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FINITENESS OF LOCAL HOMOLOGY MODULES

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Abstract. Let I be an ideal of Noetherian ring R and M a finitely generated R-module. In this paper, we introduce the concept of weakly colaskerian modules and by using this concept, we give some vanishing and finiteness results for local homology modules.

Let $I_M := \operatorname{Ann}_R(M/IM)$, we will prove that for any integer n

(i) If N is a weakly colaskerian linearly compact R-module such that $(0 :_N I_M) \neq 0$ then

$$\operatorname{width}_{I_M}(N) = \inf\{i \mid \mathrm{H}_i^{I_M}(N) \neq 0\} = \inf\{i \mid \mathrm{H}_i^{I}(M,N) \neq 0\}.$$

(ii) If $(R,\mathfrak{m})$ is a Noetherian local ring and N is an artinian R-module then

$$\cup_{i<n} \operatorname{Cos}_R\left(\mathrm{H}_i^{I_M}(N)\right) = \cup_{i<n} \operatorname{Cos}_R\left(\mathrm{H}_i^{I}(M,N)\right) = \cup_{i<n} \operatorname{Cos}_R\left(\operatorname{Tor}_i^R(M/IM,N)\right),$$

$$\inf\{i \mid \mathrm{H}_i^{I_M}(N) \text{ is not Noetherian } R\text{-module}\} = \inf\{i \mid \mathrm{H}_i^{I}(M,N) \text{ is not Noetherian } R\text{-module}\}.$$

1. Introduction

Throughout this paper assume that R is a commutative Noetherian ring, I is an ideal of R and M, N are R-modules. Cuong and Nam in [4] defined the local homology modules $\mathrm{H}_i^I(M)$ with respect to I by

$$\mathrm{H}_i^I(M) = \varprojlim_n \operatorname{Tor}_i^R(R/I^n, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenlees and May in [7] for an artinian R-module M. For basic results about local homology we refer the reader to [4], [5] and [17]; for local cohomology refer to [1]. In [12], Nam introduced the definition of generalized local homology which is an extension of the usual local homology. In fact, the i-th generalized local homology module $\mathrm{H}_i^I(M,N)$ of M, N with respect to I is defined

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by

$$\mathrm{H}_i^I(M,N) = \varprojlim_n \mathrm{Tor}_i^R(M/I^n M, N)\,.$$

Clearly, in the special case $M = R$, $\mathrm{H}_i^I(R,N) = \mathrm{H}_i^I(N)$ for all i and any R -module N. Basic facts and more information about generalized local homology can be obtained from [12], [14] and [19].

In this paper we study some properties of the local homology module $\mathrm{H}_n^I(M,N)$, where M is a finitely generated and N an artinian R-module. The colocalization is an essential tool in our investigation. Let M be an R-module and S a multiplicative set of R. The colocalization of M with respect to S is the R_S-module ${}_SM := \mathrm{Hom}_R(R_S, M)$. If $\mathfrak{p}$ is a prime ideal and $S = R - \{\mathfrak{p}\}$ then instead of ${}_SM$ we write ${}_{\mathfrak{p}}M$. When M is an artinian module it is known that $\mathrm{Hom}_R(R_S, M)$ is almost never an artinian R_S-module (see [10]). Thus the functor co-localization is not closed on the category artinian modules. To avoid this difficulty we introduce the concept of weakly colaskerian modules and we will see that if M is an artinian R-module then ${}_{\mathfrak{p}}M$ is weakly colaskerian ${}_{\mathfrak{p}}R$ -module for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Here by using the concept of weakly colaskerian we investigate finiteness of local homology modules. At first, we obtain the following main results about vanishing of local homology modules.

Theorem 1.1. *Let R be a Noetherian ring, I be an ideal of R and M a finitely generated R-module. Let n be an integer and $I_M := \mathrm{Ann}(M/IM)$. If N is a weakly colaskerian linearly compact R-module such that $(0 :_N I_M) \neq 0$ then*

$$\mathrm{width}_{I_M}(N) = \inf\{i \mid \mathrm{H}_i^{I_M}(N) \neq 0\} = \inf\{i \mid \mathrm{H}_i^I(M,N) \neq 0\}\,.$$

Then over a Noetherian local ring R we obtain the following main results.

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a Noetherian ring, I be an ideal of R and M a finitely generated R-module and an artinian R-module. Let n be an integer and $I_M := \mathrm{Ann}(M/IM)$. Then*

i) $\bigcup_{i<n} \mathrm{Cos}_R(\mathrm{H}_i^{I_M}(N)) = \bigcup_{i<n} \mathrm{Cos}_R(\mathrm{H}_i^I(M,N)) =$
$\bigcup_{i<n} \mathrm{Cos}_R(\mathrm{Tor}_i^R(M/IM,N))$,

ii) $\inf\{i \mid \mathrm{H}_i^{I_M}(N)$ *is not Noetherian R-module*$\} =$
$\inf\{i \mid \mathrm{H}_i^I(M,N)$ *is not Noetherian R-module*$\}$.

