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# GENERALIZATION OF THE $S$-NOETHERIAN CONCEPT

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Abstract. Let $A$ be a commutative ring and $\mathcal{S}$ a multiplicative system of ideals. We say that $A$ is $\mathcal{S}$-Noetherian, if for each ideal $Q$ of $A$, there exist $I \in \mathcal{S}$ and a finitely generated ideal $F \subseteq Q$ such that $IQ \subseteq F$. In this paper, we study the transfer of this property to the polynomial ring and Nagata's idealization.

## 1. Introduction

In this paper a ring means a commutative ring with unit element. Let $A$ be an integral domain with quotient field $K$. E. Hamann, E. Houston and J. Johnson in [3] defined an ideal $I$ of $A[X]$ to be almost principal, if there exist an $s \in A \setminus \{0\}$ and an $f \in I$ such that $sI \subseteq fA[X]$, and they called the polynomial ring $A[X]$ an almost principal ideal domain if each ideal of $A[X]$ that extends to a proper ideal of $K[X]$ is almost principal. In [1], D.D. Anderson and T. Dumitrescu have defined the concept of $S$-Noetherian rings as follows. Let $A$ be a ring and $S \subseteq A$ a multiplicative set. The ring $A$ is called $S$-Noetherian, if for each ideal $I$ of $A$, there exist $s \in S$ and a finitely generated ideal $F \subseteq I$ of $A$ such that $sI \subseteq F$. They have shown that if $A$ is $S$-Noetherian, then so is $A[X]$, provided $(\bigcap_{n=1}^{\infty} s^n A) \bigcap S \neq \emptyset$ for each $s \in S$. These results have been extended in [1], [4] and [5]. We extend this definition using an arbitrary multiplicative system of ideals.

Let $\mathcal{S}$ be a multiplicative system of ideals of a ring $A$. We shall call $A$ to be $\mathcal{S}$-Noetherian, if for each ideal $Q$ of $A$, there exist an ideal $I \in \mathcal{S}$ and a finitely generated ideal $F \subseteq Q$ of $A$ such that $IQ \subseteq F$. In the case when $\mathcal{S}$ consists of principal ideals, the notions $\mathcal{S}$-Noetherian and $S$-Noetherian are equivalent, where $S = \{s \in A \mid sA \in \mathcal{S}\}$. But in general we can not present a multiplicative system of ideals by a multiplicative set. In this paper, we investigate some properties of the $\mathcal{S}$-Noetherian concept. For instance, we give a Cohen-type theorem for $\mathcal{S}$-Noetherian rings. Also, we study the transfer of this property from $A$ to the polynomial ring $A[X]$ and Nagata idealization $A(+)M$, where $M$ is an $A$-module. In fact, we show that the ring $A(+)M$ is $\mathcal{S}_1$-Noetherian if and only if the ring $A$ is $\mathcal{S}$-Noetherian

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and the $A$-module $M$ is $\mathcal{S}$-finite, where $\mathcal{S}_1 = \{I(+)IM,\ I \in \mathcal{S}\}$. We give examples of $\mathcal{S}$-Noetherian rings $A$ with $\mathcal{S}$ a multiplicative system of nonprincipal ideals of $A$.

## 2. Mains results

We introduce the main concept of this paper as follows.

**Definition 2.1.** Let $A \subseteq B$ be a rings extension, $M$ an $A$-module and $\mathcal{S}$ a multiplicative system of ideals of $A$.

(1) An $A$-submodule $N$ of $M$ is said to be $\mathcal{S}$-finite, if there exist $a_1, \cdots, a_n \in N$ and $I \in \mathcal{S}$ such that $IN \subseteq \langle a_1, \cdots, a_n \rangle$.

(2) We say that $M$ is $\mathcal{S}$-Noetherian, if each submodule of $M$ is $\mathcal{S}$-finite.

(3) An ideal $Q$ of $B$ is called $\mathcal{S}$-finite, if there exist $a_1, \ldots, a_n \in Q$ and $I \in \mathcal{S}$ such that $IQ \subseteq \langle a_1, \ldots, a_n \rangle B$.

(4) We say that $B$ is an $\mathcal{S}$-Noetherian ring, if each ideal of $B$ is $\mathcal{S}$-finite.

