partial}{\partial z^j} \quad \text{and} \quad x^j_{n+1} = \xi^j . \tag{48}$$

Then

(49)

$$\begin{pmatrix} x^1_1 & \cdots & x^{n+1}_1 \\ \vdots & & \vdots \\ x^1_{n+1} & \cdots & x^{n+1}_{n+1} \end{pmatrix} \begin{pmatrix} r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} \overline{x^1_1} & \cdots & \overline{x^{n+1}_1} \\ \vdots & & \vdots \\ \overline{x^1_{n+1}} & \cdots & \overline{x^{n+1}_{n+1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \langle\langle \xi, \xi \rangle\rangle \end{pmatrix} .$$

Write

$$\begin{pmatrix} y_1^1 & \cdots & y_1^{n+1} \\ \vdots & & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^{n+1} \end{pmatrix} := \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}^{-1}$$

$$\overset{(48)}{=} \begin{pmatrix} y_1^1 & \cdots & y_1^n & -ir_1 \\ \vdots & & \vdots & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^n & -ir_{n+1} \end{pmatrix}. \tag{50}$$

Then

(51)

$$\begin{pmatrix} r_{1\bar{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\bar{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} \overline{x_1^1} & \cdots & \overline{x_1^{n+1}} \\ \vdots & & \vdots \\ \overline{x_{n+1}^1} & \cdots & \overline{x_{n+1}^{n+1}} \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} y_1^1 & \cdots & y_1^{n+1} \\ \vdots & & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^{n+1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \langle\langle\, \xi, \xi\, \rangle\rangle \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} y_1^1 & \cdots & y_1^n & \langle\langle\, \xi, \xi\, \rangle\rangle y_1^{n+1} \\ \vdots & & \vdots & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^n & \langle\langle\, \xi, \xi\, \rangle\rangle y_{n+1}^{n+1} \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-\langle\langle\, \xi, \xi\, \rangle\rangle) y_1^{n+1} x_{n+1}^1 & \cdots & -(1-\langle\langle\, \xi, \xi\, \rangle\rangle) y_1^{n+1} x_{n+1}^{n+1} \\ \vdots & & \vdots \\ -(1-\langle\langle\, \xi, \xi\, \rangle\rangle) y_{n+1}^{n+1} x_{n+1}^1 & \cdots & 1-(1-\langle\langle\, \xi, \xi\, \rangle\rangle) y_{n+1}^{n+1} x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1+(1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_1 \xi^1 & \cdots & (1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_1 \xi^{n+1} \\ \vdots & & \vdots \\ (1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_{n+1} \xi^1 & \cdots & 1+(1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_{n+1} \xi^{n+1} \end{pmatrix}$$

i.e.

$$r_{i\bar{k}} \overline{x_l^k} x_l^j = \delta_i^j + (1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_i \xi^j\,.$$

By (46), (48),

$$r_{i\bar{k}} (a^{\bar{k}j} + \xi^{\bar{k}} \xi^j) = \delta_i^j + (1-\langle\langle\, \xi, \xi\, \rangle\rangle) i r_i \xi^j\,.$$

By (45),

$$r_{i\bar{k}} a^{\bar{k}j} - i\langle\langle\, \xi, \xi\, \rangle\rangle r_i \xi^j = \delta_i^j + i r_i \xi^j - i\langle\langle\, \xi, \xi\, \rangle\rangle r_i \xi^j\,.$$

Hence,

$$-ir_i\xi^j + r_{i\bar{k}} a^{\bar{k}j} = \delta_i^j\,. \tag{52}$$

By (35), (45), (47), (52),

(53)
$$\begin{pmatrix} r & r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_1 & r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n+1} & r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} -\langle\langle\, \xi, \xi\,\rangle\rangle & -i\xi^1 & \cdots & -i\xi^{n+1} \\ i\overline{\xi^1} & a^{\overline{1}1} & \cdots & a^{\overline{1}n+1} \\ \vdots & \vdots & & \vdots \\ i\overline{\xi^{n+1}} & a^{\overline{n+1}1} & \cdots & a^{\overline{n+1}n+1} \end{pmatrix} = I\,.$$