2. The results

A Hausdorff linearly topologized R-module M is said to be linearly compact if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e the cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection. It is clear that artinian R-modules are linearly compact with the discrete topology. If $(R,\mathfrak{m})$ is a complete local ring, then finite R-modules are also linearly compact and discrete. For more facts about linearly compact modules see [8] and [20]. Let M and N two R-modules. When M is finitely generated module and N is artinian, we already note that the local homology modules $\mathrm{H}_i^I(M,N)$ are linearly compact, (see [5, Lemma 2.3 and Lemma 2.5]).

A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \mathrm{Ann}_R T$ [18]. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$.

Recall that the cosupport of M is defined by $\mathrm{Cos}_R M = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid_{\mathfrak{p}} M \neq 0\}$ (see [10]). Also, Yassemi [18] defined the co-support of an R-module M, denoted by $\mathrm{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}(L) \subseteq \mathfrak{p}$. It is well known that in case M is an artinian R-module or M is a linearly compact R-module the equality $\mathrm{Cos}_R(M) = \mathrm{Cosupp}_R(M)$ is true.

Let N be an R-module. We recall the notion of coregular sequence defined by Ooishi [15]. An element x of R is called N-coregular if $N = xN$ and a sequence $x_1, \ldots, x_r$ of elements in R is said to be an N-coregular sequence if $0\colon_N(x_1,\ldots,x_r) \neq 0$ and x_i is an $(0\colon_N(x_1,\ldots,x_{i-1}))$-coregular element for all $i = 1, \ldots, r$. We denote by $\mathrm{width}_I(N)$ the length of the longest N-coregular sequence in I. In case N is an artinian R-module, we know $\mathrm{width}_I(N)$ is finite.

In [6, Definition 2.1], the authors call an R-module N weakly Laskerian if any quotient of N has finitely many associated prime ideals. In the following, as a dual case, we introduce the class of weakly colaskerian modules.

Definition 2.1. Given an R-module M, we say that M is a weakly colaskerian R-module, if for every ideal I of R, the set $\mathrm{Coass}_R(0 :_M I)$ is finite.

It is clear that artinian R-modules are weakly colaskerian. In the next result we see that a colocalization of an artinian module is weakly colaskerian module.

Lemma 2.2. *Let M be an R-module. Then*

(i) *If M is weakly colaskerian then $(0 :_M J)$ is weakly colaskerian for any ideal J of R.*

(ii) *If M is artinian then ${}_{\mathfrak{p}}M$ is weakly colaskerian $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

(iii) *If $(R,\mathfrak{m})$ is a Noetherian local ring and M is linearly compact R-module then ${}_{\mathfrak{p}}M$ is a linearly compact $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proof. (i) Let I be an ideal of R. Then

$$(0 :_{(0:_M J)} I) \simeq (0 :_M I) \cap (0 :_M J) \simeq (0 :_M I + J)\,.$$

Assumption implies that $\mathrm{Coass}_R(0 :_M I + J)$ is finite and so $\mathrm{Coass}_R(0 :_{(0:_M J)} I)$ is finite.

(ii) Since M is artinian, $(0 :_M I)$ is artinian for every ideal I of R. Thus ${}_{\mathfrak{p}}(0 :_M I)$ is representable $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \mathrm{spec}(R)$ by [10, Theorem 3.2] and so $\mathrm{Att}_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}(0 :_M I))$ is finite. Since ${}_{\mathfrak{p}}(0 :_M I) \simeq (0 :_{{}_{\mathfrak{p}}M} IR_{\mathfrak{p}})$ it follows that $\mathrm{Coass}_{R_{\mathfrak{p}}}(0 :_{{}_{\mathfrak{p}}M} IR_{\mathfrak{p}}) = \mathrm{Att}_{R_{\mathfrak{p}}}(0 :_{{}_{\mathfrak{p}}M} IR_{\mathfrak{p}})$ is finite. This completes the proof.