With the same notations of the previous definition, clearly $B$ is $\mathcal{S}$-Noetherian if and only if it is $\mathcal{S}'$-Noetherian, where $\mathcal{S}' = \{IB \mid I \in \mathcal{S}\}$. It is clear that if $IM = 0$ for some $I \in \mathcal{S}$, then $M$ is an $\mathcal{S}$-Noetherian $A$-module.
Obviously a Noetherian ring $A$ is $\mathcal{S}$-Noetherian for every multiplicative system of ideals $\mathcal{S}$ of $A$.

**Example 2.2.** Let $A = \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$ where $p$ is a prime number, $a_1, \ldots, a_n \in A$ some finite support nonzero elements (i.e, if $a_i = (a_{i,j})_{j \in \mathbb{N}}$, then $a_{i,j} = 0$ except for a finite number of indices $j$), $I = \langle a_1, \ldots, a_n \rangle$ and $\mathcal{S} = \{I^n,\ n \geq 1\}$. For each ideal $Q$ of $A$, the ideal $IQ$ has a finite cardinality. Hence $IQ \subseteq \langle IQ \rangle \subseteq Q$, thus $Q$ is $\mathcal{S}$-finite.

So $A$ is an example of an $\mathcal{S}$-Noetherian ring which is not Noetherian.

**Proposition 2.3.** *Let $A$ be a ring, $M$ an $A$-module, $N$ a submodule of $M$ and $\mathcal{S}$ a multiplicative system of ideals of $A$. The following assertions are equivalent:*

(1) *The $A$-module $M$ is $\mathcal{S}$-Noetherian.*

(2) *The $A$-modules $N$ and $M/N$ are $\mathcal{S}$-Noetherian.*

**Proof.** (1) $\Longrightarrow$ (2) Trivial.
(2) $\Longrightarrow$ (1) Let $L$ be a submodule of $M$. Denote $\bar{L} = \{\bar{x} \in M/N \mid x \in L\}$. It is easy to check that $\bar{L}$ is a submodule of $M/N$, then it is $\mathcal{S}$-finite. Therefore, there exist $x_1, \ldots, x_n \in L$ and $I \in \mathcal{S}$ such that $I\bar{L} \subseteq \langle \bar{x_1}, \ldots, \bar{x_n} \rangle$.
Let $T = L \bigcap N$. It is clear that $T$ is a submodule of $N$, so it is $\mathcal{S}$-finite. Hence there exist $y_1, \ldots, y_k \in T$ and $J \in \mathcal{S}$ such that $JT \subseteq \langle y_1, \ldots, y_k \rangle$. For $x \in L$ fixed, we have $a\bar{x} \in \langle \bar{x_1}, \ldots, \bar{x_n} \rangle$ for each $a \in I$. Let $a \in I$, write $a\bar{x} = \sum_{i=1}^{n} \alpha_i \bar{x_i}$ with $\alpha_i \in A,\ i = 1, \ldots, n$. Then $ax - \sum_{i=1}^{n} \alpha_i x_i \in N \bigcap L = T$. Thus $J(ax - \sum_{i=1}^{n} \alpha_i x_i) \subseteq \langle y_1, \ldots, y_k \rangle$. It

yields that $Jax \subseteq \langle y_1, \dots, y_k, x_1, \dots, x_n\rangle$. Hence $(JI)L \subseteq \langle y_1, \dots, y_k, x_1, \dots, x_n\rangle$ with $y_1, \dots, y_k, x_1, \dots, x_n \in L$ and $IJ \in \mathcal{S}$. $\square$

**Corollary 2.4.** *A finite direct sum of modules is $\mathcal{S}$-Noetherian if and only if so is every term. In particular, $A^n$ is $\mathcal{S}$-Noetherian for each $n \geq 1$ provided that $A$ is an $\mathcal{S}$-Noetherian ring.*

**Corollary 2.5.** *Let $A$ be a ring, $M$ an $A$-module and $\mathcal{S}$ a multiplicative system of ideals of $A$. If $A$ is $\mathcal{S}$-Noetherian and $M$ a finitely generated $A$-module, then $M$ is an $\mathcal{S}$-Noetherian $A$-module.*

**Proof.** The $A$-module $M$ is an epimorphic image of some $A^n$. By Corollary 2.4, the $A$-module $M$ is $\mathcal{S}$-Noetherian. $\square$

