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# THE AFFINE APPROACH TO HOMOGENEOUS GEODESICS IN HOMOGENEOUS FINSLER SPACES

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Abstract. In the recent paper [Yan, Z.: Existence of homogeneous geodesics on homogeneous Finsler spaces of odd dimension, *Monatsh. Math.* **182**,1, 165–171 (2017)], it was claimed that any homogeneous Finsler space of odd dimension admits a homogeneous geodesic through any point. However, the proof contains a serious gap. The situation is a bit delicate, because the statement is correct. In the present paper, the incorrect part in this proof is indicated. Further, it is shown that homogeneous geodesics in homogeneous Finsler spaces can be studied by another method developed in earlier works by the author for homogeneous affine manifolds. This method is adapted for Finsler geometry and the statement is proved correctly.

## 1. Introduction

Let $M$ be either a pseudo-Riemannian manifold $(M,g)$, or a Finsler space $(M,F)$, or an affine manifold $(M,\nabla)$. If there is a connected Lie group $G$ which acts transitively on $M$ as a group of isometries, respectively, of affine diffeomorphisms, then $M$ is called a *homogeneous manifold*. It can be naturally identified with the *homogeneous space* $G/H$, where $H$ is the isotropy group of the origin $p \in M$.

A geodesic $\gamma(s)$ through the point $p$ is *homogeneous* if it is an orbit of a one-parameter group of isometries, respectively, of affine diffeomorphisms. More explicitly, if $s$ is an affine parameter and $\gamma(s)$ is defined in an open interval $J$, there exists a diffeomorphism $s=\varphi(t)$ between the real line and the open interval $J$ and a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector $X$ is called a *geodesic vector*. The diffeomorphism $\varphi(t)$ may be nontrivial only for null geodesics in a properly pseudo-Riemannian manifold or for geodesics in affine manifolds.

A homogeneous Riemannian manifold $(M,g)$ or a homogeneous Finsler space $(M,F)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of $H$ on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive

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decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space $T_pM$ via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product or the Minkovski norm on $T_pM$, we obtain the invariant scalar product $\langle\, ,\rangle$ or the invariant Minkowski norm denoted again by $F$ and its fundamental tensor $g$ on $\mathfrak{m}$. It will be clear from the context whether $g$ means the pseudo-Riemannian metric on the manifold or the fundamental tensor on $\mathfrak{m}$ coming from the Finsler metric. In the second case, it is used usually with the subscript in the form $g_X$. In the pseudo-Riemannian reductive case, geodesic vectors are characterized by the following *geodesic lemma*:

**Lemma 1** ([10], [8], [6]). *Let $(G/H, g)$ be a reductive homogeneous pseudo-Riemannian manifold and $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to some parameter $s$ if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = k\langle X_{\mathfrak{m}}, Z\rangle \tag{1}$$

*for all $Z \in \mathfrak{m}$ and for some constant $k \in \mathbb{R}$. If $k = 0$, then $t$ is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a properly pseudo-Riemannian space.*

In the above formula, the subscript $\mathfrak{m}$ refers to the $\mathfrak{m}$-component of vectors from $\mathfrak{g}$. The Finslerian version of this lemma was proved in [11]:

**Lemma 2** ([11]). *Let $(G/H, F)$ be a homogeneous Finsler space. The vector $X \in \mathfrak{g}$ is a geodesic vector if and only if it holds*

$$g_{X_{\mathfrak{m}}}([X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0 \tag{2}$$

*for all $Z \in \mathfrak{m}$.*

Another possible approach is to study the manifold $M$ using a more fundamental affine method, which was proposed by the present author, O. Kowalski and Z. Vlášek in [3] and [7]. It is based on the well known fact that a homogeneous manifold $M$ with the origin $p$ admits $n = \dim M$ fundamental vector fields (Killing vector fields) which are linearly independent at each point of some neighbourhood of $p$. It is well known that, in a homogeneous space $M = G/H$ with an invariant affine connection $\nabla$, each regular orbit of a 1-parameter subgroup $g_t \subset G$ on $M$ is an integral curve of an affine Killing vector field on $M$.