(iii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$. By [5, Lemma 2.5] ${}_{\mathfrak{p}}M$ is linearly compact R-module. Assume that $\{U_i\}_{i\in J}$ is a nuclear base of ${}_{\mathfrak{p}}M$ consisting of submodules. Thus

${}_{\mathfrak{p}}M \simeq \varprojlim_{i\in J} {}_{\mathfrak{p}}M/U_i$, in which ${}_{\mathfrak{p}}M/U_i$ is an artinian R-module for all $i \in J$ by [8, 4.7, 5.5]. It is easy to see that each ${}_{\mathfrak{p}}M/U_i$ is artinian over $R_{\mathfrak{p}}$ and $\{{}_{\mathfrak{p}}M/U_i\}$ can be regard as an inverse system of artinian $R_{\mathfrak{p}}$-modules. Therefore ${}_{\mathfrak{p}}M$ is linearly compact $R_{\mathfrak{p}}$-module by [8, 5.5]. □

Lemma 2.3. *Let R be a Noetherian ring, I be an ideal of R and M be an R-module with $|\operatorname{Coass}_R(M)| < \infty$. Then $IM = M$ if and only if $xM = M$ for some $x \in I$.*

Proof. ⇒) If not, $I \subseteq \cup_{\mathfrak{p}\in \operatorname{Coass}_R(M)}\mathfrak{p}$ by [18, Theorem 1.13]. Hence $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Coass}_R M$. Thus, there is a submodule N of M such that M/N is artinian and $\mathfrak{p} = \operatorname{Ann}_R(M/N)$. Since $I \subseteq \mathfrak{p}$, $IM \subseteq N \subseteq M$. But $IM = M$ and so $N = M$ which is a contradiction.

⇐) It is clear that $M = xM \subseteq IM \subseteq M$. Therefore $IM = M$. □

Lemma 2.4. *Let R be a Noetherian ring, I be an ideal of R and M be a finitely generated R-module and N an R-module with $|\operatorname{Coass}_R(N)| < \infty$. Then $M\otimes_R N = 0$ if and only if there exists an N-coregular element in $\operatorname{Ann}_R(M)$.*

Proof. By using [18, Theorem 1.9 and Theorem 1.21] we have

$$\begin{aligned} M \otimes_R N = 0 &\Leftrightarrow \operatorname{Coass}_R(M \otimes_R N) = \phi \\ &\Leftrightarrow \operatorname{Supp} M \cap \operatorname{Coass}_R(N) = \phi \\ &\Leftrightarrow \operatorname{Supp}_R(R/\operatorname{Ann}_R(M)) \cap \operatorname{Coass}(N) = \phi \\ &\Leftrightarrow \operatorname{Coass}_R(R/\operatorname{Ann}_R(M) \otimes_R N) = \phi \\ &\Leftrightarrow R/\operatorname{Ann}_R(M) \otimes_R N = 0 \\ &\Leftrightarrow N = \operatorname{Ann}_R(M)N. \end{aligned}$$

Now the result follows by Lemma 2.3. □

The following result is an extention of [15, Theorem 3.9] for weakly colaskerian modules. The proof is similar to that of [15, Theorem 3.9].

Lemma 2.5. *Let R be a Noetherian ring, I be an ideal of R and N be a weakly colaskerian R-module. Then the following are equivalent:*

(i) $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i < n$ and for any finitely generated R-module M with* $\operatorname{Supp} M \subseteq V(I)$.

(ii) $\operatorname{Tor}_i^R(R/I, N) = 0$ *for all $i < n$.*

(iii) $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i < n$ and for a finitely generated R-module M with* $\operatorname{Supp} M = V(I)$.

If in addition, $(0 :_N I) \neq 0$, then the above three conditions are equivalent to the following condition:

(iv) *There exists an N-coregular sequence $(x_1, x_2, \ldots, x_n)$ in I.*

Proof. (i) ⇒ (ii) ⇒ (iii) are trivial.

(iii) ⇒ (ii): By induction on n. Let $n = 1$ and $M \otimes_R N = 0$. Thus by Lemma 2.4 there exists $x \in \operatorname{Ann}_R(M)$ such that $xN = N$. Since $\operatorname{Supp} M = V(I)$ there exists an integer k such that $x^k \in I$. Thus $IN = N$ and so the result follows in this case. Now suppose, inductively that $n > 1$ and the result is true for $n - 1$. From the exact sequence

$$0 \longrightarrow 0 :_N x^k \longrightarrow N \xrightarrow{x^k} N \longrightarrow 0$$

we get the following exact sequence:

$$\cdots \to \operatorname{Tor}_i^R(M,N) \xrightarrow{x^k} \operatorname{Tor}_i^R(M,N) \longrightarrow \operatorname{Tor}_{i-1}^R(M, 0 :_N x^k) \to \cdots .$$

Assumption and the above long exact sequence implies that $\operatorname{Tor}_i^R(M, 0 :_N x^k) = 0$ for all $i < n-1$. By Lemma 2.2 (i) $(0 :_N x^k)$ is weakly colaskerian module and so the induction assumption implies that $\operatorname{Tor}_i^R(R/I, 0 :_N x^k) = 0$ for all $i < n-1$. On the other hand, from the above short exact sequence we have the following long exact sequence:

$$\cdots \to \operatorname{Tor}_i^R(R/I,N) \xrightarrow{x^k} \operatorname{Tor}_i^R(R/I,N) \longrightarrow \operatorname{Tor}_{i-1}^R(R/I, 0 :_N x^k) \to \cdots .$$

Thus for any integer $i < n$ we have

$$\operatorname{Tor}_i^R(R/I,N) \xrightarrow{x^k} \operatorname{Tor}_i^R(R/I,N) \longrightarrow 0 .$$

Since $x^k \in I$ the above multiplication map is a zero-map. Therefore $\operatorname{Tor}_i^R(R/I,N) = 0$ for all $i < n$.

