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SOME FUNCTORIAL PROLONGATIONS OF GENERAL CONNECTIONS

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Abstract. We consider the problem of prolongating general connections on arbitrary fibered manifolds with respect to a product preserving bundle functor. Our main tools are the theory of Weil algebras and the Frölicher-Nijenhuis bracket.

0. Introduction

Our approach to connections on an arbitrary fibered manifold $p\colon Y \to M$ is slightly different from the approach by C. Ehresmann, [2], p. 186. Roughly speaking, the fundamental idea in [2] is the development along the individual curves, while the main idea of our approach is the absolute differentiation of the sections of Y. This is explained in Chapter 1 of the present paper. But the theory of general connections on Y can be well developed even by using the concept of tangent valued form on Y. This was invented by L. Mangiarotti and M. Modugno in [7] and first systematically presented in the book [6]. We repeat the basic ideas in Chapter 2. Chapter 3 is devoted to the case of product preserving bundle functors on the category $\mathcal{M}f$ of smooth manifolds and smooth maps. Our geometrical description of them uses the language of Weil algebras, [5], [6].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [6].

1. General connections

Let $\pi_Y\colon TY \to Y$ denote the tangent bundle of a fibered manifold $p\colon Y \to M$. In [6], a general connection of Y is defined as a lifting map

(1) $$\Gamma\colon Y \times_M TM \to TY$$

linear in TM and satisfying $\pi_Y \circ \Gamma = pr_1$, $Tp \circ \Gamma = pr_2$, $Y \xleftarrow{pr_1} Y \times_M TM \xrightarrow{pr_2} TM$. If x^i, y^p are some local fiber coordinates on Y, then the equations of Γ are

(2) $$dy^p = F^p_i(x,y)\, dx^i$$

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with arbitrary smooth functions F_i^p. Every vector field X on M defines the Γ-lift $\Gamma(X)\colon Y \to TY$, $\Gamma(X)(y) = \Gamma(y, X)$. Write $\pi_M \colon TM \to M$ for the bundle projection.

Equivalently, Γ can be interpreted as a section $Y \to J^1Y$ of the first jet prolongation J^1Y of Y. It is well known that $J^1Y \to Y$ is an affine bundle with associated vector bundle $VY \otimes T^*M$, where VY is the vertical tangent bundle of Y. For a section $s\colon M \to Y$, its absolute differential $\nabla_\Gamma s$ with respect to Γ is a section $\nabla_\Gamma s\colon M \to VY \otimes T^*M$ defined by

$$\nabla_\Gamma s(x) = j^1_x s - \Gamma\big(s(x)\big) \tag{3}$$

$x \in M$. Hence the coordinate form of (3) is

$$\frac{\partial s^p}{\partial x^i} - F_i^p\big(x, s(x)\big)\,. \tag{4}$$

The curvature $C\Gamma\colon Y \times_M \Lambda^2 T^*M \to VY$ can be characterized as the obstruction for lifting the bracket

$$(C\Gamma)(y, X_1, X_2) = \big[\Gamma(X_1), \Gamma(X_2)\big](y) - \Gamma\big([X_1, X_2]\big)(y)\,. \tag{5}$$

By direct evaluation, we find that (5) depends on the values of the vector fields X_1, X_2 at $p(y)$ only and the coordinate form of (5) is

$$2\Big(\frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q}\, F_j^q\Big)\frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j\,. \tag{6}$$

Using the flow prolongation of vector fields, we construct an induced connection $\mathcal{V}\Gamma\colon VY \times_M TM \to TVY$ on VY as follows, [6]. Consider the flow $\mathrm{Fl}_t^{\Gamma(X)}$ of the vector filed $\Gamma(X)$ and its vertical flow prolongation

$$\mathcal{V}\big(\Gamma(X)\big) = \frac{\partial}{\partial t}\Big|_0 V\big(\,\mathrm{Fl}_t^{\Gamma(X)}\,\big)\colon VY \to TVY\,. \tag{7}$$

Write $\eta^p = dy^p$ for the induced coordinates on VY. Then the coordinate form of (7) is

$$\begin{aligned} dy^p &= F_i^p(x, y)\, dx^i\,,\\ d\eta^p &= \frac{\partial F_i^p}{\partial y^q}\, \eta^q\, dx^i\,, \end{aligned} \tag{8}$$

that determines a general connection $\mathcal{V}\Gamma$ on $VY \to M$. The theoretical meaning of the vertical operator $\mathcal{V}$ is underlined by the following assertion, [6].

