ARCHIVUM MATHEMATICUM (BRNO)
Tomus 57 (2021), 175–194

ALGEBRAIC RESTRICTIONS ON GEOMETRIC REALIZATIONS OF CURVATURE MODELS

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Abstract. We generalize a previous result concerning the geometric realizability of model spaces as curvature homogeneous spaces, and investigate applications of this approach. We find algebraic restrictions to realize a model space as a curvature homogeneous space up to any order, and study the implications of geometrically realizing a model space as a locally symmetric space. We also present algebraic restrictions to realize a curvature model as a homothety curvature homogeneous space up to even orders, and demonstrate that for certain model spaces and realizations, homothety curvature homogeneity implies curvature homogeneity.

1. Introduction

Let (M, g) be a smooth pseudo-Riemannian manifold of dimension n, let ∇ be the Levi-Civita connection, and let $P \in M$. The tangent space $T_P M$ of M at P is a real vector space, the metric g_P at P is an inner product on V, the Riemann curvature tensor R_P and its covariant derivatives $\nabla^k R_P$ are tensors of type $(0, 4+k)$ on this vector space that satisfy certain properties. Thus, roughly speaking, the tuple $(T_P M, g_P, R_P, \nabla R_P, \dots, \nabla^k R_P)$ is an algebraic portrait of the curvature of the manifold at the point P. Unless otherwise stated, (M, g) will always denote a smooth pseudo-Riemannian manifold of dimension n, ∇ will denote the Levi-Civita connection, and the curvature tensor R of type $(0, 4)$ on M is defined by

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)\,.$$

Conversely, given a vector space V, a nondegenerate inner product $\langle \cdot, \cdot \rangle$ on V, and tensors $A^i \in \otimes^{4+i} V^*$ $(i = 0, 1, \dots, k)$ satisfying the same properties as the curvature tensor and its covariant derivatives (these symmetries are described in detail in the next section), it is known [1] that there exists a manifold M, a metric g on M with the same signature as $\langle \cdot, \cdot \rangle$, and a point $P \in M$ with a linear isometry $\Phi\colon V \to T_P M$ satisfying $\Phi^* \nabla^i R_P = A^i$ for $i = 0, 1, \dots, k$. We say that (any subset of) the tuple $\mathcal{M}_k = (V, \langle \cdot, \cdot \rangle, A^0, A^1, \dots, A^k)$ is a *model space* (or a k-model), the tensors A^i are known as *algebraic curvature tensors*, and in this

2020 *Mathematics Subject Classification*: primary 53B30; secondary 15A69, 53B15.
Key words and phrases: curvature model, curvature homogeneous, homothethy curvature homogeneous.
Received January 4, 2021, revised May 2021. Editor J. Slovák.
DOI: 10.5817/AM2021-3-175

instance we say that (M,g) is a *geometric realization* of $\mathcal{M}_k$ at P. Two model spaces $\mathcal{M} = (V, \alpha_0, \ldots, \alpha_k)$ and $\mathcal{M}' = (W, \beta_0, \ldots, \beta_k)$ are *isomorphic* (written $\mathcal{M} \cong \mathcal{M}'$) if there is a vector space isomorphism $\Phi\colon V \to W$ with $\Phi^*\beta_i = \alpha_i$ for $i = 0, \ldots, k$, where Φ^* denotes precomposition by Φ.

There are interesting relationships between the algebraic information a model space can offer and a corresponding geometric realization. The most basic of these is the classical fact that, up to local isometry, there is a unique manifold (a space form) that geometrically realizes a 0-model of constant sectional curvature. Other examples of this study includes [2, 7, 8, 12, 17, 18], and very recently, [4].

A large area of study that examines this relationship is that of curvature homogeneity and related concepts, which can be defined using the language of model spaces. Let $\mathcal{M}_k$ be a k-model, and let $\mathcal{W}_k$ be the same model space as $\mathcal{M}$ but with the inner product omitted (sometimes referred to as a *weak* model space [9]). If (M,g) is a pseudo-Riemannian manifold and $P \in M$, let

$$\mathcal{M}_k(P) = (T_P M, g_P, R_P, \ldots, \nabla^k R_P)\,,$$

and similarly $\mathcal{W}_k(P)$ is the same as $\mathcal{M}_k(P)$ but with the metric g_p omitted. The manifold (M,g) is *curvature homogeneous up to order k (CH_k)* or *k-modeled on $\mathcal{M}_k$* if for every $P \in M$ we have $\mathcal{M}_k(P) \cong \mathcal{M}_k$. Similarly, (M,g) is *weakly curvature homogeneous up to order k (WCH_k)* or *weakly k-modeled on $\mathcal{W}_k$* if for every $P \in M$ we have $\mathcal{W}_k(P) \cong \mathcal{W}_k$. Finally, the manifold (M,g) is *homothety curvature homogeneous up to order k (HCH_k)* or *homothety k-modeled on $\mathcal{M}_k$* if there is a smooth real valued function λ so that for every $P \in M$ we have

$$\mathcal{M}_k(P) \cong (V, \langle\cdot,\cdot\rangle, \lambda(P)A_0, \lambda(P)^{3/2}A^1, \ldots, \lambda(P)^{\frac{k+2}{2}}A^k)\,.$$

The notion of curvature homogeneity originated with Singer in 1960 [16]. See [9] for more information concerning weak curvature homogeneity. Homothety curvature homogeneity originated with the work in [13] and then subsequently in [14]; see also [5, 6], and [4]. Our definition above is equivalent to the original definition given in [13], as was established in [5] or [6].

The main goal of this paper is to generalize the result in [12] (listed there as Proposition 3.3 on page 48 of [12]), which describes an algebraic obstruction for a 0-model to be geometrically realizable on a CH_0 manifold, and provide several applications of this generalization and method.

In Section 2 we describe this Proposition 3.3 in detail and offer more technical background material concerning the symmetries of higher order algebraic curvature tensors (i.e., tensors $A^i \in \otimes^{4+i}V^*$ for $i \geq 1$).

In Section 3, we establish Lemma 3.1, an observation that is the foundation of the applications to follow. The first of these applications is in Section 4: we extend the work of [12] to establish criteria for a model space to be geometrically realized as a CH_k space for $k \geq 1$ in Theorem 4.1. We rephrase this conclusion in Corollary 4.2 as a necessary condition on a model space to be geometrically realized as a CH_k space.

Every locally symmetric space (i.e., $\nabla R = 0$) is locally homogeneous, and is therefore CH_1. We study an application of our methods to locally symmetric spaces in Section 5. In Corollary 5.1, we use the results in Section 4 to provide a collection of equations that an algebraic curvature tensor must satisfy (on an orthonormal basis) to be the curvature tensor of a locally symmetric space. We study these equations in detail in dimension three, and find a certain converse to Corollary 5.1 in Theorem 5.3: each solution to that system of equations corresponds to a curvature model of some locally symmetric space in dimension three.

We exhibit applications of Lemma 3.1 relating to homothety curvature homogeneity in Section 6. After some preliminary observations relating to this situation, we again derive a set of algebraic conditions in Theorem 6.3 that must be satisfied for a k-model to be geometrically realized as an HCH_k space, where $k \geq 2$ is even. As expected, since CH_k implies HCH_k implies WCH_k, the set of algebraic restrictions becomes strictly less demanding, however we find the same number of unknowns in the associated system of equations. Remark 6.10 details this observation. We close the section and paper with an investigation of the HCH_0 situation which curiously escapes our methods in Theorem 6.8, in which we present a family of manifolds for which HCH_0 implies CH_0.

2. Preliminaries

There are two major preliminary notions before we can state our main results.