**Corollary 2.6.** *Let $A$ be a ring, $\mathcal{S}$ a multiplicative system of ideals of $A$ and $M$ an $\mathcal{S}$-finite $A$-module. If $A$ is an $\mathcal{S}$-Noetherian ring, so is the $A$-module $M$.*

**Proof.** There exist a finitely generated submodule $N$ of $M$ and $I \in \mathcal{S}$ such that $IM \subseteq N$. By Corollary 2.5, $N$ is a $\mathcal{S}$-Noetherian $A$-module. Thus $IM$ is an $\mathcal{S}$-Noetherian $A$-module. Hence, the $A$-module $M$ is $\mathcal{S}$-Noetherian by the exact sequence $0 \longrightarrow IM \longrightarrow M \longrightarrow M/IM \longrightarrow 0$. $\square$

**Theorem 2.7.** *Let $A$ be a ring and $\mathcal{S}$ a multiplicative system of ideals of $A$ such that for each $I \in \mathcal{S}$, $\bigcap_{n=1}^{\infty} I^n$ contains some ideal of $\mathcal{S}$. If $A$ is $\mathcal{S}$-Noetherian, so is $A[X]$.*

**Proof.** Let $L$ be an ideal of $A[X]$ and $L_0$ the set of leading coefficients of polynomials of $L$. It is easy to check that $L_0$ is an ideal of $A$. Since $A$ is $\mathcal{S}$-Noetherian, there exist $a_1, \dots, a_n$ and $I \in \mathcal{S}$ such that $IL_0 \subseteq \langle a_1, \dots, a_n\rangle A$. For $1 \leq i \leq n$, let $f_i \in L$ such that $a_i$ is the leading coefficient of $f_i$. We can assume that $d = deg(f_1) = \cdots = \deg(f_n)$ (it suffices to multiply by some $X^{l_i}$, $1 \leq i \leq n$). Let $M = A + AX + \cdots + AX^d$. Let $f \in L$ of degree $r + d$. Let $a_1, \dots, a_r$ be arbitrary elements of $I$. Substracting repeatedly from $f$ suitable combinations of $f_1, \dots, f_n$ we get that $a_1 \dots a_r f$ belongs to $\langle f_1, \dots, f_n\rangle + L \bigcap M$. It follows that $I^r f \subseteq \langle f_1, \dots, f_n\rangle + L \bigcap M$, thus $JL \subseteq \langle f_1, \dots, f_n\rangle + L \bigcap M$ where $J$ is some ideal of $\mathcal{S}$ contained in $\bigcap_{k=1}^{\infty} I^k$. Since $M$ is a finitely generated $A$-module, it is $\mathcal{S}$-Noetherian, by Corollary 2.5. Consequentely, $L \cap M$ is $\mathcal{S}$-finite. Then there exist $g_1, \dots, g_m \in L \bigcap M$ and $J' \in \mathcal{S}$ such that $J'(L \bigcap M) \subseteq \langle g_1, \dots, g_m\rangle A \subseteq \langle g_1, \dots, g_m\rangle A[X]$. It yields that $(J'J)f \subseteq \langle f_1, \dots f_n, g_1 \dots, g_m\rangle A[X]$. Therefore, $(J'J)L \subseteq \langle f_1, \dots, f_n, g_1, \dots, g_m\rangle$ with $J'J \in \mathcal{S}$ and $f_1, \dots, f_n, g_1, \dots, g_m \in L$. Hence $A[X]$ is an $\mathcal{S}$-Noetherian ring. $\square$

**Corollary 2.8.** *Let $A$ be a ring and $\mathcal{S}$ a multiplicative system of ideals of $A$ such that for every $I \in \mathcal{S}$, $\bigcap_{n=1}^{\infty} I^n$ contains some ideal of $\mathcal{S}$. If $A$ is $\mathcal{S}$-Noetherian, so is $A[X_1, \dots, X_n]$ for each $n \geq 1$.*

**Proof.** By induction using Theorem 2.7. □

Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings, $A = \bigcup_{n=0}^{\infty} A_n$ and $X$ an indeterminate over $A$. Recall from [4] that $\mathcal{A}[X] = \{f = \sum_{i=0}^{n} a_iX^i \in A[X] \mid n \geq 0, a_i \in A_i,\ i = 0, 1, \dots, n\}$.