Proposition 1. *$\mathcal{V}$ is the only natural operator transforming general connections on $Y \to M$ into general connections on $VY \to M$.*

Consider a section $\varphi\colon Y \to VY \otimes \Lambda^k T^*M$ with the coordinate expression

$$\eta^p = \varphi^p_{i_1\dots i_k}(x, y)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}\,.$$

According to [6], we construct its absolute exterior differential

$$d_{\mathcal{V}\Gamma}\varphi\colon Y \to VY \otimes \bigwedge^{k+1} T^*M$$

as follows. Take (at least locally) an auxiliary linear symmetric connection Λ on M. Then $\mathcal{V}\Gamma \otimes \overset{k}{\bigwedge} \Lambda^*$ is a connection on $VY \otimes \overset{k}{\bigwedge} T^*M \to Y$ and we can construct the absolute differential

$$\nabla_{\mathcal{V}\Gamma\otimes\bigwedge^k\Lambda^*}\varphi\colon Y \to V\big(VY \otimes \overset{k}{\bigwedge} T^*M\big) \otimes T^*M\,,$$

[6]. Applying antisymmetrization and natural identifications, we obtain a section $d_{\mathcal{V}\Gamma}\varphi\colon Y \to VY \otimes \Lambda^{k+1}T^*M$ independent of Λ with the coordinate expression

$$\eta^p = \Big(\frac{\partial \varphi^p_{i_1\dots i_k}}{\partial x^i} + \frac{\partial \varphi^p_{i_1\dots i_k}}{\partial y^q}\, F^q_i - \frac{\partial F^p_i}{\partial y^q}\, \varphi^q_{i_1\dots i_k}\Big)\, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}\,. \tag{9}$$

In [6], we deduced by direct evaluation

Proposition 2 (Bianchi identity)**.** *We have*

$$d_{\mathcal{V}\Gamma} C\Gamma = 0\,. \tag{10}$$

2. Tangent valued forms

Mangiarotti and Modugno studied systematically the general connections by using the concept of tangent valued forms, [7]. A tangent valued k-form P on a manifold M is a section $P\colon M \to TM \otimes \Lambda^k T^*M$, that can be also interpreted as a map

$$P\colon TM \underbrace{\times_M \cdots \times_M}_{k\text{-times}} TM \to TM\,. \tag{11}$$

If Q is another tangent valued l-form on M, Mangiarotti and Modugno defined a tangent valued $(k+l)$-form $[P,Q]$ on M by the formula

$$\begin{aligned}
&\big[P,Q\big](X_1\dots,X_{k+l})\\
&\quad= \frac{1}{k!l!}\sum_\sigma \overline{\sigma}\big[P(X_{\sigma_1},\dots,X_{\sigma_k}),Q(X_{\sigma_{(k+1)}},\dots,X_{\sigma_{(k+l)}})\big]\\
&\qquad+\frac{-1}{k!(l-1)!}\sum_\sigma \overline{\sigma} Q\big(\big[P(X_{\sigma_1},\dots,X_{\sigma_k}),X_{\sigma_{(k+1)}}\big],X_{\sigma_{(k+2)}},\dots\big)\\
&\qquad+\frac{(-1)^{kl}}{(k-1)!l!}\sum_\sigma \overline{\sigma} P\big(\big[Q(X_{\sigma_1},\dots,X_{\sigma_l}),X_{\sigma_{(l+1)}}\big],X_{\sigma_{(l+2)}},\dots\big)\\
&\qquad+\frac{(-1)^{k-1}}{(k-1)!(l-1)!2}\sum_\sigma \overline{\sigma} Q\big(P([X_{\sigma_1},X_{\sigma_2}],X_{\sigma_3},\dots),X_{\sigma_{(k+2)}},\dots\big)\\
&\qquad+\frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2}\sum_\sigma \overline{\sigma} P\big(Q([X_{\sigma_1},X_{\sigma_2}],X_{\sigma_3},\dots),X_{\sigma_{(l+2)}},\dots\big)
\end{aligned} \tag{12}$$

where $X_1,\dots,X_{k+l}$ are vector fields on M, the bracket on the right-hand side are the classical Lie bracket of vector fields, the summations are with respect to all permutations σ of $k+l$ letters and $\overline{\sigma}$ denotes the signum of σ. The tangent valued

0-forms are the vector fields and (12) reduces to the classical Lie bracket in the case $k=l=0$.

