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SUPERINTEGRABILITY AND TIME-DEPENDENT INTEGRALS

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Abstract. While looking for additional integrals of motion of several minimally superintegrable systems in static electric and magnetic fields, we have realized that in some cases Lie point symmetries of Euler-Lagrange equations imply existence of explicitly time-dependent integrals of motion through Noether's theorem. These integrals can be combined to get an additional time-independent integral for some values of the parameters of the considered systems, thus implying maximal superintegrability. Even for values of the parameters for which the systems don't exhibit maximal superintegrability in the usual sense they allow a completely algebraic determination of the trajectories (including their time dependence).

1. Introduction

The purpose of this article is to demonstrate an unusual property encountered during a search for additional integrals of motion of some classical superintegrable systems with electric and magnetic field: explicitly time-dependent integrals of motion.

We recall that the standard definition of integrability and superintegrability assumes both the Hamiltonian and integrals to be functions on the phase space, i.e. time independent. Namely, a classical Hamiltonian system in n degrees of freedom is called integrable if it admits n functionally independent integrals of motion in involution. If it admits $n+k$ functionally independent integrals of motion (where $k \leq n-1$), out of which n are in involution, it is called superintegrable.

Thus, one may not expect time-dependent integrals to arise for time-independent Hamiltonians. However, as we shall see, they do, at least for some systems.

In particular, we shall search for previously unknown integrals of the Hamiltonian systems constructed in [6] in the following way:

(1) we find Lie point symmetries of the corresponding Euler-Lagrange equations (we need second or higher order equations to be able to determine symmetry generators algorithmically; thus, Hamilton's equations are not suitable for our purpose – being first order, they possess an infinite dimensional algebra

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of point symmetry generators which solve an underdetermined system of linear PDEs);

(2) among them we look for the ones preserving not only the Euler-Lagrange equations but also the Cartan 2-form $\mathrm{d}\theta$ (defined below equation (7));

(3) we associate with them integrals of motion via geometrical version of Noether's Theorem (see Section 3).

We find that for certain systems considered in [6] some of the integrals constructed using this procedure are time dependent. For some values of the parameters, pairs of these time-dependent integrals give rise to additional globally defined time-independent integrals.

The structure of the paper is as follows: In Section 2 we recall the method for finding point symmetries of ordinary differential equations. In Section 3 we introduce a geometrical version of Noether's theorem. In Section 4 we present the results for one of the systems considered in [6]. We conclude the article with a summary and two essential questions.

For more detailed introduction to the geometrical formulation of classical mechanics and Noether's theorem, which we review in Section 3, we refer the reader to [12, 11, 1]. For recent development see [3] and references therein. Although time-dependent constants of motion in time-independent Hamiltonian systems appeared also in [9, 2, 13], their implications are not much discussed there.

2. Point symmetries of ordinary differential equations

In this paper, we shall consider point symmetries of ordinary differential equations (ODE). These are transformations acting on the space $\mathcal{J}^{(0)}$ of independent and dependent variables $(t, x_1, \ldots, x_n)$ such that they transform any graph of a solution of the given ODE to a graph of a solution.

One-parameter (local) groups of such transformations can be easily characterized through a linear condition in the following way [10, 4]. Let a vector field

$$X = \xi(t, x_1, \ldots, x_n)\frac{\partial}{\partial t} + \eta^{\alpha}(t, x_1, \ldots, x_n)\frac{\partial}{\partial x^{\alpha}} \in \mathfrak{X}(\mathcal{J}^{(0)}), \tag{1}$$

define through its flow a local 1-parameter group of transformations of $\mathcal{J}^{(0)}$. Through its action on graphs of functions and on their derivatives, it can be uniquely extended to a vector field $\mathrm{pr}^{(k)}X$ on the jet bundle $\mathcal{J}^{(k)}$ with coordinates $(t, x_\alpha, x'_\alpha, \ldots, x_\alpha^{(k)})$. Let a mapping $F : \mathcal{J}^{(k)} \to \mathbb{R}^N$ define a system of ordinary differential equations

$$F_\nu\Big(t, x_\alpha(t), \dot{x}_\alpha(t), \ldots, \frac{\mathrm{d}^k x_\alpha(t)}{\mathrm{d}t^k}\Big) = 0\,, \quad \nu = 1, \ldots, N\,. \tag{2}$$