**Theorem 2.9.** *Let $\mathcal{A} = (A_n)_{n\geq 0}$ be an increasing sequence of rings and $\mathcal{S}$ a multiplicative system of ideals of $A_0$ such that for every $I \in \mathcal{S}$, $\bigcap_{n=1}^{\infty} I^n$ contains some ideal of $\mathcal{S}$. The following conditions are equivalent :*

(1) *The ring $\mathcal{A}[X]$ is $\mathcal{S}$-Noetherian.*

(2) *The ring $A_0$ is $\mathcal{S}$-Noetherian and the $A_0$-module $A = \bigcup_{n=0}^{\infty} A_n$ is $\mathcal{S}$-finite.*

**Proof.** (1) $\Longrightarrow$ (2) Let $Q$ be an ideal of $A_0$. Then $Q\mathcal{A}[X]$ is an $\mathcal{S}$-finite ideal of $\mathcal{A}[X]$. Hence, there exist $a_1, \dots, a_n \in Q$ and $I \in \mathcal{S}$ such that $I(Q\mathcal{A}[X]) \subseteq \langle a_1, \dots, a_n\rangle\mathcal{A}[X]$. Thus $IQ \subseteq \langle a_1, \dots, a_n\rangle A_0$. Hence $A_0$ is $\mathcal{S}$-Noetherian.
Let $n \geq 1$ be an integer. The ideal $X^nA_n\mathcal{A}[X]$ of $\mathcal{A}[X]$ is $\mathcal{S}$-finite. Then there exist $a_1, \dots, a_k \in A_n$ and $I \in \mathcal{S}$ such that $I(X^nA_n\mathcal{A}[X]) \subseteq \langle a_1X^n, \dots, a_kX^n\rangle$. Let $a \in A_n$ and $b \in I$. There exist $f_1(X), \dots, f_k(X) \in \mathcal{A}[X]$ such that $b(aX^n) = \sum_{i=1}^{k} f_i(a_iX^n)$. Identifying coefficients of $X^n$, we obtain $ba = \sum_{i=1}^{k} f_i(0)a_i$ with $f_1(0), \dots, f_k(0) \in A_0$. Therefore, $A_n$ is an $\mathcal{S}$-finite $A_0$-module.
The ideal $Q$ of $\mathcal{A}[X]$ generated by $\{aX^i,\ i \in \mathbb{N}^*,\ a \in A_i\}$ is $\mathcal{S}$-finite, then there exist $I \in \mathcal{S}$, $a_1X^{\alpha_1}, \dots, a_rX^{\alpha_r},\ a_i \in A_{\alpha_i},\ \alpha_i \geq 1$ such that,

$$IQ \subseteq \langle a_kX^{\alpha_k},\ 1 \leq k \leq r\rangle\mathcal{A}[X] .$$

Let $m = \max(\alpha_1, \dots, \alpha_r)$. Then $a_1, \dots, a_r \in A_m$. For a fixed $i > m$. Let $b \in I$ and $y \in A_i$. By definition of $Q$, $yX^i \in Q$. Thus

$$byX^i \in \langle a_kX^{\alpha_k},\ 1 \leq k \leq r\rangle\mathcal{A}[X] .$$

It yields that $byX^i = \sum_{k=1}^{r} a_kX^{\alpha_k}g_k$ with $g_k = \sum_{j=0}^{n_k} g_{k,j}X^j \in \mathcal{A}[X]$. By identification, we get $by = \sum_{k=1}^{r} a_kg_{k,i-\alpha_k}$ with $g_{k,i-\alpha_k} \in A_{i-\alpha_k} \subseteq A_{i-1}$. Hence

$$bA_i \subseteq a_1A_{i-1} + \cdots + a_rA_{i-1} \subseteq A_{i-1} .$$

It follows that $IA_i \subseteq A_{i-1}$. Iterating we get $I^{m-i}A_i \subseteq A_m$. It follows that $JA_i \subseteq A_m$ for some ideal $J$ of $\mathcal{S}$ contained in $\bigcap_{n=0}^{\infty} I^n$. Consequentely, $JA_n \subseteq A_m$ for every $n \geq m$. It yields that $JA = J(\bigcup_{n=0}^{\infty} A_n) = J(\bigcup_{n=m}^{\infty} A_n) = \bigcup_{n=m}^{+\infty} JA_n \subseteq A_m$. Thus $A$ is