2.1. **Review of Proposition 3.3.** We begin by describing Proposition 3.3 in [12] and its proof:

> **Proposition 3.3** [12]. Let (M, g) be a curvature homogeneous space. Then, in a neighborhood U_P of each point $P \in M$, there exists a tensor field S of type $(1, 2)$ such that for any $m \in U_P$,

$$S_X \cdot g = 0 \quad \text{for every } X \in T_m M \tag{2.a}$$

$$\mathfrak{S}_{X,Y,Z}(S_X \cdot R)(Y, Z, U, V) = 0 \quad \text{for every } X, Y, Z, U, V \in T_m M\,. \tag{2.b}$$

> Here, $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum in X, Y, and Z, and S_X acts as a derivation on the tensor algebra.

The proof of Proposition 3.3 is easy to describe. If (M, g) is CH_0 and modeled on the 0-model $(V, \langle\cdot,\cdot\rangle, A)$, then there exists a frame $\{E_1, \ldots, E_n\}$ for the tangent bundle TU_P so that at any point $m \in U_P$,

$$g_m(E_i, E_j) = \langle e_i, e_j\rangle \quad \text{is constant, and}$$
$$R_m(E_i, E_j, E_k, E_\ell) = A(e_i, e_j, e_k, e_\ell) \quad \text{constant,}$$

where $\{e_1, \ldots, e_n\}$ is some basis for V. Define a new connection $\tilde{\nabla}$ on U_P so that this frame forms an absolute parallelism, i.e., $\tilde{\nabla}_{E_i} E_j = 0$. Because of this, and since these entries are constant, we easily see (a proof of a more general statement appears below in Lemma 3.1) that

$$\tilde{\nabla} g(E_i, E_j; E_d) = 0 \quad \text{and} \quad \tilde{\nabla} R(E_i, E_j, E_k, E_\ell; E_d) = 0\,.$$

Define $S = \nabla - \tilde{\nabla}$ as the tensor of type (1,2) in Proposition 3.3, where again, ∇ is the Levi-Civita connection. Equation (2.a) now follows because $\nabla g = 0$, and Equation (2.b) follows by the second Bianchi Identity. While the authors never specifically mention this, Proposition 3.3 holds if (M, g) is not Riemannian (i.e., pseudo-Riemannian). In addition, if one only assumes that (M, g) is WCH_0, then only Equation (2.b) is required to hold.

2.2. **Higher order curvature symmetries.** Let (M, g) be a pseudo-Riemannian manifold, and ∇ the Levi-Civita connection. The Riemann curvature tensor R satisfies the following symmetries; all capital letters (except R) are tangent vectors:

$$\begin{aligned} &R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y)\,,\\ &R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0\,, \end{aligned} \tag{2.c}$$

The symmetries of ∇R are also easy to recall:

$$\begin{aligned} \nabla R(X,Y,Z,W;D) &= -\nabla R(Y,X,Z,W;D) = \nabla R(Z,W,X,Y;D)\,,\\ 0 &= \nabla R(X,Y,Z,W;D) + \nabla R(Z,X,Y,W;D) + \nabla R(Y,Z,X,W;D)\,,\\ 0 &= \nabla R(X,Y,Z,W;D) + \nabla R(X,Y,D,Z;W) + \nabla R(X,Y,W,D;Z)\,. \end{aligned} \tag{2.d}$$

The symmetries of $\nabla^i R$ for $i \geq 2$ are more complicated. Using the nondegenerate metric g, we may characterize the associated curvature operator $\nabla^i \mathcal{R}$ of type $(1, 3+i)$ by

$$g(\nabla^i \mathcal{R}(X,Y;X_1,\ldots,X_i)Z,W) = \nabla^i R(X,Y,Z,W;X_1,\ldots,X_i)\,.$$

The following commutation relation of curvature operators is known (see Equation (1.2.d) on page 9 of [9]) for $i+1 \geq 2$. For convenience, write $\nabla^i \mathcal{R} = \mathcal{R}^i$:

$$\begin{aligned} &\mathcal{R}^{i+1}(X,Y;X_1,\ldots,X_{i-1},U,V) - \mathcal{R}^{i+1}(X,Y;X_1,\ldots,X_{i-1},V,U)\\ &\quad = \mathcal{R}(V,U)\mathcal{R}^{i-1}(X,Y;X_1,\ldots,X_{i-1})\\ &\qquad - \mathcal{R}^{i-1}(\mathcal{R}(V,U)X,Y;X_1,\ldots,X_{i-1})\\ &\qquad - \mathcal{R}^{i-1}(X,\mathcal{R}(V,U)Y;X_1,\ldots,X_{i-1})\\ &\qquad - \sum_{1\leq j\leq i-1} \mathcal{R}^{i-1}(X,Y;X_1,\ldots,\mathcal{R}(V,U)X_j,\ldots,X_{i-1})\\ &\qquad - \mathcal{R}^{i-1}(X,Y;X_1,\ldots,X_{i-1})\mathcal{R}(V,U)\,. \end{aligned} \tag{2.e}$$

It will be convenient to express this, to whatever extent possible, as an identity relating the curvature tensors of type $(0, 4+i)$. Evaluating Equation (2.e) at Z and computing the inner product of the result with W yields the following relation:

$$\begin{aligned} &\nabla^{i+1} R(X,Y,Z,W;X_1,\ldots,X_{i-1},U,V)\\ &\qquad - \nabla^{i+1} R(X,Y,Z,W;X_1,\ldots,X_{i-1},V,U)\\ &\quad = R(V,U,\nabla^{i-1}\mathcal{R}(X,Y;X_1,\ldots,X_{i-1})Z,W)\\ &\qquad - \nabla^{i-1} R(\mathcal{R}(V,U)X,Y,Z,W;X_1,\ldots,X_{i-1})\\ &\qquad - \nabla^{i-1} R(X,\mathcal{R}(V,U)Y,Z,W;X_1,\ldots,X_{i-1})\\ &\qquad - \sum_{1\leq j\leq i-1} \nabla^{i-1} R(X,Y,Z,W;X_1,\ldots,\mathcal{R}(V,U)X_j,\ldots,X_{i-1})\\ &\qquad - \nabla^{i-1} R(X,Y,\mathcal{R}(V,U)Z,W;X_1,\ldots,X_{i-1})\,. \end{aligned} \tag{2.f}$$

Now, let $\mathcal{M}_k = (V, \langle\cdot,\cdot\rangle, A^0, \dots, A^k)$ be a k-model. For convenience, write $A^0 = A$. The goal of this subsection is to describe in detail the symmetries of the tensors A^i if they are to mimic the behavior of their counterparts.

Let all lowercase letters be vectors in V. We define $A \in \otimes^4 V^*$ to satisfy

$$\begin{aligned} &A(x,y,z,w) = -A(y,x,z,w) = A(z,w,x,y)\,,\\ &A(x,y,z,w) + A(z,x,y,w) + A(y,z,x,w) = 0\,, \end{aligned} \tag{2.g}$$

Similarly, the tensor $A^1 \in \otimes^5 V^*$ is designed to mimic ∇R at a point, and is defined to satisfy the following relations:

$$\begin{aligned} A^1(x,y,z,w;d) &= -A^1(y,x,z,w;d) = A^1(z,w,x,y;d)\,,\\ 0 &= A^1(x,y,z,w;d) + A^1(z,x,y,w;d) + A^1(y,z,x,w;d)\,,\\ 0 &= A^1(x,y,z,w;d) + A^1(x,y,d,z;w) + A^1(x,y,w,d;z)\,. \end{aligned} \tag{2.h}$$