We denote the solution set of the corresponding system of algebraic equations on $\mathcal{J}^{(k)}$ as

$$\Sigma_F = \big\{(t, x_\alpha, x'_\alpha, \ldots, x_\alpha^{(k)}) \in \mathcal{J}^{(k)} \mid F_\nu(t, x_\alpha, x'_\alpha, \ldots, x_\alpha^{(k)}) = 0, \nu = 1, \ldots, N\big\}$$

and assume that $\mathrm{rank}\,(\mathrm{d}F(v)) = N$, for all $v \in \Sigma_F$. The vector field $X \in \mathfrak{X}(\mathcal{J}^{(0)})$ of the form (1) generates a local 1-parameter group of point symmetries of the differential equation (2) if and only if

$$\big(\mathrm{pr}^{(k)} X(F_\nu)\big)(v) = 0\,, \quad \forall v \in \Sigma_F,\ \forall \nu = 1,\ldots,N\,. \tag{3}$$

3. Noether's Theorem

We shall use a geometrical formulation of Lagrangian dynamics on the evolution space $TM \times \mathbb{R}$ (where M is the configuration space of our system). Assuming that the given Lagrangian L is regular, we encode the dynamics in the dynamical vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \Lambda^i(q^i, \dot{q}^i, t)\frac{\partial}{\partial \dot{q}^i}\,, \tag{4}$$

where

$$\Lambda^i(q^i, \dot{q}^i, t) = \Big(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\Big)^{-1}\Big(\frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j \partial q^k}\dot{q}^k - \frac{\partial^2 L}{\partial \dot{q}^j \partial t}\Big). \tag{5}$$

Its integral curves give us solutions of the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) - \frac{\partial L}{\partial q^i} = 0 \tag{6}$$

when projected onto the extended configuration space $M \times \mathbb{R}$ with the coordinates $[q^1, \ldots, q^n, t]$.

The key objects in our formalism are the Cartan 1-form

$$\theta = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^i}(\mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t) \in \Omega^1(TM \times \mathbb{R}) \tag{7}$$

and its exterior derivative $\mathrm{d}\theta$, which is called Cartan 2-form. The dynamical vector field Γ can then be characterized equivalently by the conditions

$$i_\Gamma \mathrm{d}\theta = 0\,, \quad \langle \Gamma, dt\rangle = 1\,, \tag{8}$$

i.e. Γ is the characteristic vector field of the Cartan 2-form $\mathrm{d}\theta$ (which by definition means $i_\Gamma \mathrm{d}\theta = 0$ and $i_\Gamma \mathrm{d}(\mathrm{d}\theta) = 0$) normalized so that the trajectories (integral curves of Γ) are parametrized by time.

A vector field $Y \in \mathfrak{X}(TM \times \mathbb{R})$ is a (generator of) dynamical symmetry of the dynamical vector field $\Gamma \in \mathfrak{X}(TM \times \mathbb{R})$ if and only if a function $g \in C^\infty(TM \times \mathbb{R})$ exists such that

$$[Y, \Gamma] = g \cdot \Gamma\,. \tag{9}$$

The flow of a dynamical symmetry Y preserves the integral curves of Γ albeit possibly reparametrized.