an $\mathcal{S}$-finite $A_0$-module.
(2) $\Longrightarrow$ (1) Since the $A_0$-module $A$ is $\mathcal{S}$-finite, there exist $a_1, \ldots, a_n \in A$ and $C \in \mathcal{S}$ such that $CA \subseteq \langle a_1, \ldots, a_n \rangle A_0$. Thus $CA[X] \subseteq \langle a_1, \ldots, a_n \rangle A_0[X]$. Hence the $A_0[X]$-module $A[X]$ is $\mathcal{S}$-finite. On the other hand, $A_0$ is $\mathcal{S}$-Noetherian and for each $I \in \mathcal{S}$, $\bigcap_{k=1}^{\infty} I^k$ contains some ideal of $\mathcal{S}$. By Theorem 2.7, the ring $A_0[X]$ is $\mathcal{S}$-Noetherian. By Corollary 2.6, the $A_0[X]$-module $A[X]$ is $\mathcal{S}$-Noetherian, and so is the submodule $\mathcal{A}[X]$. Thus the ring $\mathcal{A}[X]$ is $\mathcal{S}$-Noetherian. $\square$

**Lemma 2.10.** *Let $A$ be a ring, $\mathcal{S}$ a multiplicative system of ideals of $A$ and $M$ an $\mathcal{S}$-finite $A$-module. If $N$ is a submodule of $M$ maximal among the non-$\mathcal{S}$-finite submodules of $M$, then $[N : M]$ is a prime ideal of $A$.*

**Proof.** Denote $P = [N : M]$. Assume that $P$ is not a prime ideal. Let $a, b \in A \setminus P$ such that $ab \in P$. By maximality of $N$, $N + aM$ is $\mathcal{S}$-finite. Consequently, there exist $n_1, \ldots, n_k \in N$, $m_1, \ldots, m_k \in M$ and $I \in \mathcal{S}$ such that $I(N + aM) \subseteq \langle n_1 + am_1, \ldots, n_k + am_k \rangle$. Since $aN \subseteq N$ and $bx \in [N : a]$ for each $x \in M$ ( $N \neq M$), $N \subset [N : a]$. Then $[N : a]$ is $\mathcal{S}$-finite. It yields that there exist $q_1, \ldots, q_t \in [N : a]$ and $J \in \mathcal{S}$ such that $J[N : a] \subseteq \langle q_1, \ldots, q_t \rangle$. Let $x \in N$, $\alpha \in I$ and $\beta \in J$. We have $\alpha x = \sum_{i=1}^{k} \alpha_i (n_i + am_i)$ with $\alpha_1, \ldots, \alpha_k \in A$. Thus $a \sum_{i=1}^{k} \alpha_i m_i = \alpha x - \sum_{i=1}^{k} \alpha_i n_i \in N$. Hence $y = \sum_{i=1}^{k} \alpha_i m_i \in [N : a]$. Therefore, $\beta y = \sum_{j=1}^{t} \beta_j q_j$ with $\beta_1, \cdots, \beta_t \in A$. Thus $\beta \alpha x = \sum_{i=1}^{k} (\beta \alpha_i) n_i + \beta a y = \sum_{i=1}^{k} (\beta \alpha_i) n_i + \sum_{j=1}^{t} \beta_j (a q_j) \in \langle n_1, \ldots, n_k, aq_1, \ldots, aq_t \rangle$. Hence $JIN \subseteq \langle n_1, \ldots, n_k, aq_1, \cdots, aq_t \rangle \subseteq N$ with $JI \in \mathcal{S}$, so $N$ is $\mathcal{S}$-finite, contradiction. Therefore, $P$ is a prime ideal of $A$. $\square$

Let $A$ be a ring, $\mathcal{S}$ a multiplicative system of finitely generated ideals of $A$, $P$ a prime ideal of $A$ and $M$ an $\mathcal{S}$-finite $A$-module. It is clear that $P$ and $PM$ are $\mathcal{S}$-finite when $P$ contains some ideal in $\mathcal{S}$.