Let $\mathcal{A}^i$ be the operator associated to A^i characterized by the equation

$$\langle \mathcal{A}^i(x,y;x_1,\dots,x_i)z, w\rangle = A^i(x,y,z,w;x_1,\dots,x_i)\,,$$

and again for convenience we write $\mathcal{A}^0 = \mathcal{A}$. In view of Equation (2.f), the tensors A^{i+1} are defined to satisfy the following commutation relation:

$$\begin{aligned} &A^{i+1}(x,y,z,w;x_1,\dots,x_{i-1},u,v) - A^{i+1}(x,y,z,w;x_1,\dots,x_{i-1},v,u)\\ &\quad= A(v,u,\mathcal{A}^{i-1}(x,y;x_1,\dots,x_{i-1})z,w)\\ &\qquad - A^{i-1}(\mathcal{A}(v,u)x,y,z,w;x_1,\dots,x_{i-1})\\ &\qquad - A^{i-1}(x,\mathcal{A}(v,u)y,z,w;x_1,\dots,x_{i-1})\\ &\qquad - \sum_{1\le j\le i-1} A^{i-1}(x,y,z,w;x_1,\dots,\mathcal{A}(v,u)x_j,\dots,x_{i-1})\\ &\qquad - A^{i-1}(x,y,\mathcal{A}(v,u)z,w;x_1,\dots,x_{i-1})\,. \end{aligned} \tag{2.i}$$

For example, and for later use, A^2 must satisfy the following:

$$\begin{aligned} &A^2(x,y,z,w;u,v) - A^2(x,y,z,w;v,u)\\ &\quad= A(v,u,\mathcal{A}(x,y)z,w)\\ &\qquad -A(\mathcal{A}(v,u)x,y,z,w) - A(x,\mathcal{A}(v,u)y,z,w)\\ &\qquad -A(x,y,\mathcal{A}(v,u)z,w)\,. \end{aligned} \tag{2.j}$$

For these reasons, for $i \ge 2$ we define $A^i \in \otimes^{4+i} V^*$ to be an *algebraic curvature tensor*[1] if it and its associated operator satisfy the relation in Equation (2.i). In addition, these tensors must also be antisymmetric in the first and second slots, be symmetric in the (1,2) and (3,4) slots, and the cyclic sum in slots one through three (first Bianchi identity) and slots three through five (second Bianchi identity) must be zero. If $i = 0$ or 1, then A^i must satisfy Equations (2.g) or (2.h).

[1]The tensors A^i for $i \ge 1$ are sometimes called an *covariant derivative algebraic curvature tensors* to indicate its relation to $\nabla^i R$ (see page 18 of [9]). We simply refer to any of these as algebraic curvature tensors.

3. Main algebraic result

In this short section, we present the following observation that we will make frequent use of.

Lemma 3.1. *Suppose T is a tensor of type $(0,s)$ on (M,g). Assume:*

(1) *the components of T are constant on some moving frame field $\{F_1,\dots,F_n\}$,*

(2) *we define the connection $\tilde{\nabla}$ by declaring that $\tilde{\nabla}_{F_i}F_j=0$ for all i,j, and*

(3) *we define the tensor S of type $(1,2)$ by $S=\nabla-\tilde{\nabla}$.*

Then for all $d,i_1,\dots,i_s\in\{1,\dots,n\}$,

$$S_{F_d}\cdot T(F_{i_1},\dots,F_{i_s})=\nabla T(F_{i_1},\dots,F_{i_s};F_d)\,,$$

where S_{F_d} acts as a derivation on the tensor algebra.

Proof. By definition, $S_{F_d}F_i=\nabla_{F_d}F_i-\tilde{\nabla}_{F_d}F_i=\nabla_{F_d}F_i$. Since $T(F_{i_1},\dots,F_{i_s})$ are constant, it follows that $F_d(T(F_{i_1},\dots,F_{i_s}))=0$. Then

$$\begin{aligned}S_{F_d}\cdot T(F_{i_1},\dots,F_{i_s})&=-T(S_{F_d}F_{i_1},F_{i_2},\dots,F_{i_s})-\dots-T(F_{i_1},\dots,S_{F_d}F_{i_s})\\&=-T(\nabla_{F_d}F_{i_1},F_{i_2},\dots,F_{i_s})-\dots-T(F_{i_1},\dots,\nabla_{F_d}F_{i_s})\\&=F_d(T(F_{i_1},\dots,F_{i_s}))\\&\quad-T(\nabla_{F_d}F_{i_1},F_{i_2},\dots,F_{i_s})-\dots-T(F_{i_1},\dots,\nabla_{F_d}F_{i_s})\\&=\nabla T(F_{i_1},\dots,F_{i_s};F_d)\,.\end{aligned}$$

□

4. Algebraic restrictions for CH_k manifolds

Our first application of Lemma 3.1 presents a generalization of Proposition 3.3 of [12] to CH_k manifolds for $k\geq 1$.

Theorem 4.1. *Suppose (M,g) is CH_k. At any $P\in M$ there exists a tensor S of type $(1,2)$ on T_PM so that for all $X,Y,Z,W,U,V,X_1,\dots,X_{k-1}\in T_PM$ we have*

$$S_X\cdot g=0\quad\text{for every}\quad X\in T_PM\,,\tag{4.a}$$

$$\mathfrak{S}_{X,Y,Z}(S_X\cdot R)(Y,Z,U,V)=0\quad\text{for every}\quad X,Y,Z,U,V\in T_PM\,,\tag{4.b}$$

and for every $i=1,\dots,k$,

$$\begin{aligned}&S_V\cdot\nabla^iR(X,Y,Z,W;X_1,\dots,X_{i-1},U)\\&\qquad-S_U\cdot\nabla^iR(X,Y,Z,W;X_1,\dots,X_{i-1},V)\\&\quad=R(V,U,\nabla^{i-1}\mathcal{R}(X,Y;X_1,\dots,X_{i-1})Z,W)\\&\qquad-\nabla^{i-1}R(\mathcal{R}(V,U)X,Y,Z,W;X_1,\dots,X_{i-1})\\&\qquad-\nabla^{i-1}R(X,\mathcal{R}(V,U)Y,Z,W;X_1,\dots,X_{i-1})\\&\qquad-\sum_{1\leq j\leq i-1}\nabla^{i-1}R(X,Y,Z,W;X_1,\dots,\mathcal{R}(V,U)X_j,\dots,X_{i-1})\\&\qquad-\nabla^{i-1}R(X,Y,\mathcal{R}(V,U)Z,W;X_1,\dots,X_{i-1})\,.\end{aligned}\tag{4.c}$$

Here, $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum in X, Y, and Z, and $S.$ acts as a derivation on the tensor algebra.

The proof of Theorem 4.1 is very similar to that of Proposition 3.3 in [12], but additionally uses the curvature symmetries in Equation (2.f) and Lemma 3.1.

Proof of Theorem 4.1. Let $P \in M$. Since (M, g) is CH_k, there exists a local orthonormal frame field $\{F_1, \dots, F_n\}$ near P so that the components of the tensors $g, R, \nabla R, \dots, \nabla^k R$ are constant. Define the connection $\tilde{\nabla}$ so that $\tilde{\nabla}_{F_i} F_j = 0$ for all i, j, and the tensor S of type (1,2) as $S = \nabla - \tilde{\nabla}$. According to Lemma 3.1,

$$S_X \cdot g(U, V) = \nabla g(U, V; X) = 0\,,$$

and

$$\mathfrak{S}_{X,Y,Z}(S_X \cdot R)(Y, Z, U, V) = \mathfrak{S}_{X,Y,Z} \nabla R(Y, Z, U, V; X) = 0$$

by the second Bianchi identity. Finally, observe that by Lemma 3.1

$$\begin{aligned} S_V \cdot \nabla^i R(X, Y, Z, W; X_1, \dots, X_{i-1}, U) - S_U \cdot \nabla^i R(X, Y, Z, W; X_1, \dots, X_{i-1}, V) \\ = \nabla^{i+1} R(X, Y, Z, W; X_1, \dots, X_{i-1}, U, V) \\ - \nabla^{i+1} R(X, Y, Z, W; X_1, \dots, X_{i-1}, V, U)\,, \end{aligned}$$

and so Equation (4.c) is just a restatement of the symmetries for the tensor $\nabla^{i+1} R$ in Equation (2.f). □

If one is further aware of what a CH_k space is modeled on, then the following corollary provides a necessary condition on this model space.