(ii) $\Rightarrow$ (i): By using induction similar to the above argument.

(iii) $\Rightarrow$ (iv): We use induction on n. Let $n = 1$ and $M \otimes_R N = 0$. In this case, the result follows by an argument similar to Lemma 2.4.

Let $n > 1$. Thus there exists an N-coregular element $x_1 \in I$. By assumption $0 :_N x_1 \neq 0$ and so from the exact sequence

$$0 \longrightarrow 0 :_N x_1 \longrightarrow N \xrightarrow{x_1} N \longrightarrow 0$$

we have the following exact sequence:

$$\cdots \to \operatorname{Tor}_i^R(M,N) \xrightarrow{x_1} \operatorname{Tor}_i^R(M,N) \longrightarrow \operatorname{Tor}_{i-1}^R(M, 0 :_N x_1) \to \cdots .$$

Thus we obtain $\operatorname{Tor}_i^R(M, 0 :_N x_1) = 0$ for all $i < n-1$. By Lemma 2.2 (i) $(0 :_N x_1)$ is weakly colaskerian module and so the induction assumption implies that there exists an $(0 :_N x_1)$-coregular sequence $(x_2, \ldots, x_n)$ in I. Therefore $(x_1, x_2, \ldots, x_n)$ is a N-coregular sequence in I, as required.

(iv) $\Rightarrow$ (i): The proof is similar to the proof of [15, Theorem 3.9 (4) $\Rightarrow$ (1)]. □

Ooishi [15] prove that, if N is artinian and I is an ideal of R such that $(0 :_N I) \neq 0$ then the length of an N-coregular sequence in I is finite and

$$\operatorname{width}_I(N) = \inf\{i \mid \operatorname{Tor}_i^R(R/I,N) \neq 0\}.$$

The next result shows that this result is still true for weakly colaskerian modules.

Theorem 2.6. *Let R be a Noetherian ring, I be an ideal of R and N a weakly colaskerian R-module such that $(0 :_N I) \neq 0$. Then*

$$\operatorname{width}_I(N) = \inf\{i \mid \operatorname{Tor}_i^R(R/I,N) \neq 0\} .$$

Proof. It follows by Lemma 2.5.

Theorem 2.7. *Let R be a Noetherian ring, I an ideal of R, M a finitely generated R-module and N be an artinian R-module. Let n be an integer and $I_M := \mathrm{Ann}_R(M/IM)$. Then*

$$\bigcup_{i<n} \mathrm{Cos}_R\left(\mathrm{Tor}_i^R(R/I_M, N)\right) = \bigcup_{i<n} \mathrm{Cos}_R\left(\mathrm{Tor}_i^R(M/IM, N)\right).$$

Proof. By Lemma 2.2 (ii) ${}_{\mathfrak{p}}N$ is weakly colaskerian $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$. On the other hand, $M_{\mathfrak{p}}/(IM)_{\mathfrak{p}} \simeq M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module and so,

$$\mathrm{Supp}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}}) = \mathrm{V}(\mathrm{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}})) = \mathrm{V}(\mathrm{Ann}_R(M/IM)_{\mathfrak{p}}) = \mathrm{V}((I_M)_{\mathfrak{p}}).$$

Thus by using Lemma 2.5 (ii) $\Leftrightarrow$(iii), for all $\mathfrak{p} \in \mathrm{Spec}(R)$ we have

$$\mathrm{Tor}_i^{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(I_M)_{\mathfrak{p}}, {}_{\mathfrak{p}}N\right) = 0, \ \forall i < n \Leftrightarrow \mathrm{Tor}_i^{R_{\mathfrak{p}}}\left(M_{\mathfrak{p}}/(IM)_{\mathfrak{p}}, {}_{\mathfrak{p}}N\right) = 0, \ \forall i < n.$$