**Theorem 2.11.** *Let $A$ be a ring, $\mathcal{S}$ a multiplicative system of finitely generated ideals of $A$ and $M$ an $\mathcal{S}$-finite $A$-module. Then $M$ is an $\mathcal{S}$-Noetherian $A$-module if and only if for each prime ideal $P$ of $A$ not containing any ideal in $\mathcal{S}$, the submodule $PM$ is $\mathcal{S}$-finite.*

**Proof.** $\Longrightarrow$ Trivial.
$\Longleftarrow$ Assume that $M$ is not $\mathcal{S}$-Noetherian. Let $\mathcal{F}$ be the set of submodules of $M$ which are not $\mathcal{S}$-finite. We order $\mathcal{F}$ by inclusion. Let $(H_\alpha)_{\alpha \in \Lambda}$ be a totally ordered family of $\mathcal{F}$ and $H = \bigcup_{\alpha \in \Lambda} H_\alpha$. Assume that $H \notin \mathcal{F}$. Then there exist $a_1, \ldots, a_n \in H$ and $I \in \mathcal{S}$ such that $IH \subseteq \langle a_1, \ldots, a_n \rangle$. Since the family $(H_\alpha)_{\alpha \in \Lambda}$ is totally ordered, there exists $\alpha \in \Lambda$ such that $a_1, \ldots, a_n \in H_\alpha$. Hence $IH_\alpha \subseteq IH \subseteq \langle a_1, \ldots, a_n \rangle$. Therefore, $H_\alpha$ is $\mathcal{S}$-finite, absurd. Thus $H \in \mathcal{F}$. Therefore $\mathcal{F}$ is inductively ordered.

By Zorn's lemma, $\mathcal{F}$ has a maximal element $N$. By Lemma 2.10, $P = [N : M]$ is a prime ideal of $A$. Let $m_1, \dots, m_k \in M$ and $J \in \mathcal{S}$ such that $JM \subseteq \langle m_1, \dots, m_k \rangle$. If there exists $I \in \mathcal{S}$ such that $IM \subseteq N$, then $IJN \subseteq I\langle am_1, \dots, am_k \rangle \subseteq N$, contradiction (since $I$ is finitely generated, so is the submodule $I\langle m_1, \dots, m_k \rangle$). Therefore, for each $I \in \mathcal{S}$, $IM \nsubseteq N$. Thus $P = [N : M] \subseteq [N : \langle m_1, \dots, m_k \rangle] \subseteq [N : JM] = P : J = P$. Hence, $P = [N : \langle m_1, \dots, m_k \rangle] = [N : m_1] \cap \dots \cap [N : m_k] = [N : m_{i_0}]$ for some $1 \leq i_0 \leq k$. Since $P \neq A$, so $m_{i_0} \notin N$, hence $N + Am_{i_0}$ is $\mathcal{S}$-finite by the maximality of $N$. There exist then $n_1, \dots, n_t \in N$, $a_1, \dots, a_t \in A$ and $I \in \mathcal{S}$ such that $I(N + Am_{i_0}) \subseteq \langle n_1 + a_1 m_{i_0}, \dots, n_t + a_t m_{i_0} \rangle$. Let $x \in N$, $b \in A$ and $\alpha \in I$. There exist $\alpha_1, \dots, \alpha_t \in A$ such that $\alpha(x + bm_{i_0}) = \sum_{i=1}^{t} (\alpha_i n_i + \alpha_i a_i m_{i_0})$. Hence $(\alpha b - \sum_{i=1}^{t} \alpha_i a_i) m_{i_0} = \sum_{i=1}^{t} \alpha_i n_i - \alpha x \in N$. Thus $\alpha b - \sum_{i=1}^{t} \alpha_i a_i \in P$. It yields that $\alpha x = \sum_{i=1}^{t} \alpha_i n_i + (\sum_{i=1}^{t} \alpha_i a_i - \alpha b) m_{i_0} \in \langle n_1, \dots, n_t \rangle + PM$. Since $PM$ is $\mathcal{S}$-finite, there exist $\beta_1, \dots, \beta_r \in PM$ and $L \in \mathcal{S}$ such that $L(PM) \subseteq \langle \beta_1, \dots, \beta_r \rangle \subseteq PM \subseteq N$. Consequently, $(LI)N \subseteq \langle n_1, \dots, n_t, \beta_1, \dots, \beta_r \rangle \subseteq N$. Hence $N$ is $\mathcal{S}$-finite, absurd. Therefore, $M$ is an $\mathcal{S}$-Noetherian $A$-module. □

**Corollary 2.12.** *Let $A$ be a ring and $\mathcal{S}$ a multiplicative system of finitely generated ideals of $A$. Then the ring $A$ is $\mathcal{S}$-Noetherian, if and only if, each prime ideal of $A$ not containing any ideal in $\mathcal{S}$ is $\mathcal{S}$-finite.*

The next example shows that for each $n \geq 1$, there exists an $n$-dimensional $\mathcal{S}$-Noetherian ring which is not Noetherian.