A particular subclass of dynamical symmetries consists of $\mathrm{d}\theta$-symmetries. A vector field $Y \in \mathfrak{X}(TM \times \mathbb{R})$ is a $\mathrm{d}\theta$-symmetry of the dynamical vector field $\Gamma \in \mathfrak{X}(TM \times \mathbb{R})$ if and only if it satisfies

$$\mathscr{L}_Y \mathrm{d}\theta = 0\,. \tag{10}$$

The importance of $\mathrm{d}\theta$-symmetries stems from the following version of Noether's theorem:

Theorem 1. *Let us consider a Lagrangian system with regular Lagrangian L whose dynamics is described by the dynamical vector field Γ and the Cartan 1-form θ. Then*

(1) *to every* $\mathrm{d}\theta$*-symmetry Y corresponds an integral of motion F of the form*

$$F = f - i_Y\theta\,, \quad \text{where} \quad \mathrm{d}f = \mathscr{L}_Y\theta\,. \tag{11}$$

F is defined locally (due to use of Poincaré lemma in its construction) and is determined up to a constant (choice of f).

(2) *To each integral of motion $F \in C^\infty(TM \times \mathbb{R})$ corresponds a* $\mathrm{d}\theta$*-symmetry Y, which is unique up to $h \cdot \Gamma$, where $h \in C^\infty(TM \times \mathbb{R})$.*

(3) *To every integral of motion F corresponds a unique* $\mathrm{d}\theta$*-symmetry X such that $\langle X, \mathrm{d}t\rangle = 0$. It implies $[X, \Gamma] = 0$, i.e. the symmetry X does not change parametrization of integral curves of Γ, which are parametrized by time.*

(4) *Integral of motion F is an invariant of the* $\mathrm{d}\theta$*-symmetry Y, i.e. $Y(F) = 0$.*

For proof see [12, 1].

In comparison to the traditional Noether's theorem, this version has a wider domain of applicability as the symmetries are functions on $TM \times \mathbb{R}$, i.e. may depend on $\dot{q}$, and gives a one-to-one correspondence between the symmetry and the integral of motion.

Unfortunately, there is no general constructive algorithm for finding all $\mathrm{d}\theta$-symmetries of the given Lagrangian system. Thus we restrict our search to point symmetries and determine which of them (more precisely, their prolongations) are $\mathrm{d}\theta$-symmetries. We notice that since not all $\mathrm{d}\theta$-symmetries arise from point symmetries, some integrals of the system may not be found in this way.

4. Example of system with time-dependent integrals

Let us now apply these ideas to the system considered in [6]. We proceed as follows:

(1) we express the given Hamiltonian system in Lagrangian formulation;

(2) we find generators of point symmetries of its Euler-Lagrange equations;

(3) we extend the generators from $\mathcal{J}^{(0)}$ to $\mathcal{J}^{(1)}$ via their first prolongation to get the corresponding dynamical symmetries and establish which of them are $\mathrm{d}\theta$-symmetries;

(4) we associate to the $\mathrm{d}\theta$-symmetries the corresponding integrals of motion via Noether's Theorem (Theorem 1).

The system under consideration is Case A.2 from [6]. Its Hamiltonian reads

$$\begin{aligned} H &= \frac{1}{2}\big(\vec{p} + \vec{A}(\vec{x})\big)^2 + W(\vec{x}) \\ &= \frac{1}{2}\big(p_x^2 + p_y^2 + (p_z - \Omega_1 y - \Omega_2 x)^2\big) + \frac{\Omega_2^2}{2\kappa^2}\Big(\frac{\Omega_1\kappa^2}{\Omega_2}x - y\Big)^2, \end{aligned} \tag{12}$$

i.e. describes a particle in a constant magnetic field $\vec{B}$ and an effective potential W with

$$\vec{B}(\vec{x}) = (-\Omega_1, \Omega_2, 0), \quad \vec{A} = (0, 0, -\Omega_2 x - \Omega_1 y)\,,$$
$$W(\vec{x}) = \frac{\Omega_2^2}{2\kappa^2}\Big(\frac{\Omega_1\kappa^2}{\Omega_2}x - y\Big)^2,$$

Ω_1, Ω_2 and κ are real constants, the mass is set to 1 and electric charge to -1.

