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SEMILINEAR FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

Khalida Aissani and Mouffak Benchohra

Abstract. This paper concerns the existence of mild solutions for fractional order integro-differential equations with infinite delay. Our analysis is based on the technique of Kuratowski's measure of noncompactness and Mönch's fixed point theorem. An example to illustrate the applications of main results is given.

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus since, starting from some speculations of G. W. Leibniz (1697) and L. Euler (1730), it has been progressing up to nowadays. Fractional differential and integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport and the kinetic theory of gases and others [5, 10, 11]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [1], Baleanu et al. [6], Diethelm [13], Hilfer [18], Kilbas et al. [20], Miller and Ross [26], Podlubny [31], Samko et al. [32], and Tarasov [33], and the papers [2, 3, 8, 14, 23, 24, 28, 29, 30].

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [16], Hino et al. [19], Kolmanovskii and Myshkis [21], Lakshmikantham et al. [22], and Wu [34], and the papers [12, 15].

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In this work we discuss the existence of mild solutions for fractional order integro-differential equations with infinite delay of the form

$$
(1)\qquad \begin{aligned}
&D_t^q x(t) = Ax(t) + \int_0^t a(t,s) f(s, x_s, x(s)) ds\,, \qquad t \in J = [0,T]\,,\\
&x(t) = \phi(t)\,, \quad t \in \ (-\infty, 0]\,,
\end{aligned}
$$

where D_t^q is the Caputo fractional derivative of order $0 < q < 1$, A is a generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X, $f\colon J \times \mathcal{B} \times X \longrightarrow X$, $a\colon D \to \mathbb{R}$ ($D = \{(t,s) \in [0,T] \times [0,T] : t \geq s\}$), $\phi \in \mathcal{B}$ where $\mathcal{B}$ is called phase space to be defined in Section 2. For any function x defined on $(-\infty, T]$ and any $t \in J$, we denote by x_t the element of $\mathcal{B}$ defined by

$$
x_t(\theta) = x(t+\theta)\,, \qquad \theta \in \ (-\infty, 0]\,.
$$

Here x_t represents the history of the state up to the present time t.

In the present paper we deal with an infinite time delay. Note that in this case, the phase space $\mathcal{B}$ plays a crucial role in the study of both qualitative and quantitative aspects of theory of functional equations (see [15]).

We present an existence result of mild solutions for the problem (1) by means of the application of Mönch's fixed point theorem combined with the Kuratowski measure of noncompactness. An example illustrating the abstract theory will be presented.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space.
$C = C(J,X)$ be the space of all X-valued continuous functions on J.
$L(X)$ be the Banach space of all linear and bounded operators on X.
$L^1(J,X)$ the space of X-valued Bochner integrable functions on J with the norm

$$
\|y\|_{L^1} = \int_0^T \|y(t)\|\, dt\,.
$$

$L^\infty(J,\mathbb{R})$ is the Banach space of essentially bounded functions, normed by

$$
\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d\,,\ \text{a.e. } t \in J\}\,.
$$

Definition 2.1. A function $f\colon J \times \mathcal{B} \times X \longrightarrow X$ is said to be an Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t,\cdot,\cdot)\colon \mathcal{B} \times X \longrightarrow X$ is continuous;
(ii) for each $(v,w) \in \mathcal{B} \times X$ the function $f(\cdot, v, w)\colon J \to X$ is measurable.

Next we give the concept of a measure of noncompactness [7].

Definition 2.2. Let B be a bounded subset of a seminormed linear space Y. Kuratowski's measure of noncompactness of B is defined as

$$
\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter } \leq d\}\,.
$$

We note that this measure of noncompactness satisfies interesting regularity properties (for more information, we refer to [7]).

Lemma 2.3.

(1) *If* $A \subseteq B$ *then* $\alpha(A) \leq \alpha(B)$,

(2) $\alpha(A) = \alpha(\overline{A})$, *where* $\overline{A}$ *denotes the closure of* A,

(3) $\alpha(A) = 0 \Leftrightarrow \overline{A}$ *is compact (*A *is relatively compact),*

(4) $\alpha(\lambda A) = |\lambda| A$, *with* $\lambda \in \mathbb{R}$,

(5) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

(6) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*
$$A + B = \{x + y : x \in A, y \in B\},$$

(7) $\alpha(A + a) = \alpha(A)$ *for any* $a \in Y$,

(8) $\alpha(\overline{\mathrm{conv}}A) = \alpha(A)$, *where* $\overline{\mathrm{conv}}A$ *is the closed convex hull of* A.