Therefore

$$\begin{aligned}
\mathfrak{p} \notin \bigcup_{i<n} \mathrm{Cos}_R\left(\mathrm{Tor}_i^R(R/I_M, N)\right) &\Leftrightarrow {}_{\mathfrak{p}}\left(\mathrm{Tor}_i^R(R/I_M, N)\right) = 0 \quad \text{for all } i < n \\
&\Leftrightarrow \mathrm{Tor}_i^{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(I_M)_{\mathfrak{p}}, {}_{\mathfrak{p}}N\right) = 0 \quad \text{for all } i < n \\
&\Leftrightarrow \mathrm{Tor}_i^{R_{\mathfrak{p}}}\left(M_{\mathfrak{p}}/(IM)_{\mathfrak{p}}, {}_{\mathfrak{p}}N\right) = 0 \quad \text{for all } i < n \\
&\Leftrightarrow {}_{\mathfrak{p}}\left(\mathrm{Tor}_i^R(M/IM, N)\right) = 0 \quad \text{for all } i < n \\
&\Leftrightarrow \mathfrak{p} \notin \bigcup_{i<n} \mathrm{Cos}_R\left(\mathrm{Tor}_i^R(M/IM, N)\right).
\end{aligned}$$

□

Lemma 2.8. *Let R be a Noetherian ring, I be an ideal of R, M a finitely generated R-module and N be a linearly compact R-module. Then*

i) $\mathrm{H}_i^I(M, N) \simeq \mathrm{H}_i^{\sqrt{I}}(M, N)$ *for all $i \geq 0$,*

ii) $\mathrm{H}_i^I(M, N) \simeq \mathrm{H}_i^{I_M}(M, N)$ *for all $i \geq 0$ where $I_M = \mathrm{Ann}_R(M/IM)$.*

Proof. (i) Since R is Noetherian, there exists an integer k such that $(\sqrt{I})^k \subseteq I$. Thus for all $t > 0$ we have

$$(\sqrt{I})^{kt}(M \otimes_R N) \subseteq I^t(M \otimes_R N) \subseteq (\sqrt{I})^t(M \otimes_R N)$$

hence

$$\varprojlim_t \frac{M \otimes_R N}{I^t(M \otimes_R N)} \simeq \varprojlim_t \frac{M \otimes_R N}{(\sqrt{I})^t(M \otimes_R N)}.$$

By notation of [13, Definition 3.1] we have $\Lambda_I(M, N) \simeq \Lambda_{\sqrt{I}}(M, N)$. Thus $L_i\Lambda_I(M, N) \simeq L_i\Lambda_{\sqrt{I}}(M, N)$ where $L_i\Lambda_I(M, N)$ is the i-th derived module of $\Lambda_I(M, N)$ (see [13]). Now the result follows by [13, Theorem 3.6].

(ii) By [16, 9.23] we have $\sqrt{\mathrm{Ann}_R(M/IM)} = \sqrt{I + \mathrm{Ann}_R(M)}$ and so $\mathrm{H}_i^{I_M}(M, N) \simeq \mathrm{H}_i^{I+\mathrm{Ann}_R(M)}(M, N)$ by (i). But, by using definition, it is easy to see that $\mathrm{H}_i^{I+\mathrm{Ann}_R(M)}(M, N) \simeq \mathrm{H}_i^I(M, N)$ for all $i \geq 0$. Thus $\mathrm{H}_i^I(M, N) \simeq \mathrm{H}_i^{I_M}(M, N)$ for all $i \geq 0$. □

In the next theorem we obtain a vanishing result of generalized local homology modules.

Theorem 2.9. *Let R be a Noetherian ring, I be an ideal of R, M a finitely generated R-module and N be a weakly colaskerian linearly compact R-module. Then $\mathrm{H}_i^I(M,N)=0$ for all $i<n$ if and only if $\mathrm{Tor}_i^R(M/IM,N)=0$ for all $i<n$.*

Proof. $\Rightarrow$) By induction on n. Let $n=1$. If $\mathrm{H}_0^I(M,N)=\varprojlim_t(M/I^tM\otimes_R N)=0$ then $M/IM\otimes_R N=0$. Let $n>1$. By Lemma 2.4, there exists $x\in\mathrm{Ann}_R(M/IM)$ such that $xN=N$. Since $\varphi:N\xrightarrow{x}N$ is a continuous R-module homomorphism, and 0 is a closed submodule of N, $\ker(\varphi)=(0:_N x)$ is linearly compact R-module by [3, Lemma 2.2]. Thus $0\to 0:_N x\to N\xrightarrow{x}N\to 0$ is an exact sequence of linearly compact modules and by using [13, Corollary 3.7] we have the following exact sequences:

$$\cdots\to\mathrm{H}_i^I(M,N)\xrightarrow{x}\mathrm{H}_i^I(M,N)\to\mathrm{H}_{i-1}^I(M,0:_N x)\to\cdots,$$

$$\cdots\to\mathrm{Tor}_i^R(M/IM,N)\xrightarrow{x}\mathrm{Tor}_i^R(M/IM,N)\to\mathrm{Tor}_{i-1}^R(M/IM,0:_N x)\to\cdots.$$

By using assumption, the first sequence implies that $\mathrm{H}_i^I(M,0:_N x)=0$ for all $i<n-1$. But by Lemma 2.2 (i) $(0:_N x)$ is weakly colaskerian R-module. Therefore, by the induction assumption we conclude that $\mathrm{Tor}_i^R(M/IM,0:_N x)=0$ for all $i<n-1$. Since $x.(M/IM)=0$, the multiplication map in the second sequence is surjective and zero-map for all $i<n$. Therefore we get $\mathrm{Tor}_i^R(M/IM,N)=0$ for all $i<n$, as required.