The system (12) is known to be minimally superintegrable (see [6]) and for

$$\kappa = \frac{m}{n} \in \mathbb{Q}\,, \qquad m, n \text{ mutually prime} \tag{13}$$

even maximally superintegrable, with an additional integral of order $m + n - 1$ (cf. [7]).

The corresponding Euler-Lagrange equations read

$$\ddot{x} = \Omega_2\dot{z} - \Omega_1\Omega_2\Big(\frac{\Omega_1\kappa^2}{\Omega_2}x - y\Big), \qquad \ddot{y} = \dot{z}\Omega_1 + \frac{\Omega_2^2}{\kappa^2}\Big(\frac{\Omega_1\kappa^2}{\Omega_2}x - y\Big),$$
$$\ddot{z} = -\Omega_1\dot{y} - \Omega_2\dot{x}\,. \tag{14}$$

They possess for an arbitrary κ the following eight generators of Lie point symmetries

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial z}, \quad Y_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad Y_4 = \frac{\partial}{\partial x} + \frac{\Omega_1}{\Omega_2}\kappa^2\frac{\partial}{\partial y},$$
$$Y_5 = \sin(\omega t)\frac{\partial}{\partial x} + \frac{\Omega_2}{\omega}\cos(\omega t)\frac{\partial}{\partial z}, \quad Y_6 = \cos(\omega t)\frac{\partial}{\partial x} - \frac{\Omega_2}{\omega}\sin(\omega t)\frac{\partial}{\partial z}, \tag{15}$$
$$Y_7 = \sin\Big(\frac{\omega}{\kappa}t\Big)\frac{\partial}{\partial y} + \frac{\Omega_1\kappa}{\omega}\cos\Big(\frac{\omega}{\kappa}t\Big)\frac{\partial}{\partial z}, \quad Y_8 = \cos\Big(\frac{\omega}{\kappa}t\Big)\frac{\partial}{\partial y} - \frac{\Omega_1\kappa}{\omega}\sin\Big(\frac{\omega}{\kappa}t\Big)\frac{\partial}{\partial z},$$

where $\omega = \sqrt{\Omega_1^2\kappa^2 + \Omega_2^2}$, which are enhanced to twelve generators when $\kappa = 1$

$$\begin{aligned}
Y_9 &= z\frac{\partial}{\partial x} + z\frac{\Omega_1}{\Omega_2}\frac{\partial}{\partial y} - \Big(\frac{\Omega_1}{\Omega_2}y + x\Big)\frac{\partial}{\partial z}\,,\\
Y_{10} &= y\frac{\partial}{\partial x} + \Big(\frac{\Omega_1^2 - \Omega_2^2}{\Omega_1\Omega_2}y + x\Big)\frac{\partial}{\partial y} + \frac{\Omega_1}{\Omega_2}z\frac{\partial}{\partial z}\,,\\
Y_{11} &= \Big[\Big(\frac{\Omega_1}{\Omega_2}y + x\Big)\sin(\omega t) + \frac{\omega}{\Omega_2}z\cos(\omega t)\Big]\frac{\partial}{\partial x}\\
&\quad + \frac{\Omega_1}{\Omega_2^2}\big[(\Omega_1 y + \Omega_2 x)\sin(\omega t) + \omega z\cos(\omega t)\big]\frac{\partial}{\partial y}\\
&\quad + \frac{\omega}{\Omega_2^2}\big[(\Omega_1 y + \Omega_2 x)\cos(\omega t) - \omega^2 z\sin(\omega t)\big]\frac{\partial}{\partial z}\,,\\
Y_{12} &= \Big[\Big(\frac{\Omega_1}{\Omega_2}y + x\Big)\cos(\omega t) - \frac{\omega}{\Omega_2}z\sin(\omega t)\Big]\frac{\partial}{\partial x}\\
&\quad + \frac{\Omega_1}{\Omega_2^2}\big[(\Omega_1 y + \Omega_2 x)\cos(\omega t) - \omega z\sin(\omega t)\big]\frac{\partial}{\partial y}\\
&\quad - \frac{\omega}{\Omega_2^2}\big[(\Omega_1 y + \Omega_2 x)\sin(\omega t) + \omega^2 z\cos(\omega t)\big]\frac{\partial}{\partial z}\,.
\end{aligned} \tag{16}$$