**Lemma 3** ([7]). *The integral curve $\gamma$ of a nonvanishing Killing vector field $Z$ on $M = (G/H, \nabla)$ is geodesic if and only if*

$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)} \tag{3}$$

*holds along $\gamma$, where $k_\gamma \in \mathbb{R}$ is a constant. If $k_\gamma = 0$, then $t$ is the affine parameter of geodesic $\gamma$. If $k_\gamma \neq 0$, then the affine parameter of this geodesic is $s = e^{k_\gamma t}$.*

In the paper [9], O. Kowalski and J. Szenthe proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin. The proof is using the reductive decomposition of the Lie algebra of the isometry group and Lemma 1. The generalization to the pseudo-Riemannian (reductive and nonreductive) case was obtained by the present author in [4] in the framework of a more general result, which says that any homogeneous affine manifold $(M, \nabla)$ admits a homogeneous geodesic through the origin. Here the affine method from [7] and [3], based on the study of integral curves of Killing vector fields and Lemma 3, was used. The proof is also using differential topology, namely smooth mappings $\mathbb{S}^n \to \mathbb{S}^n$.

Recently, in the paper [12] by Z. Yan, the existence of a homogeneous geodesic in homogeneous Finsler space of odd dimension was claimed. In the proof, the same steps as in [9] were followed and Lemma 2 was used. Unfortunately, the step in generalization to the Finslerian situation was wrong.

In the present paper, the wrong part from the proof in [12] is indicated. It is further shown how the affine method developed in [4], [3] and [7] can be adapted to the Finslerian setting and the short proof of the statement in odd dimension is provided.

We should mention also the recent developments in the topic. The affine method presented in this paper was further refined by the author in the paper [5] and the existence of a homogeneous geodesic was proved in any homogeneous Berwald space and any reversible Finsler space. In the recent paper [13] by Z. Yan and L. Huang, the situation was studied in full generality. Using some ideas from the original paper [9] by O. Kowalski and J. Szenthe and a purely Finslerian construction, it was proved that any homogeneous Finsler space admits a homogeneous geodesic.

## 2. Basic settings

Recall that a *Minkowski norm* on the vector space $\mathbb{V}$ is a nonnegative function $F\colon \mathbb{V} \to \mathbb{R}$ which is smooth on $\mathbb{V} \setminus \{0\}$, positively homogeneous ($F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$) and whose Hessian $g_{ij} = (\frac{1}{2}F^2)_{y^i y^j}$ is positively definite on $\mathbb{V} \setminus \{0\}$. Here $(y^i)$ are the components of a vector $y \in \mathbb{V}$ with respect to a fixed basis $B$ of $\mathbb{V}$ and putting $y^i$ to a subscript means the patrial derivative. Then the pair $(\mathbb{V}, F)$ is called the *Minkowski space*. The tensor $g_y$ with components $g_{ij}(y)$ is the *fundamental tensor*. The *Cartan tensor* $C_y$ has components $C_{ijk}(y) = (\frac{1}{4}F^2)_{y^i y^j y^k}$. A Finsler metric on the smooth manifold $M$ is a function $F$ on $TM$ which is smooth on $TM \setminus \{0\}$ and whose restriction to any tangent space $T_xM$ is a Minkowski norm. Then the pair $(M, F)$ is called the *Finsler space*. On a Finsler space, functions $g_{ij}$ and $C_{ijk}$ depend smoothly on $x \in M$ and on $o \neq y \in T_xM$.