Corollary 4.2. *If (M, g) is CH_k and modeled on $\mathcal{M}_k = (V, \langle\cdot,\cdot\rangle, A, A^1, \dots, A^k)$, then there must exist a tensor S of type $(1,2)$ on V that solves the following equations:*

$$\langle S_x u, v\rangle + \langle u, S_x v\rangle = 0 \quad \text{for every} \quad x, u, v \in V\,, \tag{4.d}$$

$$\mathfrak{S}_{x,y,z}(S_x \cdot A)(y, z, u, v) = 0 \quad \text{for every} \quad x, y, z, u, v \in V\,, \tag{4.e}$$

and for every $i = 1, \dots, k$,

$$\begin{aligned} S_v \cdot A^i(x, y, z, w; x_1, \dots, x_{i-1}, u) - S_u \cdot A^i(x, y, z, w; x_1, \dots, x_{i-1}, v) \\ = A(v, u, \mathcal{A}^{i-1}(x, y; x_1, \dots, x_{i-1}) z, w) \\ - A^{i-1}(\mathcal{A}(v, u) x, y, z, w; x_1, \dots, x_{i-1}) \\ - A^{i-1}(x, \mathcal{A}(v, u) y, z, w; x_1, \dots, x_{i-1}) \\ - \sum_{1 \le j \le i-1} A^{i-1}(x, y, z, w; x_1, \dots, \mathcal{A}(v, u) x_j, \dots, x_{i-1}) \\ - A^{i-1}(x, y, \mathcal{A}(v, u) z, w; x_1, \dots, x_{i-1}) \end{aligned} \tag{4.f}$$

for every $x, y, z, w, u, v, x_1, \dots, x_{i-1} \in V$. Here, $\mathfrak{S}_{x,y,z}$ denotes the cyclic sum in x, y, and z, and $S.$ acts as a derivation on the tensor algebra.

Remark 4.3. There is an important difference between the CH_0 requirements in [12] and those in Corollary 4.2. Namely, on a certain frame field, the tensor S has $S_{F_i} F_j = \nabla_{F_i} F_j$, and so by observation in Equations (2.a) and (2.b), $S = 0$ will solve the corresponding system for a CH_0 manifold as given in [12]. However, as those authors point out, such a solution will not produce the correct curvature if (M, g) is not flat. The requirement for S given in Corollary 4.2 for $k \geq 1$ is generally *not* solved if $S = 0$. Thus for $k \geq 1$, a model space can be ruled out entirely algebraically prior to any consideration of the curvature.

5. An application to locally symmetric spaces

We do not attempt to analyze the system of Equations (4.d), (4.e), and (4.f) in full generality. Rather, we consider an application of our approach to locally symmetric spaces. A manifold (M, g) is *locally symmetric* if $\nabla R = 0$, and such manifolds are locally homogeneous. As such, they are CH_k for all k, and in particular, CH_1. If one knows that (M, g) is $1-$modeled on $\mathcal{M}_1 = (V, \langle\cdot,\cdot\rangle, A, 0)$, for $i = 1$ (and $A^1 = 0$) in Corollary 4.2, we must have (see also Equation (2.j))

$$\begin{aligned} 0 = {} & A\big(v, u, \mathcal{A}(x, y)z, w\big) \\ & - A\big(\mathcal{A}(v, u)x, y, z, w\big) - A\big(x, \mathcal{A}(v, u)y, z, w\big) \\ & - A\big(x, y, \mathcal{A}(v, u)z, w\big) . \end{aligned} \tag{5.a}$$

We aim to express Equation (5.a) on an orthonormal basis $\{e_1, \ldots, e_n\}$ and using only the $(0, 4)$ tensor A in Corollary 5.1 below. Let $A_{ijk\ell} = A(e_i, e_j, e_k, e_\ell)$ be the components of A relative to this basis, and suppose $\langle e_i, e_i\rangle = \epsilon_i = \pm 1$. Then

$$\mathcal{A}(e_i, e_j)e_k = \sum_p \mathcal{A}_{ijk}{}^p e_p, \text{ so } \mathcal{A}_{ijk}{}^p = \epsilon_p A_{ijkp} .$$

The following corollary is now immediate from Corollary 4.2.

Corollary 5.1. *Suppose (M, g) is a locally symmetric space that is 0-modeled on $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, A)$. Then on any orthonormal basis with $\langle e_i, e_j\rangle = \epsilon_i$, the following equation must hold for all i, j, k, ℓ, s, t:*

$$\sum_p \epsilon_p A_{ijkp} A_{tsp\ell} = \sum_p \epsilon_p A_{tskp} A_{ijp\ell} + \sum_p \epsilon_p A_{tsip} A_{pjk\ell} + \sum_p \epsilon_p A_{tsjp} A_{ipk\ell} \tag{5.b}$$

Remark 5.2. Equation (5.b) is symmetric in (i, j), (k, ℓ), and (s, t). Thus, since there are $\frac{n^2(n^2-1)}{12}$ independent entries of A, there are that many unknowns and $\binom{n}{2}^3 = \frac{n^3(n-1)^3}{8}$ equations. Thus this system has more equations than unknowns for $n \geq 3$. In [12], the authors note that there are some dependencies in the Equations ((2.a) and (2.b), although we do not investigate this possibility here in Equation (5.b).

It is the goal of this section is to provide a converse to Corollary 5.1 in the context of three-dimensional locally symmetric spaces. While Corollary 5.1 states that a three-dimensional locally symmetric space must satisfy Equation (5.b), we devote this section to proving the following:

Theorem 5.3. *Suppose $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, A)$ is a model space with $\dim(V) = 3$ satisfying Equation (5.b). Then there exists a three-dimensional manifold (M, g) which is locally symmetric and 0-modeled on $\mathcal{M}$.*

The proof of this result is broken up into two cases: the inner product is either positive definite, or Lorentzian.

5.1. **Three dimensional Riemannian locally symmetric spaces.** In the event the inner product is positive definite, each $\epsilon_i = 1$. For the purposes of solving Equation (5.b), we clear earlier notation and introduce the variables $x = A_{1221}$, $y = A_{1331}$, $z = A_{2332}$, $u = A_{1231}$, $v = A_{2132}$, and $w = A_{3123}$. In this positive definite case, instead of an arbitrary orthonormal basis we may choose a Chern basis [11]. On this basis, $u = v = w = 0$. Using this, after omitting trivialities and repetitions in the collection of Equations (5.b), we obtain the system of equations

$$\begin{aligned} xz &= yz \\ xy &= yz \\ xz &= xy \end{aligned} \tag{5.c}$$

Up to a permutation of these variables (corresponding to a permutation of our basis vectors), there are only two sorts of solutions to the Equations (5.c). Either $x = y = z$ are free variables, or $x = y = 0$ and z is a free variable. The former is the curvature model of a locally symmetric (irreducible) three dimensional Riemannian space form, and the latter is the curvature model of $\mathbb{R} \times \Sigma$, where Σ is a two-dimensional locally symmetric space. Thus in the positive definite case Theorem 5.3 is established.

5.2. **Three dimensional Lorentzian locally symmetric spaces.** We solve the system of equations in Corollary 5.1 in the event that the model space $\mathcal{M}$ has a Lorentzian inner product. First, we arrange the orthonormal basis vectors in such a way that $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$; in other words, e_3 is timelike. For the purposes of solving these equations, we use the same notation as above: $x = A_{1221}$, $y = A_{1331}$, $z = A_{2332}$, $u = A_{1231}$, $v = A_{2132}$, and $w = A_{3123}$. Unlike the positive definite case, however, a Chern basis will not be helpful in dimension three. Instead, we find that certain other curvature entries must vanish in certain cases, which we detail below.

After omitting trivialities and repetitions in the collection of Equations (5.b), we obtain the system of equations in Figure 5.2.