□

## 4. An alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$

This section gives an alternative proof of the Li-Luk formula for Webster pseudo-torsion (for definition, see (69)) for a strongly pseudoconvex pseudohermitian hypersurface in $\mathbb{C}^{n+1}$. For the convenience of readers and fixing notations, we recall some facts and definitions in the beginning. We will also use some definitions and results in Section 2. Let $M$ be a strongly pseudoconvex pseudohermitian hypersurface given by $M = \{z \in \mathbb{C}^{n+1} \mid r = 0\}$, where $r$ is a real valued defining function for $M$ and $r$ is $C^3$ in a neighborhood of $M$. Let $TM$ be the tangent bundle on $M$ and let $HM := TM \cap iTM$, the holomorphic tangent bundle on $M$. As in the previous sections, we fix the real one form $\theta$ be a pseudohermitian structure on $M$. Let $\theta^1, \ldots, \theta^n, \overline{\theta^1}, \ldots, \overline{\theta^n}$ be a local admissible coframe for $M$, $1 \le \alpha, \beta \le n$. As before we use the convention $\theta^{\overline{\alpha}} := \overline{\theta^{\alpha}}$. Webster shows that there are uniquely determined 1-forms $\omega_{\alpha}^{\ \beta}, \tau^{\beta}$ on $M$ satisfying

(54) $$d\theta = i\theta^{\gamma} \wedge \theta^{\overline{\gamma}}\,,$$

(55) $$d\theta^{\alpha} = \theta^{\beta} \wedge \omega_{\beta\cdot}^{\ \alpha} + \theta \wedge \tau^{\alpha}\,,$$

(56) $$\omega_{\alpha\cdot}^{\ \beta} + \overline{\omega_{\beta}^{\ \alpha}} = 0\,,$$

(57) $$\overline{\tau^{\alpha}} = A_{\alpha\gamma}\theta^{\gamma}\,,$$

(58) $$A_{\alpha\gamma} = A_{\gamma\alpha}\,.$$

Let $\xi, X_1, \ldots, X_n, \overline{X_1}, \ldots, \overline{X_n}$ be the dual frame satisfying

(59) $$\theta(\xi) = 1\,, \quad d\theta(\xi, \cdot) = i\theta^{\gamma} \wedge \theta^{\overline{\gamma}}(\xi, \cdot) = 0\,.$$

And we have

(60) $$-\operatorname{id}\theta(X_{\alpha}, \overline{X_{\beta}}) = i\theta^{\gamma} \wedge \theta^{\overline{\gamma}}(X_{\alpha}, \overline{X_{\beta}}) = \delta_{\alpha}^{\beta}\,.$$

The Levi form $L_{\theta}$ on $TM^{1,0}$ is defined by $L_{\theta}(\cdot, \bar{\cdot}) := -\operatorname{id}\theta(\cdot, \bar{\cdot})$. Hence,

(61) $$L_{\theta}(X_{\alpha}, \overline{X_{\beta}}) = \delta_{\alpha}^{\beta} =: \langle X_{\alpha}, \overline{X_{\beta}}\rangle\,.$$

Covariant differentiation is given by

(62) $$\nabla X_{\alpha} = \omega_{\alpha}^{\ \beta} X_{\beta}\,, \quad \nabla X_{\alpha} = \omega_{\alpha}^{\ \beta} X_{\beta}\,, \quad \nabla\xi = 0\,.$$

We also have

(63) $$\nabla_{\overline{X}_{\gamma}} X_{\alpha} = [\overline{X}_{\alpha}, X_{\alpha}]_{TM^{1,0}}\,,$$

and $\nabla_{X_\gamma} X_\alpha$ is defined by

$$\langle \nabla_{X_\gamma} X_\alpha, X_\beta \rangle = X_\gamma \langle X_\alpha, X_\beta \rangle - \langle X_\alpha, \nabla_{\overline{X}_\gamma} X_\beta \rangle \,. \tag{64}$$