**Example 2.13.** Let $A$ be a finite dimensional valuation domain, $P$ its height one prime ideal, $I \subseteq P$ a finitely generated ideal and $\mathcal{S} = \{I^n,\ n \geq 1\}$. Then $A$ is $\mathcal{S}$-Noetherian. Indeed, let $Q$ be a nonzero prime ideal of $A$. Thus $IQ \subseteq I \subseteq P \subseteq Q$. Hence $Q$ is $\mathcal{S}$-finite.

**Example 2.14.** The hypothesis that $\mathcal{S}$ consists of finitely generated ideals is necessary. Indeed, let $X_1, X_2, \dots$ be a countably family of indeterminates over a field $K$, $A = K[X_n,\ n \geq 1]/\langle X_n^n,\ n \geq 1 \rangle$, $M = \langle \bar{X_n},\ n \geq 1 \rangle A$ and $\mathcal{S} = \{M^n,\ n \geq 1\}$. The only prime ideal of $A$ is $M$. Assume that $A$ is $\mathcal{S}$-Noetherian. Then $M$ is $\mathcal{S}$-finite. Hence there exist $k,\ m \in \mathbb{N}^*$ such that $M^k M \subseteq \langle \bar{X_1}, \dots, \bar{X_m} \rangle$. Then $M^l = 0$ for some $l \geq 1$, absurd. Hence the ring $A$ is not $\mathcal{S}$-Noetherian.

**Corollary 2.15.** *Let $A \subseteq B$ be a rings extension and $\mathcal{S}$ a multiplicative system of finitely generated ideals of $A$ such that $B$ is an $\mathcal{S}$-finite $A$-module. Then the ring $A$ is $\mathcal{S}$-Noetherian if and only if $B$ is $S$-Noetherian.*

**Proof.** $\Longrightarrow$ The $A$-module $B$ is $\mathcal{S}$-finite. By Corollary 2.5, the $A$-module $B$ is $\mathcal{S}$-Noetherian. Hence, the ring $B$ is $\mathcal{S}$-Noetherian.
$\Longleftarrow$ By Theorem 2.11, the $A$-module $B$ is $\mathcal{S}$-Noetherian. Therefore, the ring $A$ is $\mathcal{S}$-Noetherian (as an $A$-submodule of $B$). □

Let $A$ be a ring and $M$ an $A$-module. Recall that Nagata introduced the extension ring of $A$ called the idealization of $M$ in $A$, denoted here by $A(+)M$, whose underlying abelian group is $A \times M$ and multiplication defined by:

$$(a, x)(a', x') = (aa', ax' + a'x), \quad \text{for every} \quad (a, x), (a', x') \in A(+)M\,.$$

It is well known that $A(+)M$ is a commutative ring with identity element $(1, 0)$. (It is also called the trivial extension of $A$ by $M$.) For more details see [2] and [4].

Let $A$ be an ideal of $A$. Note that $I(+)IM$ is the extension of $I$ in $A(+)M$, so $\mathcal{S}_1 = \{I(+)IM,\ I \in \mathcal{S}\}$ is clearly a multiplicative system of ideals of $A(+)M$. As $A \subseteq A(+)M$, we get $A(+)M$ is $\mathcal{S}$-Noetherian if and only if $A(+)M$ is $\mathcal{S}_1$-Noetherian.

**Proposition 2.16.** *Let $A$ be a ring, $\mathcal{S}$ a multiplicative system of finitely generated ideals of $A$ and $M$ an $A$-module. Denote $\mathcal{S}_1 = \{I(+)IM,\ I \in \mathcal{S}\}$. Then the ring $A(+)M$ is $\mathcal{S}_1$-Noetherian if and only if the ring $A$ is $\mathcal{S}$-Noetherian and the $A$-module $M$ is $\mathcal{S}$-finite.*