$(i,j)(k,j)(s,t)$	Equation (5.b)
$(1,2)(1,2)(1,3)$	$0 = vy + uw$
$(1,2)(1,2)(2,3)$	$0 = vw + uz$
$(1,2)(1,3)(1,3)$	$0 = uv + wx$
$(1,2)(1,3)(2,3)$	$0 = yz + xz - v^2 - w^2$
$(1,2)(2,3)(1,3)$	$0 = xy + yz - u^2 - w^2$
$(1,3)(2,3)(1,2)$	$0 = xy - xz - u^2 + v^2$

FIG. 1: The system of equations for the curvature model of a locally symmetric Lorentzian space.

Maple helps us quickly solve these equations in Figure 5.2, which reveal four cases. We see that Case 1 is the curvature model of the locally symmetric space

Case 1:	$u = v = w = y = z = 0$, x free.
Case 2:	$v = w = z = 0$, u, y are free, $xy = u^2$.
Case 3:	$u = v = w = 0$, z is free, $x = -z$ and $y = z$.
Case 4:	v, w, z are free, $uz = -vw$, $xz = v^2$, $yz = w^2$.

FIG. 2: There are four types of solutions to Equation (5.b) in the three dimensional Lorentzian case.

$\Sigma \times \mathbb{R}$ where Σ is a two-dimensional Riemannian space form. We see that Case 2 is the curvature model (after a suitable change of basis) of a locally symmetric space as well as discovered by Calvaruso: see Theorem 5.1 and Equations (5.2) and (5.3) in [3]. Case 3 is the curvature model of a three-dimensional irreducible locally symmetric space form. We presently show that under a suitable orthonormal change of basis, solutions in Case 4 correspond to the curvature model of a locally symmetric space.

In dimension three the curvature tensor is determined by the Ricci tensor. The corresponding Ricci operator Q falls into one of four Jordan decompositions relative to an orthonormal basis with the last basis vector timelike, known as Segre types (see [3]): Segre type $\{11, 1\}$, $\{1z\bar{z}\}$, $\{21\}$, and $\{3\}$. The curvature tensor A described in Case 4 above must correspond to one of these types.

5.2.1. *Segre type* $\{11, 1\}$. Suppose first that the Ricci operator of A has Segre type $\{11, 1\}$:

$$Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

In this case, we find that $u = v = w = 0$, and so in Case 4 we find that $x = y = 0$ or $z = 0$: the former is the curvature model of the locally symmetric space $\mathbb{R} \times \Sigma_1$, where Σ_1 is a two dimensional Lorentzian space form. In the latter, a review of the equations in Figure 5.2, we see that $xy = 0$, which is again the curvature model of the locally symmetric space $\Sigma \times \mathbb{R}$ where Σ is a two-dimensional Riemannian space form.

5.2.2. *Segre type* $\{1z\bar{z}\}$. Now suppose the Ricci operator Q has Segre type $\{1z\bar{z}\}$:

$$Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{bmatrix}$$

In this case we must have c (the imaginary part of the complex eigenvalue) is nonzero. In this case we find that $v = w = 0$, $u = c$, $x = -y = \frac{a}{2}$, and $z = \frac{1}{2}(a - 2b)$. The second equation in Figure 5.2 forces $z = 0$ since $u = c \neq 0$. However, the last equation in Figure 5.2 now reads

$$0 = -\frac{a^2}{4} - c^2,$$

which is not possible since $c \neq 0$. Thus, solutions in the Case 4 category do not correspond to a curvature model whose Ricci operator has Segre type $\{1z\bar{z}\}$. We conclude that a Lorentzian locally symmetric space cannot have a curvature model whose Ricci operator has Segre type $\{1z\bar{z}\}$.

5.2.3. *Segre type* $\{21\}$. Now suppose the Ricci operator Q has Segre type $\{21\}$:

$$Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & -c \\ 0 & c & b+2c \end{bmatrix}$$

It follows that in situation $v = w = 0$, $u = c$, $x = \frac{1}{2}(a-2c)$, $y = \frac{1}{2}(-a-2c)$, and $z = \frac{1}{2}(a-2b-2c)$. The second equation in Figure 5.2 reveals that $z = 0$, reducing the remaining equations to $xy = c^2$. Thus in this situation, Case 4 reduces to Case 2, which is already known to be the curvature model of a locally symmetric space.

5.2.4. *Segre type* $\{3\}$. Finally, suppose the Ricci operator Q has Segre type $\{3\}$:

$$Q = \begin{bmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{bmatrix}$$

In this situation, $u = 0$, $w = -v = a$, and $x = -y = -z = \frac{b}{2}$. The second equation in Figure 5.2 forces $a = 0$, which reduces this situation to Case 3. This completes the proof of Theorem 5.3.

6. Algebraic restrictions for HCH_k manifolds

We now turn our attention to applications of our results to homothety curvature homogeneous manifolds of order k. We first establish the existence of algebraic restrictions for an HCH_k manifold to be modeled on $\mathcal{M}_k$ in the event $k \geq 2$ is even. We then consider the HCH_0 situation. The main idea will be to identify some tensor(s) that have constant components on some frame, and then use Lemma 3.1 to generate the corresponding algebraic requirements, as we did in Theorem 4.1 and Corollary 4.2. For notational convenience, in this section we replace the inner product $\langle \cdot, \cdot \rangle$ with φ.

6.1. HCH_k manifolds, where $k \geq 2$ is even.

Lemma 6.1. *Suppose (M, g) is HCH_k, where $k = 2a \geq 2$. At any point $P \in M$ there exists a frame field $\{F_1, \ldots, F_n\}$ near P so that the following tensors have constant components relative to this frame:*

$$R,\ \nabla^2 R \otimes g,\ \nabla^4 R \otimes g \otimes g,\ \ldots,\ \nabla^{2a} R \otimes \underbrace{g \otimes \cdots \otimes g}_{a \ \text{times}}.$$

Proof. Since (M, g) is HCH_k, near any point $P \in M$ there is an orthonormal moving frame $\{E_1, \ldots, E_n\}$ so that

$$\begin{aligned} R(E_i, E_j, E_k, E_\ell) &= \lambda A_{ijk\ell}, \text{ and} \\ \nabla^p R(E_i, E_j, E_k, E_\ell; E_{q_1}, \ldots, E_{q_p}) &= \lambda^{\frac{1}{2}(p+2)} {A^p}_{ijk\ell;q_1,\ldots,q_p}, \end{aligned} \tag{6.a}$$

where $\lambda\colon M \to \mathbb{R}$ is a smooth positive real valued function on M, and $A_{ijk\ell}$ and ${A^p}_{ijk\ell;q_1,\ldots,q_p}$ are a collection of constants. Define

$$F_i = \frac{1}{\sqrt[4]{\lambda}} E_i \,.$$

The components of R on the frame field $\{F_1, \ldots, F_n\}$ are constant[2]:

$$R(F_i, F_j, F_k, F_\ell) = A_{ijk\ell} \,. \tag{6.b}$$

From Equation (6.a) and for $i \geq 1$, the components of $\nabla^{2i}R$ on this frame are

$$\begin{aligned} \nabla^{2i}R(F_i, F_j, F_k, F_\ell; F_{q_1}, \ldots, F_{q_{2i}}) &= \Big(\frac{1}{\sqrt[4]{\lambda}}\Big)^{2i+4} \lambda^{\frac{1}{2}(2i+2)} {A^{2i}}_{ijk\ell;q_1,\ldots,q_{2i}} \\ &= \lambda^{\frac{i}{2}} {A^{2i}}_{ijk\ell;q_1,\ldots,q_{2i}} \,. \end{aligned} \tag{6.c}$$