We have

$$\nabla_\xi X_\alpha = [\xi, X_\alpha]_{TM^{1,0}} \,. \tag{65}$$

The torsion tensor is defined by $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ for $X, Y \in \mathbb{C}TM$. We have

$$T(X_\alpha, \overline{Y}_\beta) = i\delta_\alpha^\beta \xi \,, \tag{66}$$

$$T(X_\alpha, X_\beta) = 0 \,, \tag{67}$$

$$T(\xi, X\alpha) = A_{\alpha\beta} \overline{X}_\beta \,. \tag{68}$$

The Webster pseudo-torsion is defined as [3],

$$\mathrm{Tor}(z)(U,V) = i(A_{\overline{\alpha}\bar{\beta}} \overline{u}^\alpha \overline{v}^\beta - A_{\alpha\beta} u^\alpha v^\beta) \,, \tag{69}$$

where $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$. We will use following notations.

$$J(r) := - \begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{vmatrix} \,, \tag{70}$$

$$H(r) := (r_{j\overline{k}}) \,. \tag{71}$$

We shall prove the following theorem.

**Theorem 4.1** ([3])**.** *Let $M$ be a $C^4$ strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$. Let $r$ be a defining function for $M$ which is $C^3$ in a neigborhood of $M$. Consider the pseudohermitian structure defined by $\theta = -i\partial r$ on $M$. Then for any $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$, we have*

$$\mathrm{Tor}(z)(U,V) = 2\,\mathrm{Re}\left( \frac{\overline{u^l v^k}}{J(r)} \left(N - \det H(r)\right) r_{\overline{lk}} \right) , \tag{72}$$

*where*

$$N = \sum_i (-1)^{j+i} r_{\overline{i}} \begin{vmatrix} - & \begin{matrix} | \\ \mathbf{r}_{\mathbf{i}\bar{\mathbf{j}}} \\ | \end{matrix} & - \end{vmatrix} \frac{\partial}{\partial z^{\overline{j}}} \,. \tag{73}$$

We will need some preliminaries to prove this theorem. First, by (53), we have

$$1 = r(-\langle\langle\, \xi, \xi \,\rangle\rangle) + r_1(-i\xi^1) + r_2(-i\xi^2) + \cdots + r_{n+1}(-i\xi^{n+1}) \,. \tag{74}$$

Expanding $-J(r)$ by the 1st column, we have

(75)

$$\begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{vmatrix} = r \begin{vmatrix} r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} - r_1 \begin{vmatrix} r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ \mathbf{r_{1\overline{1}}} & \cdots & \mathbf{r_{1\overline{n+1}}} \\ \vdots & & \vdots \\ r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} + \cdots$$

$$+ (-1)^{n+1} r_{n+1} \begin{vmatrix} r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ \mathbf{r_{n+1\overline{1}}} & \cdots & \mathbf{r_{n+1\overline{n+1}}} \end{vmatrix} .$$

Hence, by (74), (75), we have

$$-\langle\langle \xi, \xi \rangle\rangle = \frac{|r_{j\overline{k}}|}{\begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{vmatrix}} = -\frac{\det H(r)}{J(r)} \tag{76}$$

and

(77)

$$-i\xi^j = \frac{(-1)^j}{\begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{vmatrix}} \begin{vmatrix} r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\overline{1}}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix}$$

$$= \frac{(-1)^j}{-J(r)} \left( r_{\overline{1}} \begin{vmatrix} \mathbf{r_{1\overline{1}}} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\overline{1}}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ \mathbf{r_{n+1\overline{1}}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} - r_{\overline{2}} \begin{vmatrix} r_{1\overline{1}} & \mathbf{r_{1\overline{2}}} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\overline{1}}} & \mathbf{r_{j\overline{2}}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\overline{1}} & \mathbf{r_{n+1\overline{2}}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} \right.$$

$$\left. +(-1)^n r_{\overline{n+1}} \begin{vmatrix} r_{1\overline{1}} & \cdots & \mathbf{r_{1\overline{n+1}}} \\ \mathbf{r_{j\overline{1}}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\overline{1}} & \cdots & \mathbf{r_{n+1\overline{n+1}}} \end{vmatrix} \right)$$

$$= \sum_{k=1}^{n+1} \frac{(-1)^j(-1)^{k+1}}{-J(r)} r_{\overline{k}} \begin{vmatrix} r_{1\overline{1}} & | & r_{1\overline{n+1}} \\ - & \mathbf{r_{j\overline{k}}} & - \\ r_{n+1\overline{1}} & | & r_{n+1\overline{n+1}} \end{vmatrix}$$