$\Leftarrow$) We use induction on n. Let n=1. If $M/IM\otimes_R N=0$ then $M/I^tM\otimes_R N=0$ for all $t\geq 0$ and so $\varprojlim_t(M/I^tM\otimes_R N)=\mathrm{H}_0^I(M,N)=0$. Thus the result follows in this case. Now suppose, inductively that $n>1$ and the result is true for $n-1$. Let $I_M:=\mathrm{Ann}_R(M/IM)$. Lemma 2.4 implies that there exists $x\in I_M$ such that $xN=N$. From the following exact sequence

$$0\longrightarrow 0:_N x\longrightarrow N\xrightarrow{x}N\longrightarrow 0$$

we have the following long exact sequences

$$\cdots\to\mathrm{Tor}_i^R(M/IM,N)\xrightarrow{x}\mathrm{Tor}_i^R(M/IM,N)\to\mathrm{Tor}_{i-1}^R(M/IM,0:_N x)\to\cdots,$$

$$\cdots\to\mathrm{H}_i^{I_M}(M,N)\xrightarrow{x}\mathrm{H}_i^{I_M}(M,N)\to\mathrm{H}_{i-1}^{I_M}(M,0:_N x)\to\cdots.$$

The first above sequence implies that $\mathrm{Tor}_i^R(M/IM,0:_N x)=0$ for all $i<n-1$. Thus by the induction assumption, we get $\mathrm{H}_i^I(M,0:_N x)=0$ for all $i<n-1$ and so by Lemma 2.8(ii) we have $\mathrm{H}_i^{I_M}(M,0:_N x)=0$ for all $i<n-1$. Thus the second long exact sequence implies that $\mathrm{H}_i^{I_M}(M,N)=x\,\mathrm{H}_i^{I_M}(M,N)$. Since $x\in I_M$ by [14, Proposition 2.3(i)] we have $\cap_{t>0}x^t\,\mathrm{H}_i^{I_M}(M,N)=0$. Thus $\mathrm{H}_i^{I_M}(M,N)=0$ for all $i<n$. By Lemma 2.8(ii) we conclude that $\mathrm{H}_i^I(M,N)=0$ for all $i<n$. □

Corollary 2.10. *Let R be a Noetherian ring, I be an ideal of R, M a finitely generated R-module and N a weakly colaskerian linearly compact R-module. Let*

$I_M := \operatorname{Ann}(M/IM)$ *such that* $(0 :_N I_M) \neq 0$. *Then*

$$\operatorname{width}_{I_M}(N) = \inf\{i \mid \mathrm{H}_i^{I_M}(N) \neq 0\} = \inf\{i \mid \mathrm{H}_i^{I}(M,N) \neq 0\}\,.$$

Proof. By Theorem 2.6 and Lemma 2.5

$$\operatorname{width}_{I_M}(N) = \inf\{i \mid \operatorname{Tor}_i^R(R/I_M, N) \neq 0\} = \inf\{i \mid \operatorname{Tor}_i^R(M/IM, N) \neq 0\}$$

But by Theorem 2.9 we have

$$\inf\{i \mid \operatorname{Tor}_i^R(R/I_M, N) \neq 0\} = \inf\{i \mid \mathrm{H}_i^{I_M}(N) \neq 0\}$$

and

$$\inf\{i \mid \operatorname{Tor}_i^R(M/IM, N) \neq 0\} = \inf\{i \mid \mathrm{H}_i^{I}(M,N) \neq 0\}\,.$$

and so the proof is complete. □

Theorem 2.11. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring, I be an ideal of R, M a finitely generated R-module and N be an artinian R-module. Let n be an integer. Then*

$$\bigcup_{i<n} \operatorname{Cos}_R(\operatorname{Tor}_i^R(M/IM, N)) = \bigcup_{i<n} \operatorname{Cos}_R(\mathrm{H}_i^I(M,N)).$$