Notice that $\{F_1, \ldots, F_n\}$ is no longer an orthonormal frame, however,

$$g(F_i, F_j) = \frac{1}{\sqrt{\lambda}} \epsilon_i \delta_{ij} \,, \tag{6.d}$$

where $\epsilon_i = \pm 1$, and δ_{ij} is the Kronecker delta function. We complete the proof of this lemma by using Equations (6.c) and (6.d) in considering the following components:

$$\begin{aligned} &\big(\nabla^{2i}R \otimes \underbrace{g \otimes \cdots \otimes g}_{i \text{ times}}\big)(F_i, F_j, F_k, F_\ell; F_{q_1}, \ldots, F_{q_{2i}}, F_{a_1}, F_{b_1}, \ldots, F_{a_i}, F_{b_i}\big) \\ &\quad = \nabla^{2i}R(F_i, F_j, F_k, F_\ell; F_{q_1}, \ldots, F_{q_{2i}}) g(F_{a_1}, F_{b_1}) \cdots g(F_{a_i}, F_{b_i}) \\ &\quad = \lambda^{\frac{i}{2}} {A^{2i}}_{ijk\ell;q_1,\ldots,q_{2i}} \Pi_{j=1}^{i} \Big(\frac{1}{\sqrt{\lambda}} \epsilon_{a_j} \delta_{a_j b_j}\Big) \\ &\quad = {A^{2i}}_{ijk\ell;q_1,\ldots,q_{2i}} \Pi_{j=1}^{i} \big(\epsilon_{a_j} \delta_{a_j b_j}\big) \,, \end{aligned} \tag{6.e}$$

which we now see are constant. □

Remark 6.2. There are other constructions involving $\nabla^{2i+1}R$ and g that have constant components on a certain frame. For example, the tensors

$$\nabla R \otimes \nabla R \otimes g,\ \nabla^3 R \otimes \nabla R \otimes g \otimes g,\ \nabla^3 R \otimes \nabla^3 R \otimes g \otimes g \otimes g,\ \text{etc.},$$

also have constant components on the frame field $\{F_1, \ldots, F_n\}$ in Lemma 6.1, but we could not find a way to exploit this in what follows to derive a necessary algebraic condition on the model space involved.

We use the previous lemma and Lemma 3.1 to derive a necessary algebraic condition for the model space of an HCH_{2a} manifold. We find in Theorem 6.3 that these conditions are the exact same as those to be CH_k at even levels, however, the condition on the metric is omitted.

[2]This establishes that HCH_0 implies WCH_0. See comment (2) in Remark 6.4 below.

Theorem 6.3. *Suppose (M,g) is HCH_{2a} for $a \geq 1$, and modeled on $\mathcal{M}_{2a} = (V,\varphi,A,A^1,\dots,A^{2a})$. At any point $P \in M$ there exists a tensor S of type $(1,2)$ on T_PM that satisfies Equation (4.b), and Equation (4.c) for $i = 2,4,\dots,2a$. Consequently, there must exist a tensor S of type $(1,2)$ on V that satisfies Equation (4.e) and Equation (4.f) for $i = 2,4,\dots,2a$.*

Proof. Suppose (M,g) is HCH_{2a} that is modeled on $\mathcal{M}_{2a}$. By Lemma 6.1, there is a frame $\{F_1,\dots,F_n\}$ so that the following tensors have constant components:

$$R,\ \nabla^2 R\otimes g,\ \nabla^4 R\otimes g\otimes g,\ \dots,\ \nabla^{2a}R\otimes \underbrace{g\otimes\cdots\otimes g}_{a\ \text{ times}}\,.$$

Define the connection $\tilde{\nabla}$ and the tensor S of type $(1,2)$ as in Lemma 3.1. As in Equation (6.b), $R(F_i,F_j,F_k,F_\ell) = A_{ijk\ell}$, and so the first assertion (1) above now follows from the second Bianchi identity and Lemma 3.1.

We now establish the second assertion. As in Equation (6.e) in Lemma 6.1, for each $i = 1,\dots,a$ we have the constant entries

$$\begin{aligned}(\nabla^{2i}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\ \text{ times}})&(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}},F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i})\\ &= {A^{2i}}_{ijk\ell;q_1,\dots,q_{2i}}\Pi_{j=1}^{i}\left(\epsilon_{a_j}\delta_{a_jb_j}\right)\,.\end{aligned}$$

By Lemma 3.1, the fact that any connection obeys the product rule with respect to tensor products (Lemma 4.6(c) on Page 53 of [15]), and the fact that $\nabla g = 0$, we have

$$\begin{aligned}&S_{F_d}\cdot(\nabla^{2i}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\ \text{ times}})(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}},F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i})\\ &= \nabla_{F_d}(\nabla^{2i}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\ \text{ times}})(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}},F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i})\\ &= \nabla^{2i+1}R(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}},F_d)\Pi_{j=1}^{i}g(F_{a_j},F_{b_j})\\ &\quad+\nabla^{2i}R(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}})(\nabla g)(F_{a_1},F_{b_1};F_d)\cdot\cdots\cdot g(F_{a_i},F_{b_i})\\ &\quad+\cdots+\nabla^{2i}R(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}})g(F_{a_1},F_{b_1})\cdot\cdots\cdot(\nabla g)(F_{a_i},F_{b_i};F_d)\\ &= \nabla^{2i+1}R(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i}},F_d)\Pi_{j=1}^{i}g(F_{a_j},F_{b_j})\,.\end{aligned}$$

Multiplying by $\Pi_{j=1}^{i}g(F_{a_j},F_{b_j})$ in Equation (2.f), we conclude

$$\begin{aligned}&S_{F_u}\cdot(\nabla^{2i}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\ \text{ times}})(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i-1}},F_v,F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i})\\ &\qquad- S_{F_v}\cdot(\nabla^{2i}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\ \text{ times}})(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,F_{q_{2i-1}},F_u,F_{a_1},F_{b_1},\\ &\qquad\qquad\dots,F_{a_i},F_{b_i})\\ &\quad= R(F_v,F_u,\nabla^{2i-1}\mathcal{R}(F_i,F_j;F_{i_1},\dots,F_{2i-1})F_k,F_\ell)\Pi_{j=1}^{i}g(F_{a_j},F_{b_j})\end{aligned}$$

(6.f)

$$
\begin{aligned}
&- \nabla^{2i-1}R(\mathcal{R}(F_v,F_u)F_i,F_j,F_k,F_\ell;F_{i_1},\dots,F_{q_{2i-1}})\Pi_{j=1}^i g(F_{a_j},F_{b_j})\\
&- \nabla^{2i-1}R(F_i,\mathcal{R}(F_v,F_u)F_j,F_k,F_\ell;F_{i_1},\dots,F_{q_{2i-1}})\Pi_{j=1}^i g(F_{a_j},F_{b_j})\\
&- \sum_{1\le j\le 2i-1} \nabla^{2i-1}R(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,\mathcal{R}(F_v,F_u)F_{q_j},\dots,F_{q_{2i-1}})\\
&\qquad\qquad \times \Pi_{j=1}^i g(F_{a_j},F_{b_j})\\
&- \nabla^{2i-1}R(F_i,F_j,\mathcal{R}(F_v,F_u)F_k,F_\ell;F_{q_1},\dots,F_{q_{2i-1}})\Pi_{j=1}^i g(F_{a_j},F_{b_j})\\
= &\ (R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\text{ times}})(F_v,F_u,\nabla^{2i-1}\mathcal{R}(F_i,F_j;F_{i_1},\dots,F_{2i-1})F_k,F_\ell,F_{a_1},F_{b_1},\\
&\qquad\qquad \dots,F_{a_i},F_{b_i}))\\
&- (\nabla^{2i-1}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\text{ times}})(\mathcal{R}(F_v,F_u)F_i,F_j,F_k,F_\ell;F_{i_1},\dots,F_{q_{2i-1}},F_{a_1},F_{b_1},\\
&\qquad\qquad \dots,F_{a_i},F_{b_i}))\\
&- (\nabla^{2i-1}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\text{ times}})(F_i,\mathcal{R}(F_v,F_u)F_j,F_k,F_\ell;F_{i_1},\dots,F_{q_{2i-1}},F_{a_1},F_{b_1},\\
&\qquad\qquad \dots,F_{a_i},F_{b_i}))\\
&- \sum_{j=1}^{2i-1}(\nabla^{2i-1}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\text{ times}})(F_i,F_j,F_k,F_\ell;F_{q_1},\dots,\mathcal{R}(F_v,F_u)F_{q_j},\\
&\qquad\qquad \dots,F_{q_{2i-1}},F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i}))\\
&- (\nabla^{2i-1}R\otimes \underbrace{g\otimes\cdots\otimes g}_{i\text{ times}})(F_i,F_j,\mathcal{R}(F_v,F_u)F_k,F_\ell;F_{q_1},\\
&\qquad\qquad \dots,F_{q_{2i-1}},F_{a_1},F_{b_1},\dots,F_{a_i},F_{b_i}))
\end{aligned}
$$