**Proof of Theorem 4.1.**

*Step 1.* We first find a relation between the torsion tensor $T$ and the Webster torsion Tor. Let $U = \mu^\alpha X_\alpha, V = \nu^\beta X_\beta \in T^{1,0}M$. We have

$$\begin{aligned} \mathrm{Tor}(U,V) &= \mathrm{Tor}(\mu^\alpha X_\alpha, \nu^\beta X_\beta) \\ &= 2\mathrm{Re}(i\overline{A_{\alpha\beta}}) \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, X_\alpha)}, X_\beta\rangle)\overline{\mu^\alpha}\overline{\nu^\beta} \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, \mu^\alpha X_\alpha)}, \nu^\beta X_\beta\rangle) \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, U)}, V\rangle)\,. \end{aligned} \tag{78}$$

*Step 2.* We compute

$$\begin{aligned} T(\xi,U) &= \nabla_\xi U - \nabla_U \xi - [\xi,U] \\ &= [\xi,U]_{T^{1,0}M} - [\xi,U] \\ &= -[\xi,U]_{T^{0,1}M} \\ &= -\left[\xi^j \frac{\partial}{\partial z^j} + \overline{\xi^j}\frac{\partial}{\partial \overline{z^j}}, U\right]_{T^{0,1}M} \\ &= (U\overline{\xi^j})\frac{\partial}{\partial \overline{z^j}}\,. \end{aligned} \tag{79}$$

We check that $(U\overline{\xi^j})\frac{\partial}{\partial \overline{z^j}} \in T^{1,0}M$ as follows. Using $U = u^j \frac{\partial}{\partial z^j}$, we have

$$(U\overline{\xi^j})r_{\bar{j}} = U(\overline{\xi^j}r_{\bar{j}}) - \overline{\xi^j}Ur_{\bar{j}} = -u^k r_{k\bar{j}}\overline{\xi^j} = 0.$$

*Step 3.* Let $U = u^j \frac{\partial}{\partial z^j}, V = v^k \frac{\partial}{\partial z^k}$ such that $u^j r_j = 0$, $v^k r_k = 0$. Using (78), (79), we have

$$\begin{aligned} \mathrm{Tor}(U,V) &= 2\mathrm{Re}\left(i\left\langle \overline{(U\overline{\xi^j})\frac{\partial}{\partial \overline{z^j}}}, v^k\frac{\partial}{\partial z^k}\right\rangle\right) \\ &= 2\mathrm{Re}\left(i\left\langle \overline{u^l}\frac{\partial \xi^j}{\partial \overline{z^l}}\frac{\partial}{\partial z^j}, v^k\frac{\partial}{\partial z^k}\right\rangle\right) \\ &= 2\mathrm{Re}\left(i\overline{u^l}\frac{\partial \xi^j}{\partial \overline{z^l}} r_{j\bar{k}}\overline{v^k}\right) \\ &= 2\mathrm{Re}\left(i\overline{u^l v^k}\left(\frac{\partial}{\partial \overline{z^l}}(\xi^j r_{j\bar{k}}) - \xi^j r_{j\overline{kl}}\right)\right) \\ &= 2\mathrm{Re}\left(\overline{u^l v^k}\left(\frac{\partial}{\partial \overline{z^l}}(a r_{\bar{k}}) - i\xi^j \frac{\partial r_{\overline{kl}}}{\partial z^j}\right)\right) \\ &= 2\mathrm{Re}\left(\overline{u^l v^k}\left(-\langle\langle\, \xi,\xi\,\rangle\rangle r_{\overline{lk}} - i\xi^j\frac{\partial r_{\overline{kl}}}{\partial z^j}\right)\right)\,. \end{aligned} \tag{80}$$