Proof. By Lemma 2.2 ${}_{\mathfrak{p}}N$ is weakly colaskerian linearly compact $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus by using [13, Proposition 3.13], Theorem 2.9 and [11, Proposition 3.5] we have:

$$\begin{aligned}
\mathfrak{p} \notin \bigcup_{i<n} \operatorname{Cos}\left(\mathrm{H}_i^I(M,N)\right) &\Leftrightarrow {}_{\mathfrak{p}}\left(\mathrm{H}_i^I(M,N)\right) = 0 \quad \text{for all } i<n \\
&\Leftrightarrow \left(\mathrm{H}_i^{IR_{\mathfrak{p}}}(M_{\mathfrak{p}},\, {}_{\mathfrak{p}}N)\right) = 0 \quad \text{for all } i<n \\
&\Leftrightarrow \left(\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}},\, {}_{\mathfrak{p}}N)\right) = 0 \quad \text{for all } i<n \\
&\Leftrightarrow {}_{\mathfrak{p}}\left(\operatorname{Tor}_i^R(M/IM,N)\right) = 0 \quad \text{for all } i<n \\
&\Leftrightarrow \mathfrak{p} \notin \bigcup_{i<n} \operatorname{Cos}\left(\operatorname{Tor}_i^R(M/IM,N)\right).
\end{aligned}$$

□

Corollary 2.12. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring, I an ideal of R, M a finitely generated R-module and N be an artinian R-module. Let n be an integer and let* $I_M := \operatorname{Ann}(M/IM)$. *Then* $\bigcup_{i<n} \operatorname{Cos}_R(\mathrm{H}_i^{I_M}(N)) = \bigcup_{i<n} \operatorname{Cos}_R(\mathrm{H}_i^I(M,N))$.

Proof. It follows by Theorems 2.11 and 2.7. □

In the remainder, we obtain some results about Noetherianness of local homology modules over local rings.

In the following proof we need the concepet of coatomic modules. Recall that an R-module M is called coatomic, if every proper submodule of M is contained in a maximal submodule of M. This property can also be expressed by $\operatorname{Coass}_R(M) \subseteq \operatorname{Max} R$. Coatomic modules have been studied by Zöschinger [21].

Theorem 2.13. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R, M a finitely generated R -module, N be an artinian R-module and n an integer. Then $\mathrm{Cos}_R(\mathrm{H}_i^I(M,N)) \subseteq \{\mathfrak{m}\}$ for all $i < n$ if and only if $\mathrm{H}_i^I(M,N)$ is a Noetherian R-module for all $i < n$.*

Proof. $\Rightarrow$) At first, note that by [12, Lemma 2.3] $\mathrm{H}_i^I(M,N)$ is linearly compact R-module for all $i \geq 0$.

We use induction on n. Let $n = 1$. There exists an epimorphism $N \to N/I^tN \to 0$ for all $t > 0$ and so we have an epimorphism $N \to \mathrm{H}_0^I(N) \to 0$ for all $t > 0$. Hence $\mathrm{H}_0^I(N)$ is an artinian R-module. By using [5, 2.7] it is easy to see that $\mathrm{H}_0^I(M,N) \simeq M \otimes_R \mathrm{H}_0^I(N)$. It follows that $\mathrm{H}_0^I(M,N)$ is artinian R-module. But $\mathrm{Cos}_R(\mathrm{H}_0^I(M,N)) \subseteq \{\mathfrak{m}\}$ and so $\mathrm{H}_0^I(M,N)$ is a Noetherian R-module by [10, Proposition 7.4].

Now, let $n > 1$. Since N is artinian there is a positive integer u such that $\mathfrak{m}^tN = \mathfrak{m}^uN$ for all $t \geq u$. Set $K = \mathfrak{m}^uN$. The short exact sequence $0 \to K \to N \to N/K \to 0$ induces an exact sequence of generalized local cohomology modules

$$\cdots \to \mathrm{H}_{i+1}^I(M,N/K) \to \mathrm{H}_i^I(M,K) \to \mathrm{H}_i^I(M,N) \to \mathrm{H}_i^I(M,N/K) \to \cdots .$$

Clearly N/K is complete in the $\mathfrak{m}$-adic topology, and $I \subseteq \mathfrak{m}$. Thus N/K is complete in the I-adic topology and so by [14, Lemma 2.7] $\mathrm{H}_i^I(M,N/K) \simeq \mathrm{Tor}_i^R(M,N/K)$ for all $i \geq 0$. Since N/K is artinian and $\mathfrak{m}^u(N/K) = 0$ it follows that N/K is of finite length. Thus $\mathrm{Tor}_i^R(M,N/K)$ is of finite length and so $\mathrm{H}_i^I(M,N/K)$ is an R-module of finite length for all $i \geq 0$. So $\mathrm{Cos}_R(\mathrm{H}_i^I(M,N/K)) \subseteq \{\mathfrak{m}\}$ for all $i \geq 0$ by [10, Proposition 7.4]. Now by the above long exact sequence and assumption $\mathrm{Cos}_R(\mathrm{H}_i^I(M,K)) \subseteq \{\mathfrak{m}\}$ for all $i < n$ also it follows that $\mathrm{H}_i^I(M,K)$ is Noetherain if and only if $\mathrm{H}_i^I(M,N)$ is Noetherian for all $i < n$. Thus it is sufficient to show that $\mathrm{H}_i^I(M,K)$ is Noetherian R-module for all $i < n$.