We arrive at the desired conclusion after contracting this tensorial expression in the final i pairs of indices and dividing by n^i. □

Remark 6.4. The following are interesting observations concerning the algebraic requirements listed in Theorem 6.3:

(1) An HCH_{2a} manifold satisfies roughly half of the required equations that a CH_{2a} manifold would have to satisfy. In fact, if one were to replace the CH_{2a} condition with the weaker condition that omits the requirement that there be an orthonormal basis on which the components of $\nabla^{2i-1}R$ are constant[3], then such a space would share the exact same set of algebraic curvature requirements as an HCH_{2a} space.

(2) It is easy to see that HCH_k implies WCH_k for $k = 0$ (see the footnote in the proof of Lemma 6.1), however the extent of this relationship is not

[3]This condition has yet to be well-studied, however it seems to originate in [6]. See also what could be a collection of hypotheses in what is called *variable curvature homogeneity* as defined in Definition 1.1.1 of [4].

known for $k > 0$. If (M, g) is a WCH_{2a} manifold, then one could simply repeat the proof of Theorem 4.1 to find the algebraic requirements of Equation (4.f) must hold for $i = 2, 4, \ldots, 2a$ on its model space. This seems to suggest an algebraic relationship between the WCH_{2a} and HCH_{2a} properties.

6.2. HCH_0 **manifolds.** We now turn our attention to deriving algebraic conditions for the model space of an HCH_0 manifold. Since HCH_0 implies WCH_0, we can find a frame field on which the components of R are constant, as we did in Equation (6.b) of Lemma 6.1. The components of the metric may not be constant on this frame, however, such a frame is related enough to the metric in Equation (6.d) to generate a set of equations similar to that of Theorem 4.1. Recall that the Ricci tensor ρ has components $\rho_{ij} = \sum_{p,q} g^{pq} R_{ipqj}$.

Lemma 6.5. *Suppose (M, g) is HCH_0. Then for any $P \in M$, there is a frame field $\{F_1, \ldots, F_n\}$ near P so that the components of $\rho \otimes g$ are constant.*

Proof. Similarly to the beginning of the proof of Lemma 6.1, near any point P there is an orthonormal frame field $\{E_1, \ldots, E_n\}$ so that

$$g(E_i, E_i) = \epsilon_i = \pm 1, \text{ and}$$
$$R(E_i, E_j, E_k, E_\ell) = \lambda A_{ijk\ell}\,,$$

where $A_{ijk\ell}$ is a collection of constants, and λ is a smooth and positive real valued function on M. Thus the Ricci tensor

$$\rho(E_i, E_j) = \sum_p \epsilon_p R(E_i, E_p, E_p, E_j) = \lambda \sum_p \epsilon_p A_{ippj}\,.$$

Again, as in the proof of Lemma 6.1, define

$$F_i = \frac{1}{\sqrt[4]{\lambda}} E_i\,.$$

On this frame,

$$\begin{aligned} g(F_k, F_\ell) &= \frac{1}{\sqrt{\lambda}} \epsilon_k \delta_{k\ell}, \\ \rho(F_i, F_j) &= \sqrt{\lambda} \sum_p \epsilon_p A_{ippj}, \qquad \text{and} \\ (\rho \otimes g)(F_i, F_j, F_k, F_\ell) &= \epsilon_k \delta_{k\ell} \sum_p \epsilon_p A_{ippj} \end{aligned} \tag{6.g}$$

is constant. □

Lemma 6.6. *Suppose (M, g) is HCH_0 and modeled on $\mathcal{M} = (V, \varphi, A)$. Suppose also that there exists an orthonormal basis of V on which the Ricci tensor q of $\mathcal{M}$ is diagonalized. Then near any point of M, there exists a frame field $\{F'_1, \ldots, F'_n\}$ so that*

(1) $g(F'_k, F'_\ell) = \frac{1}{\sqrt{\lambda}} \epsilon_k \delta_{k\ell}$ *for* $\epsilon_k = \pm 1$,

(2) $\rho(F'_i, F'_j) = \delta_{ij} \sqrt{\lambda}\, q_{ij}$ *is diagonalized at* P, *and*

(3) *the components of $\rho \otimes g$ are constant on this frame.*

Here, q_{ij} are the components of the Ricci tensor q on a basis of V that diagonalizes q.

Proof. Choose any $P \in M$. In the proof of Lemma 6.5 we created a frame field $\{F_1, \ldots, F_n\}$ near P satisfying Conditions (1) and (3). We only need to show that we can find such a frame field that diagonalizes ρ.

Since (M, g) is HCH_0 and modeled on $\mathcal{M}$, there is a basis $\{e_1, \ldots, e_n\}$ for V so that $\varphi(e_k, e_\ell) = \delta_{k\ell}\epsilon_k$ and the components of A on this basis are $A(e_i, e_j, e_k, e_\ell) = A_{ijk\ell}$. Since q is orthogonally diagonalizable, there is a change of basis $e'_i = Te_i$ for which $\varphi(e'_k, e'_\ell) = g(e_k, e_\ell) = \delta_{k\ell}\epsilon_k$, and that

$$q(e'_i, e'_j) = \delta_{ij} \sum_p \epsilon_p A(e'_i, e'_p, e'_p, e'_j) \quad \text{is diagonalized.}$$

Here, $T = [T_{ij}]$ is an $n \times n$ orthogonal matrix consisting of real numbers.

Returning now to the frame field $\{F_1, \ldots, F_n\}$ above, create the new frame field $F'_i = \sum_p T_{ip} F_p$, that is, a *constant* change of frame. We claim that this frame field satisfies the required conditions of the lemma. Since T is orthogonal we have

$$g(F'_k, F'_\ell) = g(F_k, F_\ell) = \frac{1}{\sqrt{\lambda}} \epsilon_k \delta_{k\ell}\,,$$

so Condition (1) is satisfied. The components of $\rho \otimes g$ on this new frame field are just constant linear combinations of the constant components on the old frame field, and are therefore constant. So, Condition (3) is satisfied.

To complete the proof and show that Condition (2) is satisfied, we use the fact that $\{e'_1, \ldots, e'_n\}$ diagonalizes q. That is,

$$q(e'_i, e'_j) = \sum_p \epsilon_p A(e'_i, e'_p, e'_p, e'_j)\,,$$

and is equal to 0 if $i \neq j$. Since $R(F_i, F_j, F_k, F_\ell) = \lambda A_{ijk\ell}$, and the F'_i are a constant linear combination of the F_i, we find that $R(F'_i, F'_j, F'_k, F'_\ell) = \lambda A'_{ijk\ell}$, where $A'_{ijk\ell} = A(e'_i, e'_j, e'_k, e'_\ell)$ is a different collection of constants. We then have

$$\begin{aligned} \rho(F'_i, F'_j) &= \sqrt{\lambda} \sum_p \epsilon_p R(F'_i, F'_p, F'_p, F'_j) \\ &= \sqrt{\lambda} \sum_p \epsilon_p A(e'_i, e'_p, e'_p, e'_j) \\ &= \delta_{ij} \sqrt{\lambda} q_{ij}\,. \end{aligned}$$

Here, $q_{ij} = q(e'_i, e'_j) = \sum_p \epsilon_p A(e'_i, e'_p, e'_p, e'_j)$, which now satisfies Condition (2). $\square$

Remark 6.7. An easy corollary of the previous lemma is the observation that if a manifold is HCH_0 and modeled on a model space whose Ricci tensor is diagonalizable, then the Ricci tensor of the manifold is also diagonalizable at every point. More generally, the same techniques show that the Ricci operator of the model space and the Ricci operator of the manifold must share the same Jordan type.