Further, we recall that the *slit tangent bundle* $TM_0$ is defined as $TM_0 = TM \setminus \{0\}$. Using the restriction of the natural projection $\pi\colon TM \to M$ to $TM_0$, we naturally construct the pullback vector bundle $\pi^*TM$ over $TM_0$, as indicated in the following diagram:

$$\begin{array}{ccc} \pi^*TM & & TM \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ TM_0 & \xrightarrow{\ \pi\ } & M \end{array}$$

For a given local coordinate system $(x^1, \dots, x^n)$ on $U \subset M$, at any $x \in M$, one has a natural basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of $T_xM$. It is natural to express tangent vectors

$y \in T_xM$ with respect to this basis. Then $(x^i, y^i)$ is the *natural coordinate system* on $TU_0$. The *Chern connection* is the unique linear connection on the vector bundle $\pi^*TM$ which is torsion free and almost $g$-compatible, see some monograph, for example [1] by D. Bao, S.-S. Chern and Z. Shen or [2] by S. Deng for details. If we fix a nowhere vanishing vector field $V$ on $M$, we obtain an affine connection $\nabla^V$ on $M$. In the local chart, it is expressed with respect to arbitrary vector fields $W_1 = W_1^i \frac{\partial}{\partial x^i}$ and $W_2 = W_2^i \frac{\partial}{\partial x^i}$ by the formula

$$\nabla^V_{W_1} W_2|_x = \big[W_1(W_2^i) + W_2^j {W_1}^k \Gamma^i_{jk}(x, V)\big] \frac{\partial}{\partial x^i}\,, \tag{4}$$

where Christoffel symbols $\Gamma^i_{jk}(x, V)$ are determined by the Finsler metric. The affine connection $\nabla^V$ on $M$ is torsion free and almost metric compatible, which means

$$\begin{aligned} \nabla^V_{W_1} W_2 - \nabla^V_{W_2} W_1 &= [W_1, W_2]\,, \\ W g_V(W_1, W_2) &= g_V(\nabla^V_W W_1, W_2) + g_V(W_1, \nabla^V_W W_2) \\ &\quad + 2C_V(\nabla^V_W V, W_1, W_2)\,, \end{aligned} \tag{5}$$

for arbitrary vector fields $W$, $W_1$, $W_2$. Using the affine connection $\nabla^V$, we define the derivative along a curve $\gamma(t)$ with velocity vector field $T$. Let $W_1$, $W_2$ be vector fields along $\gamma$, we define

$$D_{W_1} W_2 = \nabla^{T'}_{W_1'} W_2'\,, \tag{6}$$

where the vector fields $T'$, $W_1'$ and $W_2'$ on the right-hand side are smooth extensions of $T$, $W_1$ and $W_2$ to the neighbourhood of $\gamma(t)$. The definition above does not depend on the particular extension. A regular smooth curve $\gamma$ with tangent vector field $T$ is a *geodesic* if $D_T(\frac{T}{F(T)}) = 0$. In particular, a geodesic of constant speed satisfies $D_T T = 0$.

## 3. Wrong part in [12]

In the paper [12], the existence of a homogeneous geodesic in a homogeneous Finsler space $(M, F)$ of odd dimension is proved using the method from [9]. The reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and the Killing form $K$ of $\mathfrak{g}$ are considered. The case $\mathrm{rad}(K) = \mathfrak{m}$ is rather easy, so the case $\mathrm{rad}(K) \subsetneq \mathfrak{m}$ is of interest.

For each unit vector $X \in \mathrm{S}^{n-1} \subset \mathfrak{m}$ (with respect to an invariant scalar product on $\mathfrak{m}$), the operator $\alpha^X : \mathfrak{m} \to \mathfrak{m}$ is defined by the formula

$$g_X(\alpha^X U, V) = K(U, V) \quad \forall U, V \in \mathfrak{m}\,.$$

If there was a vector $\bar{X} \in \mathfrak{m}$ such that the eigenvector $Y^{\bar{X}}$ of the operator $\alpha^{\bar{X}}$ corresponding to a nonzero eigenvalue $\lambda^{\bar{X}}$ satisfies $Y^{\bar{X}} = \bar{X}$, we could use similar steps as in [9]. We could write