Hence, we have

$$\begin{aligned} \mathrm{Tor}(U,V) &= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)}\left(-|r_{i\bar{j}}|r_{\overline{lk}} + \sum_i (-1)^{j+i} r_{\bar{i}}\begin{vmatrix} & | & \\ - & \mathbf{r_{i\bar{j}}} & - \\ & | & \end{vmatrix} r_{j\overline{lk}}\right)\right) \\ &= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)}\left(\sum_i (-1)^{j+i} r_{\bar{i}}\begin{vmatrix} & | & \\ - & \mathbf{r_{i\bar{j}}} & - \\ & | & \end{vmatrix}\frac{\partial}{\partial z^j} - \det H(r)\right) r_{\overline{lk}}\right) \\ &= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)}\left(N - \det H(r)\right) r_{\overline{lk}}\right)\,. \end{aligned}$$

□

## 5. A formula for Webster pseudo-torsion for on the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$

In this section we derive a formula for the Webster pseudo-torsion on the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$. Let $M := \{f = 0\} \cap \{r = 0\}$ where $r$ is a defining function of the sphere of radius $\epsilon$, centered at the origin and $f$ is a holomorphic function away from the origin, we assume that $\partial f \wedge dr \neq 0$ along $M$. Then $M$ is a strongly pseudoconvex CR manifold of real hypersurface type, of dimension $2n-1$. We will use the result in the last section to find an explicit formula for Webster torsion of $M$. The key idea is to express the components of the characteristic vector field $\xi$ in terms of the derivatives of $f$ and $r$.

Let $\mathcal{N} := \{z \in \mathbb{C}^{n+1} | f = 0\}$ where $f(0) = 0$, $\overline{\partial} f = 0$, $\partial f \neq 0$. Let $S := \{z \in \mathbb{C}^{n+1} | r = |z^1|^2 + |z^2|^2 + \cdots + |z^{n+1}|^2 - \epsilon = 0\}$ for some $\epsilon > 0$. Let $M := \mathcal{N} \cap S$, we assume $\partial f \wedge dr \neq 0$ along $M$. The complexified tangent bundles for $S$ and $M$ are denoted by $\mathbb{C}TS$ and $\mathbb{C}TM$ respectively. Let the pseudohermitian structure of $S$ be given by $\theta = i\overline{\partial} r = -i\partial r$ on $\mathbb{C}TS$. Then, the pseudohermitian structure of $M$ is given by $\theta|_M$. We will denote $\theta|_M$ by $\theta$. Throughout this section the ranges of indices are : $1 \leq A, B, \cdots \leq n+1$, $1 \leq j, k, \cdots \leq n$, $1 \leq \alpha, \beta, \cdots \leq n-1$, and we will use the summation convention. Let $\theta, \theta^\alpha, \theta^{\overline{\alpha}}$ be a local basis of $\mathbb{C}TM^*$ such that $d\theta = i\theta^\alpha \wedge \theta^{\overline{\alpha}}$. Let $\xi, X_\alpha, X_{\overline{\alpha}}$ be the dual basis. We may write

$$\xi = \xi^A \frac{\partial}{\partial z^A} + \overline{\xi^A} \frac{\partial}{\partial \overline{z^A}}\,, \tag{81}$$

$$X_\alpha = x_\alpha^A \frac{\partial}{\partial z^A}\,. \tag{82}$$

We have

$$\xi \rfloor \theta = 1 \Rightarrow \xi^A r_A = i\,, \tag{83}$$

$$\xi \rfloor \partial f = 0 \Rightarrow \xi^A f_A = 0\,, \tag{84}$$

$$X_\alpha \rfloor \theta = 0\,, \tag{85}$$

$$X_\alpha \rfloor \partial f = 0\,, \tag{86}$$

$$X_\alpha \rfloor \theta^\beta = \delta_\alpha^\beta\,, \tag{87}$$

$$\xi \rfloor \theta^\beta = 0\,, \tag{88}$$

$$\xi \rfloor d\theta = 0\,, \tag{89}$$

and

$$d\theta = i r_{\overline{A}B} dz^{\overline{A}} \wedge dz^B = i\delta_{\overline{A}B} dz^{\overline{A}} \wedge dz^B = i dz^{\overline{A}} \wedge dz^A\,. \tag{90}$$