The Lie algebras of these vector fields are summarized in Table 1 for both $\kappa \neq 1$ and $\kappa = 1$. For $\kappa \neq 1$ it is a solvable algebra with an Abelian nilradical spanned by Y_2, Y_4, Y_5, Y_6, Y_7, Y_8 on which ad_{Y_3} acts as $-\mathbb{I}$ and Y_1 generates rotation in Y_5, Y_6 and Y_7, Y_8 subspaces.

For $\kappa = 1$ we have Levi decomposable algebra with $\mathfrak{sl}(2)$ Levi factor spanned by Y_9, Y_{11}, Y_{12} and the radical spanned by the elements $Y_1 + \frac{\Omega_2}{2}Y_9, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_{10}$. The radical is in turn decomposed into an Abelian nilradical spanned by $Y_2, Y_4, Y_5, Y_6, Y_7, Y_8$ and an Abelian subalgebra spanned by $Y_1 + \frac{\Omega_2}{2}Y_9, Y_3, Y_{10}$, which acts on the nilradical in a rather nontrivial way.

Among the generators $Y_1, \dots, Y_8$ all except Y_3 give rise to dθ-symmetries through their prolongation. The integrals obtained through Theorem 1 read

$$
\begin{aligned}
& E = \frac{1}{2}\dot{\vec{x}}^2 + \frac{\Omega_2^2}{2\kappa^2}\left(\frac{\Omega_1}{\Omega_2}\kappa^2 x - y\right)^2, \quad F_2 = -p_z = -\dot{z} - \Omega_1 y - \Omega_2 x\,, \\
& F_4 = -\frac{\Omega_1}{\Omega_2}\kappa^2\dot{y} - \dot{x} + \frac{\omega^2}{\Omega_2}z\,, \\
& F_5 = -\frac{\Omega_2}{\omega}p_z\cos(\omega t) + \omega x\cos(\omega t) - \dot{x}\sin(\omega t)\,, \\
(17) \qquad & F_6 = \frac{\Omega_2}{\omega}p_z\sin(\omega t) - \omega x\sin(\omega t) - \dot{x}\cos(\omega t)\,, \\
& F_7 = -\frac{\Omega_1\kappa}{\omega}p_z\cos\left(\frac{\omega}{\kappa}t\right) + \frac{\omega}{\kappa}y\cos\left(\frac{\omega}{\kappa}t\right) - \dot{y}\sin\left(\frac{\omega}{\kappa}t\right), \\
& F_8 = \frac{\Omega_1\kappa}{\omega}p_z\sin\left(\frac{\omega}{\kappa}t\right) - \frac{\omega}{\kappa}y\sin\left(\frac{\omega}{\kappa}t\right) - \dot{y}\cos\left(\frac{\omega}{\kappa}t\right).
\end{aligned}
$$

Inspired by [5], we can consider F_5 and F_6 (F_7 and F_8, respectively) as real and imaginary parts of complex integrals

$$
(18) \qquad J_5 = \left(\omega x - \frac{\Omega_2}{\omega}p_z + \mathrm{i}\dot{x}\right)\mathrm{e}^{\mathrm{i}\omega t}, \quad J_7 = \left(\frac{\omega}{\kappa}y - \frac{\Omega_1\kappa}{\omega}p_z + \mathrm{i}\dot{y}\right)\mathrm{e}^{\mathrm{i}\frac{\omega}{\kappa}t}.
$$

Two time-independent integrals can be constructed as squares of their norms. After simplification, they read

$$
\tilde{F}_5 = \dot{x}^2 + \left(\frac{\Omega_2(\dot{z} + \Omega_1 y) - \Omega_1^2\kappa^2 x}{\omega}\right)^2, \ \tilde{F}_7 = \dot{y}^2 + \left(\frac{\Omega_1\kappa^2(\dot{z} + \Omega_2 x) - \Omega_2^2 y}{\kappa\omega}\right)^2,
$$

out of which only one is independent of E, p_z and F_4 since they are related to the energy through $\tilde{F}_5 + \tilde{F}_7 = 2E$.