Later it was clarified, [6], that (12) was introduced in a quite different situation by Frölicher-Nijenhuis, so that this bracket is related with their names today.

The identity of TM is a special tangent valued 1-form on M and we have

$$[\mathrm{id}_{TM}, P] = 0 \tag{13}$$

for every tangent valued form P. By [6],

$$[P,Q] = -(-1)^{kl}[Q,P] \tag{14}$$

and the graded Jacobi identity holds

$$\big[P_1,[P_2,P_3]\big] = \big[[P_1,P_2],P_3\big] + (-1)^{k_1k_2}\big[P_2,[P_1,P_3]\big] \tag{15}$$

for tangent valued k_i-forms P_i, $i=1,2,3$.

A general connection $\Gamma\colon Y\times_M TM \to TY$ defines a tangent valued 1-form ω_Γ on Y

$$\omega_\Gamma(Z) = \Gamma\big(y, Tp(Z)\big)\,, \quad Z\in T_yY\,. \tag{16}$$

Even $C\Gamma$ can be interpreted as a tangent valued 2-form C_Γ on Y,

$$C_\Gamma(Z_1,Z_2) = C\Gamma\big(y, Tp(Z_1), Tp(Z_2)\big)\,, \quad Z_1,Z_2\in T_yY\,. \tag{17}$$

Proposition 3. *We have* $C_\Gamma = \frac{1}{2}[\omega_\Gamma,\omega_\Gamma]$.

Proof. This follows directly from Lemma 8.13 in [6]. □

Consider an arbitrary tangent valued 1-form ψ of Y. Put $P_1=P_2=P_3=\psi$ into (14) and (15) This yields

$$\big[\psi,[\psi,\psi]\big] = 0\,.$$

If $\psi=\omega_\Gamma$, we obtain

Proposition 4. *We have* $[\omega_\Gamma,[\omega_\Gamma,\omega_\Gamma]]=0$.

A simple evaluation shows that this relation coincides with the identity from Proposition 2. This gives a simple geometric proof of the Bianchi identity of a general connection Γ on Y.

3. Weilian prolongations

We recall that Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A=\mathbb{R}\times N$, where N is the ideal of all nilpotent elements of A. Since A is finite dimensional, there exists an integer r such that $N^{r+1}=0$. The smallest r with this property is called the order of A. On the other hand, the dimension wA of the vector space N/N^2 is the width of A, [8]. Using systematically our point of view, we say that a Weil algebra of width k and order r is a Weil (k,r)-algebra, [5].

The simpliest example of a Weil (k,r)-algebra is

$$\mathbb{D}_k^r = \mathbb{R}[x_1,\dots,x_k]\big/\langle x_1,\dots,x_k\rangle^{r+1} = J_0^r(\mathbb{R}^k,\mathbb{R})\,.$$

For $k=r=1$, $\mathbb{D}_1^1=\mathbb{D}$ is the algebra of Study numbers. In [3] we deduced

Lemma 1. *Every Weil (k,r)-algebra is a factor algebra of $\mathbb{D}^r_k$. If ϱ, $\sigma\colon \mathbb{D}^r_k \to A$ are two algebra epimorphisms, then there exists an algebra isomorphism $\chi\colon \mathbb{D}^r_k \to \mathbb{D}^r_k$ such that $\varrho = \sigma \circ \chi$.*

We are going to present the covariant approach to Weil functors, [5].

Definition 1. Two maps γ, $\delta\colon \mathbb{R}^k \to M$ determine the same A-velocity $j^A\gamma = j^A\delta$, if for every smooth function $\varphi\colon M \to \mathbb{R}$,

$$\varrho\big(j^r_0(\varphi\circ\gamma)\big) = \varrho\big(j^r_0(\varphi\circ\delta)\big)\,. \tag{18}$$

By Lemma 1, this is independent of the choice of ϱ. We say that

$$T^A M = \{j^A\gamma; \gamma\colon \mathbb{R}^k \to M\} \tag{19}$$

is the bundle of all A-velocities on M. For every smooth map $f\colon M \to N$, we define $T^A f\colon T^A M \to T^A M$ by

$$T^A f(j^A\gamma) = j^A(f\circ\gamma)\,. \tag{20}$$

Clearly, $T^A\mathbb{R} = A$.

We say that (19) and (20) represent the covariant approach to Weil functors. The following result is a fundamental assertion, see [6] or [5] for a survey.

