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ON THE GEOMETRY OF VERTICAL WEIL BUNDLES

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Abstract. We describe some general geometric properties of the fiber product preserving bundle functors. Special attention is paid to the vertical Weil bundles. We discuss namely the flow natural maps and the functorial prolongation of connections.

The main purpose of the present paper is to describe some geometric properties of the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and local fibered morhisms with local diffeomorphisms as base maps. Special attention is paid to the vertical Weil functors V^A.

In Section 1 we present the covariant approach to the Weil functors on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps. We mention the fundamental theoretical result that the classical Weil functors T^A coincide with the product preserving bundle functors on $\mathcal{M}f$. In Section 2 we introduce the Weil fields as the sections of Weil bundles and we describe their basic properties. Section 3 is devoted to the concept of flow natural map, that represents a suitable tool for constructing the flow prolongation of a projectable vector field on a fibered manifold $Y \to M$. The last section describes the functorial prolongation of connections with respect to a fiber product preserving bundle functor on $\mathcal{FM}_m$.

Unless otherwise specified, we use the terminology and notation from [6]. All manifolds and maps are assumed to be infinitely differentiable.

1. Fiber product preserving bundle functors

We recall that a Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where N is the ideal of all nilpotent elements of A. There exists an integer r such that $N^{r+1} = 0$, the smallest r with this property is called the order of A. On the other hand, the dimension wA of the vector space N/N^2 is the width of A. We say that a Weil algebra of width k and order r is a Weil (k,r)-algebra, [5].

The simpliest example of a Weil algebra is

$$\mathbb{D}^r_k = \mathbb{R}[x_1, \dots, x_k] \big/ \langle x_1, \dots, x_k \rangle^{r+1} = J^r_0(\mathbb{R}^k, \mathbb{R}) \,.$$

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In particular, $\mathbb{D}_1^1 =: \mathbb{D}$ is the algebra of Study numbers. In [5], we deduced

Lemma 1. *Every Weil (k,r)-algebra is a factor algebra of $\mathbb{D}_k^r$. If ϱ, $\sigma\colon \mathbb{D}_k^r \to A$ are two algebra epimorphisms, then there is an algebra isomorphism $\chi\colon \mathbb{D}_k^r \to \mathbb{D}_k^r$ such that $\varrho = \sigma \circ \chi$.*

Definition 1. Two maps $\gamma, \delta\colon \mathbb{R}^k \to M$ determine the same A-velocity $j^A\gamma = j^A\delta$, if for every smooth function $\varphi\colon M \to \mathbb{R}$

$$\varrho\big(j_0^r(\varphi\circ\gamma)\big) = \varrho\big(j_0^r(\varphi\circ\delta)\big)\,. \tag{1}$$

By Lemma 1 this is independent of the choice of ϱ. One verifies easily, [6], that the bundle of all A-velocities

$$T^AM = \{j^A\gamma, \gamma\colon \mathbb{R}^k \to M\} \tag{2}$$

coincides with the bundle of infinitely near points of type A on M introduced by A. Weil, [9]. For every smooth map $f\colon M \to N$, we define $T^Af\colon T^AM \to T^AN$ by

$$T^Af(j^A\gamma) = j^A(f\circ\gamma)\,. \tag{3}$$

Clearly, $T^A\mathbb{R} = A$.

We say that (2) and (3) represents the covariant approach to Weil bundles. The following result is a fundamental assertion, see [6] for a survey.

Theorem 1. *The product preserving bundle functors on $\mathcal{M}f$ are in bijection with T^A. The natural transformations $T^{A_1} \to T^{A_2}$ are in bijection with the algebra homomorphisms $\mu\colon A_1 \to A_2$.*

We write $\mu_M\colon T^{A_1}M \to T^{A_2}M$ for the value of $\mu\colon A_1 \to A_2$ on M. If A is a Weil (k_i, r_i)-algebra, $i = 1,2$, then there exists a polynomial map $\bar\mu\colon \mathbb{R}^{k_2} \to \mathbb{R}^{k_1}$ such that

$$\mu_M(j^{A_1}\gamma) = j^{A_2}(\gamma\circ\bar\mu)\,, \qquad \gamma\colon \mathbb{R}^{k_1} \to M\,. \tag{4}$$

The iteration $T^{A_2}T^{A_1}$ corresponds to the tensor product of A_1 and A_2. The algebra exchange homomorphism $ex\colon A_1\otimes A_2 \to A_2\otimes A_1$ defines a natural exchange transformation $i_M^{A_1,A_2}\colon T^{A_1}T^{A_2}M \to T^{A_2}T^{A_1}M$. We have $T = T^{\mathbb{D}}$.