$$\begin{aligned} g_{\bar X}(\bar X,[\bar X,Z]_{\mathfrak m}) &= g_{\bar X}(Y^{\bar X},[\bar X,Z]_{\mathfrak m}) = \frac{1}{\lambda^{\bar X}} g_{\bar X}(\alpha^{\bar X}(Y^{\bar X}),[\bar X,Z]_{\mathfrak m}) \\ &= \frac{1}{\lambda^{\bar X}} g_{\bar X}(\alpha^{\bar X}(\bar X),[\bar X,Z]_{\mathfrak m}) = \frac{1}{\lambda^{\bar X}} K(\bar X,[\bar X,Z]_{\mathfrak m}) \\ &= \frac{1}{\lambda^{\bar X}} K(\bar X,[\bar X,Z]) = \frac{1}{\lambda^{\bar X}} K([\bar X,\bar X],Z]) = 0 \quad \forall Z\in\mathfrak m \end{aligned}$$

and $\bar X$ would be a geodesic vector. The crucial step here are the first equality and the third equality, where the vector $\bar X$ and the eigenvector $Y^{\bar X}$ are exchanged. For the construction of the vector $\bar X$ with the desired property $Y^{\bar X}=\bar X$, in the paper [12], the mapping $v\colon \mathrm S^{n-1}\to\mathrm S^{n-1}$ is constructed in the following way: For each unit vector $X\in\mathrm S^{n-1}\subset\mathfrak m$ as above, define $v(X)$ as the eigenvector of the operator $\alpha^X$ corresponding to the eigenvalue with the maximal absolute value. The mapping $v$ is claimed to be continuous on the definition domain $\mathrm S^{n-1}$ and the fixed point theorem is used. However, this part of the proof is not well justified. In general, for the family of operators $\alpha^X$, the assignment $X\mapsto v(X)$ as above is not a continuous mapping.

We demonstrate this fact with a counterexample. Consider the one-parameter family of operators $\alpha(t)$ represented by the diagonal matrices with $(1-t,t)$ on the diagonal, for $0<t<1$. Clearly, the eigenvalues of these operators are $\lambda_1(t)=1-t$ and $\lambda_2(t)=t$. For $t=1/2$, we obtain $\lambda_1=\lambda_2=1/2$ and any vector in the plane is the eigenvector of the operator $\alpha(1/2)$. The image of the mapping $v$ defined as above is the vector $(1,0)$ for $t<1/2$ and the vector $(0,1)$ for $t>1/2$. At $t=1/2$, $v$ is a multivalued mapping and it is not continuous. This is a serious gap in the proof, because there is not an obvious way how to correct it.

## 4. Affine method adapted to Finsler spaces

We are now going to adapt the affine method, developed in the papers [7], [4] and [3] for affine homogeneous manifold, to Finsler geometry. We shall prove the existence of a homogeneous geodesic in a homogeneous Finsler space $(M,F)$ of odd dimension using this method. First, let us formulate simple observations which follow from homogeneity of the Finsler metric $F$.

**Proposition 4.** *Let $(M,F)$ be a homogeneous Finsler space, $g$ be the corresponding fundamental tensor, $G$ be a group of isometries acting transitively on $M$, $X^*$ be a Killing vector field generated by a vector $X\in\mathfrak g$, $\phi(t)=\exp(tX)$ and $\gamma(t)$ be the integral curve of $X^*$ through $p\in M$. Along the curve $\gamma(t)$, it holds*

$$\begin{aligned} \phi(t)(p) &= \gamma(t)\,, \\ \phi(t)_*(X^*(p)) &= X^*(\gamma(t)) \end{aligned} \tag{7}$$

*and*

$$\begin{aligned} F(\phi(t)(p),\phi(t)_*V) &= F(p,V)\,, \\ g_{(\gamma(t),X^*(\gamma(t)))}(\phi(t)_*U,\phi(t)_*V) &= g_{(p,X^*(p))}(U,V)\,, \end{aligned} \tag{8}$$