Hence, we have

$$\overline{x^A_\alpha} r_{\overline{A}} = 0\,, \tag{91}$$

$$\overline{x^A_\alpha}\,\overline{f_A} = 0\,, \tag{92}$$

$$\overline{x^A_\alpha} r_{\overline{A}B}\xi^B = 0 \Rightarrow \overline{x^A_\alpha}\xi^A = 0\,. \tag{93}$$

We consider (93) as a system of linear equations in unknowns $\xi^A$. The matrix $(\overline{x^A_\alpha})$ has rank $n-1$. So (93) has only 2 independent solutions. On the other hand the matrix $\begin{pmatrix} \overline{f_1} & \cdots & \overline{f_{n+1}} \\ r_{\overline{1}} & \cdots & r_{\overline{n+1}} \end{pmatrix}$ has rank 2. Hence, we may write

$$\xi^A = a\overline{f_A} + br_{\overline{A}}\,, \tag{94}$$

for $a$, $b \in \mathbb{C}$. Contracting (94) with $\overline{\xi^A}$, using (83), (86) we obtain $\|\xi\|^2 = -ib$ where $\|\xi\|^2 := \xi^A\overline{\xi^A}$. Hence,

$$b = i\|\xi\|^2\,. \tag{95}$$

Contracting (94) with $f_A$, we obtain $0 = a\overline{f_A}f_A + br_{\overline{A}}f_A$. So,

$$a = -\frac{br_{\overline{A}}f_A}{\overline{f_C}f_C}\,. \tag{96}$$

By (94), (95), (96), we have

$$\xi^A = -i\|\xi\|^2\frac{r_{\overline{B}}f_B\overline{f_A}}{\overline{f_C}f_C} + i\|\xi\|^2 r_{\overline{A}}\,. \tag{97}$$

Contracting (97) with $r_A$, using (83),

$$i = r_A\xi^A = -i\|\xi\|^2\left(-\frac{r_{\overline{B}}f_B\overline{f_D}r_D}{\overline{f_C}f_C} + r_{\overline{D}}r_D\right)\,. \tag{98}$$

We solve for $\|\xi\|^2$ in (98) and using (97), we obtain

$$\begin{aligned}\xi^A &= \frac{i\left(-\frac{r_{\overline{B}}f_B\overline{f_A}}{\overline{f_C}f_C} + r_{\overline{A}}\right)}{\frac{r_{\overline{B}}f_B\overline{f_D}r_D}{\overline{f_C}f_C} - r_{\overline{D}}r_D} \\ &= \frac{i\left(-\frac{z^Bf_B\overline{f_A}}{\overline{f_C}f_C} + z^A\right)}{\frac{z^Bf_B\overline{f_D}z^{\overline{D}}}{\overline{f_C}f_C} - \epsilon}\,.\end{aligned} \tag{99}$$

Now, we are ready to show:

**Theorem 5.1.** *Let* $\mathcal{N} := \{z \in \mathbb{C}^{n+1} \mid f = 0\}$ *where* $f(0) = 0, \overline{\partial}f = 0, \partial f \neq 0$. *Let* $S := \{z \in \mathbb{C}^{n+1} \mid r = |z^1|^2 + |z^2|^2 + \cdots + |z^{n+1}|^2 - \epsilon = 0\}$ *for some* $\epsilon > 0$. *Let* $M := \mathcal{N} \cap S$, *we assume* $\partial f \wedge dr \neq 0$ *along* $M$. *Consider the pseudohermitian* *structure defined by* $\theta = -i\partial r$ *on* $M$. *Then for any* $U = u^A \frac{\partial}{\partial z_A}, V = v^B \frac{\partial}{\partial z_B} \in H_z M$ *and* $z \in M$, *we have*

$$\mathrm{Tor}(z)(U,V) = 2\,\mathrm{Re}\left( i\overline{u^B v^A} \frac{\partial \xi^A}{\partial \overline{z^B}} \right) \tag{100}$$

*where*

$$\xi^A = \frac{i\left( -\frac{z_B f_B \overline{f_A}}{\overline{f_C} f_C} + z_A \right)}{\frac{z_B f_B \overline{f_D} z_{\overline{D}}}{\overline{f_C} f_C} - \epsilon} .$$