The canonical exchange $\varkappa_M^A\colon T^ATM \to TT^AM$ is called flow natural. Indeed, if Fl_t^X is the flow of a vector field $X\colon M \to TM$, then the flow prolongation of X is defined by

$$\mathcal{T}^AX = \frac{\partial}{\partial t}\Big|_0 T^A(Fl_t^X)\colon T^AM \to TT^AM\,. \tag{5}$$

One deduces easily, [6],

$$\mathcal{T}^AX = \varkappa_M^A \circ T^AX\,. \tag{6}$$

Further, consider a bundle functor F on $\mathcal{FM}_m$ that preserves fiber products. Examples are the r-th jet prolongation J^rY of a fibered manifold $p\colon Y \to M$, $\dim M = m$, the vertical A-prolongation $V^AY = \bigcup\limits_{x\in M} T^A(Y_x)$, the vertical r-jet prolongation $\bigcup\limits_{x\in M} J_x^r(M, Y_x)$ and iterations.

We say F is of the base order r, if for two $\mathcal{FM}_m$-morphisms φ, $\psi\colon Y \to Y'$ of $p\colon Y \to M$ into $p'\colon Y' \to M'$ with base maps $\underline{\varphi}$, $\underline{\psi}\colon M \to M'$, $j^r_x\underline{\varphi} = j^r_x\underline{\psi}$ and $\varphi \mid Y_x = \psi \mid Y_x$ imply $F\varphi \mid F_xY = F\psi \mid F_xY$, $x \in M$.

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. The construction of product fibered manifolds defines an injection $\iota\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, $\iota(M,N) = (M \times N \to M)$, $\iota(f_1,f_2) = f_1 \times f_2$, $f_1\colon M \to M'$, $f_2\colon N \to N'$.

W. Mikulski and the author deduced, [7], that the bundle functors $\Phi = F \circ \iota$ on $\mathcal{M}f_m \times \mathcal{M}f$ are in bijection with the pairs (A,H), where A is a Weil algebra and $H\colon G^r_m \to \operatorname{Aut} A$ is a group homomorphism of the r-jet group in dimension m into the group of all automorphisms of A. Since $H(g)\colon A \to A$ is an algebra automorphism for every $g \in G^r_m$, we have the induced action $H_N(g) = H(g)_N\colon T^AN \to T^AN$ of G^r_m on T^AN. Then $\Phi(M,N)$ is the associated fiber bundle $P^rM[T^AN]$. For a local diffeomorphism $f_1\colon M \to M'$ and a smooth map $f_2\colon N \to N'$,

$$\Phi(f_1,f_2) = P^rf_1[T^Af_2]\colon \Phi(M,N) \to \Phi(M',N')\,, \tag{7}$$

where $P^rf_1\colon P^rM \to P^rM'$ is the induced local isomorphisms of principal bundles and $T^Af_2\colon T^AN \to T^AN'$ is a G^r_m-equivariant map, [5].

Then the functor F is determined by adding an equivariant algebra homomorphism $t\colon \mathbb{D}^r_m \to A$, where $\operatorname{Aut}\mathbb{D}^r_m = G^r_m$. We have

$$FY = \big(\{u,Z\} \in P^rM[T^AY], t_M(u) = T^Ap(Z), u \in P^r_xM, Z \in T^AY\big)\,, \tag{8}$$

where $t_M\colon T^r_mM \to T^AM$ and $P^rM \subset T^r_mM$. For an $\mathcal{FM}_m$-morphism $f\colon Y \to Y'$ over $\underline{f}\colon M \to M'$, Ff is the restriction of $\Phi(\underline{f},f)$ to FY. In the product case $Y = M \times N$, we have

$$F(M \times N) = P^rM[T^AN]\,. \tag{9}$$

If we consider another fibered manifold $Y' \to M$ over M and $\underline{f} = \mathrm{id}_M$, we have

$$Ff\big(\{u,Z\}\big) = \big\{u, T^Af(Z)\big\}\,. \tag{10}$$

Further, t induces a natural map

$$\widetilde{t}_Y\colon J^rY \to FY,\ \{u,Z\} \mapsto \big\{u, t_Y(Z)\big\},\ u \in P^rM,\ Z \in T^r_mY\,. \tag{11}$$

Geometrically, we interpret a section $s\colon M \to Y$ as a morphism $\widetilde{s}\colon \widetilde{M} \to Y$, where $\widetilde{M} = (M \xrightarrow{\mathrm{id}} M)$ is the "doubled" manifold. Then $F\widetilde{s}$ is identified with j^rs and $\widetilde{t}_Y(j^r_xs) = (F\widetilde{s})(x)$.

Remark 1. We remark that W. Mikulski has recently described another construction of $F = (A,H,t)$, [8].