We can adapt our techniques to prove the following fact, showing that in a certain circumstance, HCH_0 implies CH_0. We recall that a manifold has a *cyclic parallel* Ricci tensor if $\mathfrak{S}_{X,Y,Z}\nabla\rho(X,Y;Z)=0$ for any vectors X,Y, and Z, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum. If $X=Y=Z$, then it follows that $\nabla\rho(X,X;X)=0$.

Theorem 6.8. *Suppose (M,g) is HCH_0, has a cyclic parallel Ricci tensor, and is modeled on $\mathcal{M}=(V,\varphi,A)$. Let q be the Ricci tensor of $\mathcal{M}$, and suppose*

(1) *q is diagonalizable, and*

(2) *the eigenvalues of q are nonzero.*

Then (M,g) is CH_0.

Proof. Choose any $P\in M$. According to Lemma 6.6, near any point $P\in M$ there is a frame field $\{F_1',\dots,F_n'\}$ so that

(1) $g(F_k',F_\ell')=\frac{1}{\sqrt{\lambda}}\epsilon_k\delta_{k\ell}$ for $\epsilon_k=\pm1$,

(2) $\rho(F_i',F_j')=\delta_{ij}\sqrt{\lambda}\,q_{ij}$, for q_{ij} as in Lemma 6.6, is diagonalized at P, and

(3) the components of $\rho\otimes g$ are constant on this frame.

As in Lemma 3.1, we define the connection $\tilde\nabla$ so that $\tilde\nabla_{F_i'}F_j'=0$, and the tensor S of type $(1,2)$ by $S=\nabla-\tilde\nabla$. We recall that $S_{F_i'}F_j'=\nabla_{F_i'}F_j'$. Define the components of S on this frame as $S_{F_i'}F_j'=\sum_p S_{ij}{}^pF_p'$. Since the components of $\rho\otimes g$ are constant on this frame, Lemma 3.1 applies. Also, since ∇ obeys the product rule, since $\nabla\rho(F_i',F_i';F_I')=0$, and since $\nabla g=0$, we have

$$\begin{aligned}S_{F_i'}\cdot(\rho\otimes g)(F_i',F_i',F_i',F_i')&=\nabla(\rho\otimes g)(F_i',F_i',F_i',F_i';F_i')\quad\text{(Lemma 3.1)}\\&=(\nabla\rho)(F_i',F_i';F_i')g(F_i',F_i')\\&=0\,.\end{aligned}$$

On the other hand, by the definition of the action of $S_{F_d'}$ as a derivation, and the fact that both g and ρ are diagonalized on this frame,

$$\begin{aligned}S_{F_i'}\cdot(\rho\otimes g)(F_i',F_i',F_i',F_i')&=-4S_{ii}^i\rho(F_i',F_i')g(F_i',F_i')\\&=-4S_{ii}^i\sqrt{\lambda}q_{ii}g(F_i',F_i')\,.\end{aligned}$$

Since $g(F_i',F_i')\neq0$, the eigenvalues q_{ii} are nonzero, and $\sqrt{\lambda}\neq0$, it follows that $S_{ii}^i=0$.

But $\nabla g=0$ and $g(\nabla_{F_i'}F_i',F_i')=S_{ii}^i\epsilon_i\frac{1}{\sqrt{\lambda}}$, so

$$\begin{aligned}0&=(\nabla g)(F_i',F_i';F_i')\\&=F_i'\left(\epsilon_i\frac{1}{\sqrt{\lambda}}\right)-2S_{ii}^i\epsilon_i\frac{1}{\sqrt{\lambda}}\,,\end{aligned}$$

so

$$\begin{aligned} 2S_{ii}^i\epsilon_i\frac{1}{\sqrt{\lambda}} &= F_i'\left(\epsilon_i\frac{1}{\sqrt{\lambda}}\right) \\ &= \epsilon_i\frac{-1}{2\sqrt{\lambda^3}}F_i;(\lambda), \qquad \text{so} \\ S_{ii}^i &= -\frac{1}{4}\frac{1}{\lambda}F_i'(\lambda)\,. \end{aligned}$$

It must then be the case that $F_i'(\lambda) = 0$ for all i, and it follows that λ is constant. Now, $E_i = \sqrt[4]{\lambda}F_i'$ is an orthonormal frame on which the entries of the curvature tensor are constant, so (M, g) is CH_0. □

There are several relevant remarks to make about Theorem 6.8.

Remark 6.9. The first condition of Theorem 6.8 is automatically satisfied if (M, g) is Riemannian.

Remark 6.10. Theorem 6.8 shows that within the set of manifolds whose Ricci tensor is cyclic parallel, there is a large class of model spaces that cannot be geometrically realized by an HCH_0 manifold that is not already CH_0. The crucial mechanism that controls this is the component S_{di}^i, which is highlighted in the proof. One sees in [12] that this component must vanish in the Riemannian signature: in the CH_0 case the frame involved $\{E_1, \ldots, E_n\}$ is an orthonormal one, hence, the metric entries on this frame are constant and Lemma 3.1 applies. As a result, $S_{di}^j\epsilon_j = -S_{dj}^i\epsilon_i$ (and in particular, $S_{di}^i = 0$ when $\epsilon_i = \epsilon_j$) on this frame. In the HCH_0 case, the frame $\{F_1, \ldots, F_n\}$ above is not orthonormal, however it is orthogonal. A quick calculation shows

$$0 = \nabla g(F_i, F_j; F_d) = F_d\left(\frac{1}{\sqrt{\lambda}}\epsilon_i\delta_{ij}\right) - S_{di}^j\frac{1}{\sqrt{\lambda}}\epsilon_j - S_{dj}^i\frac{1}{\sqrt{\lambda}}\epsilon_i\,.$$

So if $i \neq j$ the first term vanishes, and we still find that $S_{di}^j\epsilon_j = -S_{dj}^i\epsilon_i$. In this way (when $i = j$), the only difference in the components of S in the HCH_0 case is the entry S_{di}^i, which we compute above as

$$S_{di}^i = -\frac{1}{4}\frac{1}{\lambda}F_d'(\lambda)\,,$$

which is zero for every d precisely when λ is constant, i.e., the manifold is CH_0. While we prove here that there are certain situations where HCH_0 implies CH_0, it would be interesting to know if there is a model space that cannot be geometrically realized as a CH_0 manifold, but *can* be geometrically realized as an HCH_0 manifold; if such a model space exists, the tensor component S_{di}^i must be nonzero and plays a crucial role.

Remark 6.11. If a manifold has a cyclic parallel Ricci tensor, then the scalar curvature is constant (see page 262 of [10]). This fact would provide a proof of Theorem 6.8 with the additional assumption that the scalar curvature is nonzero. Even in this specialized case, Theorem 6.8 is somewhat more general in that we allow for the possibility that the scalar curvature could vanish.

Acknowledgement. The authors appreciate helpful discussions with Jeff Meyer concerning this work. In addition, the authors are grateful to the referee for the detailed review of this work, and for a helpful suggested reference in Remark 6.11. This research was jointly funded by California State University, San Bernardino, and the NSF grant DMS-1758020.

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