If $\kappa = \frac{m}{n} \in \mathbb{Q}$, we obtain an additional integral. We combine J_5 and J_7 to get

$$
(19) \qquad J_{57} = J_5^n \bar{J}_7^m = \left(\omega x - \frac{\Omega_2}{\omega}p_z + \mathrm{i}\dot{x}\right)^n\left(\frac{n\omega}{m}y - \frac{m\Omega_1}{n\omega}p_z - \mathrm{i}\dot{y}\right)^m,
$$

where bar means complex conjugation. Its real or imaginary part is the fifth independent integral, the other four being E, p_z, F_4 and $\tilde{F}_5$ (or $\tilde{F}_7$). Explicit expressions for the real and imaginary part of (19) can be obtained from the following formulas

$$
\begin{aligned}
\operatorname{Re} w_1^n \bar{w}_2^m &= |w_1|^{n-1}|w_2|^{m-1}\Big[|w_1||w_2| T_n\Big(\frac{\operatorname{Re} w_1}{|w_1|}\Big) T_m\Big(\frac{\operatorname{Re} w_2}{|w_2|}\Big) \\
&\quad + \operatorname{Im} w_1 \operatorname{Im} w_2\, U_{n-1}\Big(\frac{\operatorname{Re} w_1}{|w_1|}\Big) U_{m-1}\Big(\frac{\operatorname{Re} w_2}{|w_2|}\Big)\Big], \\
\operatorname{Im} w_1^n \bar{w}_2^m &= |w_1|^{n-1}|w_2|^{m-1}\Big[|w_2| \operatorname{Im} w_1\, U_{n-1}\Big(\frac{\operatorname{Re} w_1}{|w_1|}\Big) T_m\Big(\frac{\operatorname{Re} w_2}{|w_2|}\Big) \\
&\quad - |w_1| \operatorname{Im} w_2\, T_n\Big(\frac{\operatorname{Re} w_1}{|w_1|}\Big) U_{m-1}\Big(\frac{\operatorname{Re} w_2}{|w_2|}\Big)\Big],
\end{aligned}
\tag{20}
$$

where

$$
w_1 = \omega x - \frac{\Omega_2}{\omega} p_z + \mathrm{i}\dot{x}\,, \quad w_2 = \frac{n\omega}{m} y - \frac{m\Omega_1}{n\omega} p_z + \mathrm{i}\dot{y}
\tag{21}
$$

and T_n, U_n are Chebyshev polynomials of the first and second type, respectively.

Thus, we have recovered through the point symmetry approach the four known time-independent integrals which were already found in [6] and imply minimal superintegrability of our system as well as the fifth integral for rational κ as in [7]. For $\kappa = 1$ the only additional dθ-symmetry is Y_9. The corresponding integral reads

$$
F_9 = \frac{\omega^2 z^2 + (\Omega_1 y + \Omega_2 x)^2 + 2\Omega_1(y\dot{z} - \dot{y}z) + 2\Omega_2(x\dot{z} - \dot{x}z)}{2\Omega_2}\,.
$$