2. Prolongation of Weil fields

Write $\pi_{A,M}\colon T^AM \to M$ for the bundle projection.

Definition 2 ([1])**.** A section $\xi\colon M \to T^AM$ is called an A-field on M.

Consider another Weil algebra B. Let $X \in T^B(T^A M)$, $X = j^B\varphi$, where $\varphi\colon \mathbb{R}^l \to T^A M$, $l =$ the width of B. Every $\varphi(t) \in T^A M$, $t \in \mathbb{R}^l$ is of the form $j^A\psi(\tau,t)$, $\tau \in \mathbb{R}^k$, where ψ is a map $\mathbb{R}^k \times \mathbb{R}^l \to M$. Hence $X = j^B(j^A\psi(\tau,t))$ and the exchange diffeomorphism $i_M^{B,A}\colon T^B(T^A M) \to T^A(T^B M)$ is of the form

$$i_M^{B,A}(X) = j^A\big(j^B\psi(\tau,t)\big)\,. \tag{12}$$

Consider the bundle projection $\pi_{B,T^A M}\colon T^B T^A M \to T^A M$ and the induced map $T^B\pi_{A,M}\colon T^B T^A M \to T^B M$. One verifies easily that $i_M^{B,A}$ exchanges the related projections, i.e.

$$T^A\pi_{B,M} \circ i_M^{B,A} = \pi_{B,T^A M}\,, \quad \pi_{A,T^B M} \circ i_M^{B,A} = T^B\pi_{A,M}\,. \tag{13}$$

Definition 3. Let $\xi\colon M \to T^A M$ be an A-field on M. The A-field $i_M^{B,A} \circ T^B\xi\colon T^B M \to T^A(T^B M)$ on $T^B M$ will be called the B-prolongation of ξ.

The flow prolongation of a projectable vector field η on a fibered manifold $p\colon Y \to M$ with respect to F is defined by a formula analogous to (5)

$$\mathcal{F}\eta = \frac{\partial}{\partial t}\Big|_0 F\big(Fl_t^\eta\big)\colon FY \to TFY\,. \tag{14}$$

Now we discuss the special case of the vertical Weil bundle $V^A Y \to M$ of a fibered manifold $p\colon Y \to M$. Consider the subbundles $V^B(T^A Y \to T^A M) \subset T^B T^A Y$ and $T^A(V^B Y) \subset T^A T^B Y$.

Lemma 2. *$i_Y^{B,A}$ maps $V^B(T^A Y \to T^A M)$ into $T^A(V^B Y)$.*

Proof. By locality, it suffices to consider a product bundle $Y = (M \times N) \to M$. We have

$$T^A Y = T^A M \times T^A N\,, \quad V^B(T^A Y \to T^A M) = T^A M \times T^B T^A N\,,$$

$$V^B Y = M \times T^B N\,, \quad T^A(V^B Y) = T^A M \times T^A T^B N\,.$$

In this situation, $i_Y^{B,A}$ is reduced to the exchange diffeomorphism $i_N^{B,A}\colon T^B T^A N \to T^A T^B N$. □

The restricted and corestricted map, that will be denoted by

$$i_{Y,V}^{B,A}\colon V^B(V^A Y \to M) \to V^A(V^B Y \to M)\,, \tag{15}$$

represents the exchange diffeomorphism applied fiberwise. For $B = \mathbb{D}$, we write

$$\varkappa_{Y,V}^A\colon V(V^A Y \to M) \to V^A(VY \to M)\,. \tag{16}$$

Let η be a vertical vector field on Y and $\mathcal{V}^A\eta$ be its flow prolongation. Analogously to (6), we obtain

$$\mathcal{V}^A\eta = \varkappa_{Y,V}^A \circ V^A\eta\,. \tag{17}$$

Remark 2. It is remarkable that we also have a canonical exchange

$$J^r(V^A Y \to M) \to V^A(J^r Y \to M)\,, \quad j_x^r j^A\varphi(\tau,t) \mapsto j^A j_x^r\varphi(\tau,t)\,, \tag{18}$$

$\tau \in M$, $t \in \mathbb{R}^l$, [5]. Indeed, locally is Y isomorphic to $U \times N$, $U \subset \mathbb{R}^m$. In such a situation, (18) is reduced to the canonical exchange transformation $i_N^{\mathbb{D}_k^r, A}$ corresponding to (12).