*for all $t \in \mathbb{R}$ and for all $U, V \in T_pM$.*

**Proposition 5.** *With the same assumptions as in Proposition 4, along the curve $\gamma(t)$, it holds*

$$g_{(\gamma(t),X^*(\gamma(t)))}(D_{X^*}X^*\big|_{\gamma(t)}, \phi(t)_*U) = g_{(p,X^*(p))}(D_{X^*}X^*\big|_p, U)\,, \tag{9}$$

*for all $t \in \mathbb{R}$ and for all $U \in T_pM$. Consequently, if*

$$D_{X^*}X^*\big|_p = 0\,, \tag{10}$$

*then the curve $\gamma(t)$ is a homogeneous geodesic.*

We shall now give a correct proof for odd dimension.

**Theorem 6.** *Let $(M, F)$ be a homogeneous Finsler space of odd dimension and $p \in M$. Then $M$ admits a homogeneous geodesic through $p$.*

**Proof.** Let us consider the Killing vector fields $K_1, \dots, K_n$ which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of $p$ and denote by $B$ the basis $\{K_1(p), \dots, K_n(p)\}$ of $T_pM$. Any tangent vector $X \in T_pM$ has coordinates $(x_1, \dots x_n)$ with respect to the basis $B$. These coordinates determine the Killing vector field $X^* = x_1K_1 + \cdots + x_nK_n$ and an integral curve $\gamma$ of $X^*$ through $p$. We are going to show that there exists a vector $\bar{X} \in T_pM$ such that the integral curve $\gamma$ of $\bar{X}^*$ through $p$ is geodesic.

Let us consider the sphere $\mathbb{S}^{n-1}$ of vectors $X \in T_pM$ whose coordinates $(x_1, \dots, x_n)$ with respect to $B$ have the norm equal to 1 with respect to the standard Euclidean scalar product $\langle , \rangle$ on $\mathbb{R}^n$. In other words, the scalar product $\langle , \rangle$ is chosen in a way that the above basis $B$ is orthonormal. We stress that this scalar product does not come from any Finslerian product $g$ used so far. For each $X \in \mathbb{S}^{n-1}$, we denote by $v(X)$ the derivative $D_{X^*_{\gamma(t)}}X^*|_{t=0}$. Further, we denote by $t(X)$ the vector $v(X) - \langle v(X), X\rangle X$. Then, for each $X \in \mathbb{S}^{n-1}$, $t(X) \perp X$ with respect to the above Euclidean scalar product. Clearly, the map $X \mapsto t(X)$ defines a smooth tangent vector field on the sphere $\mathbb{S}^{n-1}$. If $n$ is odd, according to a well known fact from differential topology, there is a vector $\bar{X}$ such that $t(\bar{X}) = 0$.

To finish the proof, we use formula (5) and the standard fact that

$$C_{X^*}(X^*, X^*, X^*) = 0\,.$$

We observe that, for each $X \in \mathbb{S}^{n-1} \subset T_pM$, it holds

$$g_{(p,X)}(v(X), X) = g_{(p,X)}(D_{X^*_{\gamma(t)}}X^*\big|_{t=0}, X^*_p) = 0$$

and hence $v(X)$ lies in the orthogonal complement of $X$ in $T_pM$ with respect to the scalar product $g_{(p,X)}$. The vector $t(X)$ is the projection of $v(X)$ to another complementary subspace to span$(X)$ in $T_pM$ and hence $v(X) = 0$ if and only if $t(X) = 0$. If follows that

$$D_{\bar{X}^*}\bar{X}^*\big|_p = v(\bar{X}) = 0\,.$$

Now we see, using Proposition 5 and formula (10), that the integral curve of the vector field $\bar{X}^*$ through $p$ is a homogeneous geodesic. □

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