$[.,.]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
Y_1	0	0	0	0	ωY_6	$-\omega Y_5$	$\frac{\omega}{\kappa} Y_8$	$-\frac{\omega}{\kappa} Y_7$	0	0	ωY_{12}	$-\omega Y_{11}$
Y_2		0	Y_2	0	0	0	0	0	Y_4	$\frac{\Omega_1}{\Omega_2} Y_2$	$\omega \frac{\Omega_1}{\Omega_2^2} Y_8 + \frac{\omega}{\Omega_2} Y_6$	$-\omega \frac{\Omega_1}{\Omega_2^2} Y_7 - \frac{\omega}{\Omega_2} Y_5$
Y_3			0	$-Y_4$	$-Y_5$	$-Y_6$	$-Y_7$	$-Y_8$	0	0	0	0
Y_4				0	0	0	0	0	$-\frac{\omega^2}{\Omega_2^2} Y_2$	$\frac{\Omega_1}{\Omega_2} Y_4$	$\omega^2 \frac{\Omega_1}{\Omega_2^3} Y_7 + \frac{\omega^2}{\Omega_2^2} Y_5$	$\omega^2 \frac{\Omega_1}{\Omega_2^3} Y_8 + \frac{\omega^2}{\Omega_2^2} Y_6$
Y_5					0	0	0	0	$\frac{\Omega_1}{\omega} Y_8 + \frac{\Omega_2}{\omega} Y_6$	Y_7	Y_4	$\frac{\omega}{\Omega_2} Y_2$
Y_6						0	0	0	$-\frac{\Omega_1}{\omega} Y_7 - \frac{\Omega_2}{\omega} Y_5$	Y_8	$\frac{\omega}{\Omega_2} Y_2$	Y_4
Y_7							0	0	$\frac{\Omega_1^2}{\omega\Omega_2} Y_8 + \frac{\Omega_1}{\omega} Y_6$	$Y_5 + (\frac{\Omega_1}{\Omega_2} - \frac{\Omega_2}{\Omega_1}) Y_7$	$\frac{\Omega_1}{\Omega_2} Y_4$	$-\omega \frac{\Omega_1}{\Omega_2^2} Y_2$
Y_8								0	$-\frac{\Omega_1^2}{\omega\Omega_2} Y_7 - \frac{\Omega_1}{\omega} Y_5$	$Y_6 + (\frac{\Omega_1}{\Omega_2} - \frac{\Omega_2}{\Omega_1}) Y_8$	$\omega \frac{\Omega_1}{\Omega_2^2} Y_2$	$\frac{\Omega_1}{\Omega_2} Y_4$
Y_9									0	0	$-2 \frac{\omega}{\Omega_2} Y_{12}$	$2 \frac{\omega}{\Omega_2} Y_{11}$
Y_{10}										0	0	0
Y_{11}											0	$2 \frac{\omega^3}{\Omega_2^3} Y_9$

TAB. 1: Algebra of symmetries $Y_1, \ldots, Y_{12}$ from (15). The generators $Y_9, \ldots, Y_{12}$ are present only for $\kappa = 1$.

It is independent of E, p_z, F_4 and $\tilde{F}_5$, i.e. the system is maximally superintegrable. However, the integral F_9 is linear in velocities, thus in momenta, and was already known in [6].

It is also worth noting that while $\mathrm{d}\theta$-symmetry generators Y_1, Y_2, $Y_4, \ldots, Y_8$ form an ideal in the generic algebra of the point symmetries for $\kappa \neq 1$, $\mathrm{d}\theta$-symmetry generators for $\kappa = 1$ form a subalgebra but not an ideal in the full point symmetry algebra, cf. Table 1.

We observe that time-dependent integrals $F_5, \ldots, F_8$ in (17) are useful even for $\kappa \notin \mathbb{Q}$. The integrals (17) involve six independent functions on the 7-dimensional manifold $TM \times \mathbb{R}$, i.e. their values determined by the initial conditions restrict the dynamics to a curve in $TM \times \mathbb{R}$ and allow us to find the trajectories algebraically as follows:

By eliminating $\dot{x}$ from F_5 and F_6 we get

$$F_6 \sin(\omega t) - F_5 \cos(\omega t) = -\omega x + \frac{\Omega_2}{\omega} p_z \,,$$

from which follows that

$$x(t) = \frac{F_5 \cos(\omega t) - F_6 \sin(\omega t)}{\omega} + \frac{\Omega_2}{\omega^2} p_z \,. \tag{22}$$