3. The flow natural map

In the case of a fiber product preserving bundle functor $F = (A, H, t)$ on $\mathcal{FM}_m$, we have the following analogy of the flow natural map from Section 1. Consider a vector field ξ on M. Its flow prolongation $\mathcal{P}^r\xi$ is a right invariant vector field on the r-th order frame bundle P^rM, whose value at every $u \in P^r_xM$ depends on $j^r_x\xi$ only. This defines a map

$$\nu^r_M \colon P^rM \times_M J^rTM \to TP^rM\,. \tag{19}$$

For a fibered manifold $p\colon Y \to M$, we will consider TY as a fibered manifold $TY \to M$. Then $Tp\colon TY \to TM$ is a base preserving morphism, that induces $FTp\colon FTY \to FTM$. Taking into account the natural transformation $\widetilde{t}_{TM}\colon J^rTM \to FTM$, we construct the fiber product

$$J^rTM \times_{FTM} FTY\,. \tag{20}$$

By (8), we have $FTY \subset P^rM[T^ATY]$. Consider $(X, \{u, Z\})$ in (20), $X \in J^r_xTM$, $u \in P^r_xM$, $Z = T^ATY$. Write $\nu^r_M(u, X) = (\partial/\partial t)_0\gamma(t)$, $\gamma\colon \mathbb{R} \to P^rM$. By Section 1, $\varkappa^A_Y(Z) \in TT^AY$ can be expressed as $(\partial/\partial t)_0\xi(t)$, where $\xi\colon \mathbb{R} \to T^AY$ satisfies $t_Y(\gamma(t)) = T^Ap(\xi(t))$ for all t. So $\{\gamma(t), \xi(t)\}$ is a curve on FY and we define

$$\psi^F_Y\big(X, \{u, Z\}\big) = \frac{\partial}{\partial t}\Big|_0 \big\{\gamma(t), \xi(t)\big\}\,. \tag{21}$$

By right invariancy, this is independent of the choice of u. Hence we obtain a map

$$\psi^F_Y\colon J^rTM \times_{FTM} FTY \to TFY\,. \tag{22}$$

A projectable vector field η on Y over ξ on M can be interpreted as a base preserving morphism $\eta\colon Y \to TY$. Then we construct its functorial prolongation $F\eta\colon FY \to FTY$ as well as the r-th jet prolongation $j^r\xi\colon M \to J^rTM$. The values of $j^r\xi \times_{\mathrm{id}_M} F\eta$ with respect to ψ^F_Y are in TFY. The proof of the following assertion can be found in [4].

Proposition 1. *The flow prolongation $\mathcal{F}\eta$ satisfies*

$$\mathcal{F}\eta = \psi^F_Y(j^r\xi \times_{\mathrm{id}_M} F\eta)\,. \tag{23}$$

In the case of $F = V^A$, the base order of V^A is zero, so that (23) is defined on the space

$$TM \times_{V^ATM} V^ATY \approx V^ATY\,. \tag{24}$$

For a vertical field η on Y, we have $\xi = O_M$. By (23), we rededuce

$$\mathcal{V}^A\eta = \varkappa^A_{Y,V} \circ V^A\eta\,.$$

4. Functorial prolongation of connections

It was clarified in several concrete problems that if F is of base order r, we need an auxiliarly linear splitting $\Lambda\colon TM \to J^rTM$ to construct an induced connection $\mathcal{F}(\Gamma,\Lambda)\colon FY \times_M TM \to TFY$ from a general connection on Y, which is considered as a lifting map $\Gamma\colon Y \times_M TM \to TY$ linear in TM, [6], [3]. Consider a vector field $\xi\colon M \to TM$ and its Γ-lift $\Gamma\xi\colon Y \to TY$. The flow prolongation $\mathcal{F}(\Gamma\xi)\colon FY \to TFY$ depends on $j^r\xi$ only. This defines $\mathcal{F}\Gamma\colon J^rTM \times_M FY \to TFY$ linear in J^rTM. Then $\mathcal{F}(\Gamma,\Lambda) = \mathcal{F}\Gamma \circ (\Lambda \times_{\mathrm{id}_M} \mathrm{id}_{FY})$.

It is useful to describe this construction by using the flow natural map ψ^F_Y. In Section 1, we constructed $\widetilde{t}_{TM}\colon J^rTM \to FTM$. Consider a projectable vector field $\eta\colon Y \to TY$ over $\xi\colon M \to TM$. If we interpret η as an $\mathcal{FM}_m$-morphism $\eta\colon Y \to TY$ over id_M, we can construct $F\eta\colon FY \to FTY$. By (23), we have $\mathcal{F}\eta = \psi^F_Y(j^r\xi \times_{\mathrm{id}_M} F\eta)$ for every η. This map is linear in J^rTM. Hence

$$\mathcal{F}(\Gamma,\Lambda) = \psi^F_Y\big(\Lambda(X), v\big) = \mathcal{F}\Gamma \circ (\Lambda \times_{\mathrm{id}_M} \mathrm{id}_{FY})\,, \tag{25}$$

$X \in T_xM$, $v \in F_xY$.

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