Theorem. *The product preserving bundle functors on $\mathcal{M}f$ are in bijection with T^A. The natural transformations $T^{A_1} \to T^{A_2}$ are in bijection with the algebra homomorphisms $\mu\colon A_1 \to A_2$.*

We write $\mu_M\colon T^{A_1}M \to T^{A_2}M$ for the value of $\mu\colon A_1 \to A_2$ on M.

The iteration $T^{A_2}\circ T^{A_1}$ corresponds to the tensor product of A_1 and A_2. The algebra exchange homomorphism $\mathrm{ex}\colon A_1\otimes A_2 \to A_2\otimes A_1$ defines a natural exchange transformation $T^{A_2}T^{A_1} \to T^{A_1}T^{A_2}$. We have $T = T^{\mathbb{D}}$.

The canonical exchange $\varkappa^A_M\colon T^A TM \to TT^A M$ is called flow natural. Indeed, if Fl^X_t is the flow of a vector field $X\colon M \to TM$, then

$$\mathcal{T}^A X = \frac{\partial}{\partial t}\Big|_0 T^A(\mathrm{Fl}^X_t)\colon T^A M \to TT^A M$$

is the flow prolongation of X. It is related with the functorial prolongation $T^A X\colon T^A M \to T^A TM$ by

$$\mathcal{T}^A X = \varkappa^A_M \circ T^A X\,. \tag{21}$$

Consider a tangent valued k-form P on a manifold M

$$P\colon TM\times_M\cdots\times_M TM \to TM\,.$$

Applying functor T^A, we obtain

$$T^A P\colon T^A TP\times_M\cdots\times_M T^A TP \to T^A TP\,.$$

Using the flow natural exchange $\varkappa^A_M$, we construct

$$\mathcal{T}^A P = \varkappa^A_M \circ T^A P \circ \big((\varkappa^A_M)^{-1}\times\cdots\times(\varkappa^A_M)^{-1}\big)\,. \tag{22}$$

This is an antisymmetric tensor field of type $(1,k)$, so a tangent valued k-form on $T^A M$.

In [1], the following result is deduced.

Proposition 5. *The Frölicher-Nijenhuis bracket is preserved under $\mathcal{T}^A$, i.e. for every tangent valued k-form P and every tangent valued l-form Q on the same manifold M, we have*

$$\mathcal{T}^A([P,Q]) = [\mathcal{T}^A P, \mathcal{T}^A Q]\,. \tag{23}$$

Further, consider a tangent valued k-form P on a manifold M, a tangent valued k-form Q on a manifold N and a smooth map $f\colon M \to N$. We say that P and Q are f-related, if the following diagram commutes

$$\begin{array}{ccc} \Lambda^k TM & \xrightarrow{\;P\;} & TM \\ {\scriptstyle \Lambda^k Tf}\big\downarrow & & \big\downarrow{\scriptstyle Tf} \\ \Lambda^k TN & \xrightarrow{\;Q\;} & TN \end{array}$$

In [6], p. 74, one has deduced

Proposition 6. *Consider a smooth map $f\colon M \to N$. Let P_1, Q_1 or P_2, Q_2 be two f-related pairs of k-forms or l-forms, respectively. Then the Frölicher-Nijenhuis brackets $[P_1, Q_1]$ and $[P_2, Q_2]$ are also f-related.*

Consider a general connection Γ on Y in the lifting form $\Gamma\colon Y \times_M TM \to TY$. Applying T^A, $\varkappa^A_M$ and $\varkappa^A_Y$, [4, 5], we can construct the induced connection on $T^A Y \to T^A M$

$$\mathcal{T}^A\Gamma\colon T^A Y \times_{T^A M} TT^A M \to TT^A Y\,. \tag{24}$$

Consider the connection form $\omega_\Gamma\colon TY \to TY$ of Γ. Then Proposition 5 and (24) imply

$$\mathcal{T}^A C_\Gamma = \frac{1}{2}\big[\mathcal{T}^A\omega_\Gamma, \mathcal{T}^A\omega_\Gamma\big]\,. \tag{25}$$

Hence the curvature of $\mathcal{T}^A\Gamma$ is the $\mathcal{T}^A$-prolongation of the curvature of Γ.

Further, the Bianchi identity of $\mathcal{T}^A\Gamma$ is the $\mathcal{T}^A$-prolongation of the Bianchi identity of Γ.

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