**Proof.** $\Longrightarrow$ The map $\phi\colon A(+)M \longrightarrow A$ defined by $\phi(a, x) = a$ for every $(a, x) \in A(+)M$ is a surjective homomorphism of rings. Since $A(+)M$ is $\mathcal{S}_1$-Noetherian, the ring $A$ is $\phi(\mathcal{S}_1) = \mathcal{S}$-Noetherian.
The ideal $\{0\}(+)M$ of $A(+)M$ is $\mathcal{S}_1$-finite. Then there exist $m_1, \dots m_k \in M$ and $I \in \mathcal{S}$ such that $(I(+)IM)(\{0\}(+)M) \subseteq \langle (0, m_1), \dots, (0, m_k)\rangle A(+)M$. Therefore, $IM \subseteq \langle m_1, \dots, m_k\rangle A$. It yields that the $A$-module $M$ is $\mathcal{S}$-finite.
$\Longleftarrow$ It is clear that the extension $A \subseteq A(+)M$ is $\mathcal{S}$-finite. Then $A$ is $\mathcal{S}$-Noetherian if and only if $A(+)M$ is $\mathcal{S}$-Noetherian by Corollary 2.15. Thus the ring $A(+)M$ is $\mathcal{S}_1$-Noetherian. □

**Example 2.17.** Let $A$ be an $n$-dimensional nonNoetherian integral domain. Assume that $P = \cap\{Q \mid (0) \neq Q \in \mathrm{Spec}(A)\}$ is a nonzero ideal of $A$ and let $I \subseteq P$ be a finitely generated nonprincipal ideal of $A$. Set $\mathcal{S} = \{I^k,\ k \geq 1\}$. Clearly $A$ is an $\mathcal{S}$-Noetherian ring (since each nonzero prime ideal of $A$ contains $I$). Then for each $\mathcal{S}$-finite $A$-module $M$, the ring $A(+)M$ is $\mathcal{S}_1$-Noetherian, by Proposition 2.16, where $\mathcal{S}_1 = \{I(+)IM,\ I \in \mathcal{S}\}$.

Let $A$ be a ring and $P \in \mathrm{Spec}(A)$. Denote $\mathcal{S}_P = \{I \text{ ideal of } A \text{ such that } I \nsubseteq P\}$. $\mathcal{S}_P$ is clearly a multiplicative system of ideals of $A$.

**Theorem 2.18.** *The following assertions are equivalent for an $A$-module $E$ :*

1. *The module $E$ is Noetherian.*
2. *The module $E$ is $\mathcal{S}_P$-Noetherian for every $P \in \mathrm{Spec}(A)$.*
3. *The module $E$ is $\mathcal{S}_M$-Noetherian for every $M \in \mathrm{Max}(A)$.*

**Proof.** The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are simple.
(3) $\Longrightarrow$ (1) Let $N$ be a submodule of $E$. For each $M \in \mathrm{Max}(A)$, there exist $I_M \in \mathcal{S}_M$ and a finitely generated submodule $F_M \subseteq N$ of $E$ such that $I_M N \subseteq F_M$. Let $Q = \langle I_M,\ M \in \mathrm{Max}(A)\rangle$. Since $I_M \nsubseteq M$ for each maximal ideal $M$ of $A$, we get $Q = A$. Therefore there exist $M_1, \cdots, M_r \in \mathrm{Max}(A)$ such that $A = \langle I_{M_1}, \dots, I_{M_r}\rangle$. Hence $N = AN = \langle I_{M_1}, \cdots, I_{M_r}\rangle N = I_{M_1}N + \cdots + I_{M_r}N \subseteq F_{M_1} + \cdots + F_{M_r} \subseteq N$. Thus $N = F_{M_1} + \cdots + F_{M_r}$ is finitely generated. □

**Corollary 2.19.** *The following assertions are equivalent for a ring A :*

1. *The ring A is Noetherian.*
2. *The ring A is $\mathcal{S}_P$-Noetherian for every $P \in \mathrm{Spec}(A)$.*
3. *The ring A is $\mathcal{S}_M$-Noetherian for every $M \in \mathrm{Max}(A)$.*

*Questions.* We end this paper by posing two questions.

1. Let $A$ be an integral domain with quotient field $K$ and $\mathcal{S}$ a multiplicative system of ideals of $A$ such that $A$ is $\mathcal{S}$-Noetherian. Does it follow that the generalized fraction ring $A_{\mathcal{S}} = \{x \in K;\ xH \subseteq A$ for some $H \in \mathcal{S}\}$ is Noetherian?
2. Under the hypothesis of Theorem 2.7, is the power series ring $A[[X]]$ $\mathcal{S}$-Noetherian?

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