Since $\mathrm{H}_i^I(M,N)$ is linearly compact, by [2, 4.2] $\mathrm{Coass}_R(\mathrm{H}_i^I(M,N)) \subseteq \mathrm{Cos}_R(\mathrm{H}_i^I(M,N))$ and so assumption implies that $\mathrm{Coass}_R(\mathrm{H}_i^I(M,N)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Thus $\mathrm{Coass}_R(\mathrm{H}_i^I(M,N))$ is coatomic R-module for all $i < n$ and so by [21, Satz 2.4] we can find an integer $t \geq 1$ such that $\mathfrak{m}^t\,\mathrm{H}_i^I(M,N)$ is Noetherian for all $i < n$. But $\mathfrak{m}K = K$ and so $\mathfrak{m}^tK = K$. Thus there is an element $x \in \mathfrak{m}^t$ such that $xK = K$ by [9, 2.8]. The short exact sequence

$$0 \to 0 :_K x \to K \xrightarrow{x} K \to 0$$

induces an exact sequence

$$\cdots \to \mathrm{H}_i^I(M,K) \xrightarrow{x} \mathrm{H}_i^I(M,K) \to \mathrm{H}_{i-1}^I(M,0 :_K x) \to \mathrm{H}_{i-1}^I(M,K) \to \cdots .$$

From the above long exact sequence we conclude that $\mathrm{Cos}_R(\mathrm{H}_i^I(M,0 :_K x)) \subseteq \{\mathfrak{m}\}$ for all $i < n-1$. By the inductive hypothesis, $\mathrm{H}_i^I(M,0 :_K x)$ is Noetherian R-module for all $i < n-1$. Hence $\mathrm{H}_i^I(M,K)/x\,\mathrm{H}_i^I(M,K)$ is Noetherian R-module for all $i < n$. But $x \in \mathfrak{m}^t$. Thus $\mathrm{H}_i^I(M,K)/\mathfrak{m}^t\,\mathrm{H}_i^I(M,K)$ is Noetherian R-module for all $i < n$. On the other hand, since $\mathfrak{m}^t\,\mathrm{H}_i^I(M,N)$ is Noetherian for all $i < n$ it follows that $\mathrm{H}_i^I(M,K)$ is a Noetherian R-module for all $i < n$, as required.

$\Leftarrow$) Since $\mathrm{H}_i^I(M,N)$ is linearly compact by [11, Theorem 3.8] we have $\mathrm{Cos}_R(\mathrm{H}_i^I(M,N)) = \mathrm{Cosupp}_R(\mathrm{H}_i^I(M,N))$ for each i. Now the result follows by [18, 2.10]. □

Corollary 2.14. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R, M be a finitely generated R-module and N an artinian R-module. Then*

$$\inf\{i|\, \mathrm{Cos}_R(\mathrm{H}_i^I(M,N)) \nsubseteq \{\mathfrak{m}\}\} = \inf\{i|\, \mathrm{H}_i^I(M,N) \textit{ is not a Noetherian } R\textit{-module}\}.$$

Proof. It follows by Theorem 2.13. □

Corollary 2.15. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and N be an artinian R-module. Then*

$$\inf\{i \mid \mathrm{Cos}_R(\mathrm{H}_i^I(N)) \nsubseteq \{\mathfrak{m}\}\} = \inf\{i \mid \mathrm{H}_i^I(N) \textit{ is not a Noetherian } R\textit{-module}\}\,.$$

Proof. It follows by Corollary 2.14 by using $M = R$. □

Corollary 2.16. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R, M be a finitely generated R-module and N an artinian R-module. Let $I_M := \mathrm{Ann}_R(M/IM)$. Then*

$$\inf\{i \mid \mathrm{H}_i^{I_M}(N) \textit{ is not Noetherian } R\textit{-module}\} = \inf\{i \mid \mathrm{H}_i^I(M,N) \textit{ is not Noetherian } R\textit{-module}\}\,.$$

Proof. By Corollary 2.12

$$\inf\{i \mid \mathrm{Cos}_R(\mathrm{H}_i^{I_M}(N)) \nsubseteq \{\mathfrak{m}\}\} = \inf\{i \mid \mathrm{Cos}_R(\mathrm{H}_i^I(M,N)) \nsubseteq \{\mathfrak{m}\}\}\,.$$

Now by using Corollaries 2.14 and 2.15 we obtain the result. □

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