**Proof of Theorem 5.1.**
*Step 1.* We first find a relation between the torsion tensor $T$ and the Webster torsion Tor. Let $U = \mu^\alpha X_\alpha, V = \nu^\beta X_\beta \in T^{1,0}M$. By computation similar to (78), we have

$$\mathrm{Tor}(U,V) = 2\mathrm{Re}(i\langle \overline{T(\xi,U)}, V\rangle) . \tag{101}$$

*Step 2.* By computation similar to (79), we have

$$T(\xi,U) = (U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}} . \tag{102}$$

We check that $(U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}} \in T^{1,0}M$ as follows. Using $U = u^A \frac{\partial}{\partial z^A}$, we have

$$(\overline{U}\xi^A) f_A = \overline{U}(\xi^A f_A) - \xi^A \overline{U}(f_A) = 0 .$$

*Step 3.* Let $U = u^A \frac{\partial}{\partial z^A}, V = v^A \frac{\partial}{\partial z^A}$ such that $u^A r_A = 0, u^A f_A = 0, v^A r_A = 0, v^A f_A = 0$. Using (101), (102), we have

$$\begin{aligned}
\mathrm{Tor}(U,V) &= 2\mathrm{Re}\left( i \left\langle \overline{(U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}}}, v^A \frac{\partial}{\partial z^A} \right\rangle \right) \\
&= 2\mathrm{Re}\left( i \left\langle \overline{u^B} \frac{\partial \xi^C}{\partial \overline{z^B}} \frac{\partial}{\partial z^C}, v^A \frac{\partial}{\partial z^A} \right\rangle \right) \\
&= 2\mathrm{Re}\left( i\overline{u^B v^A} \frac{\partial \xi^A}{\partial \overline{z^B}} \right) .
\end{aligned}$$

□

**Example 5.2.** Let $f = (z^3)^2 - z^1 z^2$. Let $M := \{f = 0\} \cap \{|z^1|^2 + |z^2|^2 + |z^3|^2 = 1\}$. We may see that the the codimension 3 real hypersurface $M$ is spherical as follows. Using the map $F$ given by

$$\begin{aligned}
\tilde{z^1} &= -\frac{1}{\sqrt{2}}(z^1 - iz^2) , \\
\tilde{z^2} &= \frac{1}{\sqrt{2}}(z^1 + iz^2) , \\
\tilde{z^3} &= z^3 ,
\end{aligned}$$

the $CR$ manifold $M_0$ given by

$$\begin{cases} (z^1)^2 + (z^2)^2 + (z^3)^2 = 0\,, \\ |z^1|^2 + |z^2|^2 + |z^3|^2 = 1 \end{cases}$$

is mapped to

$$\begin{cases} 2\tilde{z^1}\tilde{z^2} - (\tilde{z^3})^2 = 0\,, \\ |\tilde{z^1}|^2 + |\tilde{z^2}|^2 + |\tilde{z^3}|^2 = 1\,. \end{cases}$$

Together with the map $\phi\colon S^3 \to M_0$ given by

$$(\zeta,\eta) \mapsto \left(\frac{\zeta^2-\eta^2}{\sqrt{2}}, \frac{i(\zeta^2+\eta^2)}{\sqrt{2}}, \frac{2\zeta\eta}{\sqrt{2}}\right) =: (z^1, z^2, z^3)$$

where $S^3 := \{(\zeta,\eta) \in \mathbb{C}^2 : |\zeta|^2 + |\eta|^2 - 1 = 0\}$. $\phi$ is well defined, holomorphic, onto. By [2], $M_0$ is $CR$ diffeomorphic to $S^3/G$ where $G = \{I, -I\}$, so that $M_0$ is locally biholomorphic to $S^3$. Hence, $M$ is locally biholomorphic to $S^3$. Then $z^B f_B = 0$. By (100) $\mathrm{Tor}(z)(U,V) = 0$, $\forall z \in M$.

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Applied Mathematics Department,
Hong Kong Polytechnic University
*E-mail*: choryinhope@gmail.com