The expression for $y(t)$ arising from F_8 and F_7 in the same way is

$$y(t) = \frac{\left[F_7 \cos\left(\frac{\omega}{\kappa} t\right) - F_8 \sin\left(\frac{\omega}{\kappa} t\right)\right]\kappa}{\omega} + \frac{\Omega_1 \kappa^2}{\omega^2} p_z \,. \tag{23}$$

The following expression for $z(t)$ is obtained from F_4, cf. (17), through substitution for $\dot{x}$ and $\dot{y}$ from F_5, F_6 and F_7, F_8, respectively,

$$z(t) = \frac{\Omega_2 F_4}{\omega^2} - \frac{\Omega_1 \kappa^2}{\omega^2}\Big(F_7 \sin\left(\frac{\omega}{\kappa} t\right) + F_8 \cos\left(\frac{\omega}{\kappa} t\right)\Big) - \frac{\Omega_2}{\omega^2}(F_5 \sin(\omega t) + F_6 \cos(\omega t)) \,. \tag{24}$$

As a side note, let us mention that the time-dependent integrals $F_5, \ldots, F_8$ are integrated Euler-Lagrange equations for x and y (14) with a suitable integrating factor, e.g.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} F_5 &= \Omega_2 p_z \sin(\omega t) + \omega \dot{x} \cos(\omega t) - \omega^2 x \sin(\omega t) - \ddot{x} \sin(\omega t) - \omega \dot{x} \cos(\omega t) \\ &= -\sin(\omega t)\left(\ddot{x} + \omega^2 x - \Omega_2 p_z\right),\end{aligned}$$

because p_z is an integral of motion, i.e. constant.

5. Conclusions

We have considered an approach to the construction of integrals of motion through the Lie point symmetry analysis of the Euler-Lagrange equations and Noether's theorem. For the considered system, the obtained integrals allow algebraic determination of trajectories and for some values of the system's parameters also allow the construction of an additional independent integral, making the system maximally superintegrable.

The example just presented (and a few others) lead us to two essential questions:

– Is the presence of time-dependent integrals of time-independent systems just an indication that the system is in some way trivial?

  All the time-dependent integrals found so far were in pairs, involving trigonometric functions sin and cos of time t and could be combined into one of the following type of integrals: $(\omega x^j + \mathrm{i}\dot{x}^j)e^{\mathrm{i}\omega t}$ or $(\dot{x}^j - \mathrm{i}\dot{x}^k)e^{\mathrm{i}\omega t}$ for $j \neq k$ and some constant ω. The first type is connected with harmonic motion in one degree of freedom, the other with elliptical motion connecting two degrees of freedom. We have also seen that these integrals are actually once integrated linear equations of motion. Such triviality of time-dependent integrals was already encountered in the literature, cf. [8], where it was related to a (gauge) freedom in the choice of Lagrangian for the given system.

– Does the presence of time-dependent integrals give us some new information about superintegrability?

  The system considered here is known to be maximally superintegrable for $\kappa = \frac{m}{n} \in \mathbb{Q}$, with an additional integral of order $m + n - 1$ in momenta (or, equivalently, velocities). However, in the Lie point symmetry analysis of Euler–Lagrange equations, there was no difference in the number or structure of the symmetries between κ rational and irrational, except for the particular value $\kappa = 1$.

  On the other hand, the integrals of the system (12) restrict the motion to an algebraically computable curve in the extended phase space. For the values of parameters for which the projection of the curve on the phase space is closed and the integrals can be combined to an time–independent one we found an additional integral which makes the system maximally superintegrable. Since maximal superintegrability implies closed trajectories, time-dependent integrals and their implications for the shape of the trajectories restrict the range of parameters for which maximal superintegrability is possible. Moreover, they hint at maximal superintegrability for the allowed parameters.

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