For $H \subset C(J, X)$, we define

$$\int_0^t H(s)ds = \Big\{ \int_0^t u(s)ds : u \in H \Big\} \quad \text{for} \quad t \in J\,, \tag{2}$$

where $H(s) = \{u(s) \in X : u \in H\}$.

Lemma 2.4 ([7]). *If* $H \subset C(J, X)$ *is a bounded, equicontinuous set, then*

$$\alpha(H) = \sup_{t \in J} \alpha(H(t))\,. \tag{3}$$

Lemma 2.5 ([17]). *If* $\{u_n\}_{n=1}^{\infty} \subset L^1(J, X)$ *and there exists* $m \in L^1(J, \mathbb{R}^+)$ *such that* $\|u_n(t)\| \leq m(t)$, *a.e.* $t \in J$, *then* $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ *is integrable and*

$$\alpha\Big(\Big\{ \int_0^t u_n(s)\, ds \Big\}_{n=1}^{\infty}\Big) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})\, ds\,. \tag{4}$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [15]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|.\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1): If $x : (-\infty, T\,] \longrightarrow X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x_t is continuous in $t \in J$ and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}}\,, \tag{5}$$

where $C \geq 0$ is a constant.

(A2): There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s \in [0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{6}$$

for $t \in [0, T]$ and x as in $(A1)$.

(A3): The space $\mathcal{B}$ is complete.

Remark 2.6. Condition (5) in (A1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

For our purpose we will only need the following fixed point theorem.

Theorem 2.7 ([4, 27])**.** *Let U be a bounded, closed and convex subset of a Banach space such that $0 \in U$, and let N be a continuous mapping of U into itself. If the implication*

$$V = \overline{\text{conv}}N(V) \ \text{or} \ V = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0$$

holds for every subset V of U, then N has a fixed point.

Let Ω be a set defined by

$$\Omega = \left\{x : (-\infty, T] \to X \quad \text{such that} \quad x|_{(-\infty,0]} \in \mathcal{B},\ x|_J \in C(J, X)\right\} .$$

3. Existence of mild solutions

Following [25, 14] we will introduce now the definition of mild solution to (1).

Definition 3.1. A function $x \in \Omega$ is said to be a mild solution of (1) if x satisfies

$$x(t) = \begin{cases} \phi(t)\,, & t \in \ (-\infty, 0]\,; \\ -Q(t)\phi(0) + \displaystyle\int_0^t \int_0^s R(t-s)a(s,\tau)f(\tau, x_\tau, x(\tau))\, d\tau ds\,, & t \in J\,, \end{cases} \tag{7}$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma\,, \qquad R(t) = q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)d\sigma$$

and ξ_q is a probability density function defined on $(0, \infty)$ such that

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0\,,$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q)\,, \quad \sigma \in (0, \infty)\,.$$

Remark 3.2. Note that $\{S(t)\}_{t\geq 0}$ is a uniformly bounded semigroup, i.e,

$$\text{there exists a constant} \quad M > 0 \quad \text{such that} \quad \|S(t)\| \leq M \quad \text{for all} \quad t \in [0, T]\,.$$

Remark 3.3. According to [25], a direct calculation gives that

$$\|R(t)\| \leq C_{q,M}t^{q-1}\,, \quad t > 0\,, \tag{8}$$

where $C_{q,M} = \dfrac{qM}{\Gamma(1+q)}$.

We make the following assumptions.

(H1) $f\colon J\times\mathcal{B}\times X\to X$ satisfies the Carathéodory conditions, and there exist two positive functions $\mu_i(\cdot)\in L^1(J,\mathbb{R}^+)$ $(i=1,2)$ with

$$\|\mu_2\|_{L^1(J,\mathbb{R}^+)} < \frac{q}{T^q a C_{q,M}}\,, \tag{9}$$

such that

$$\|f(t,v,w)\|\le \mu_1(t)\|v\|_{\mathcal{B}}+\mu_2(t)\|w\|,\quad (t,v,w)\in J\times\mathcal{B}\times X\,. \tag{10}$$

(H2) For any bounded sets $D_1\subset\mathcal{B}, D_2\subset X$, and $0\le s\le t\le T$, there exists an integrable positive function η such that

$$\alpha\big(R(t-s)f(\tau,D_1,D_2)\big)\le \eta_t(s,\tau)\big(\alpha(D_2)+\sup_{-\infty<\theta\le 0}\alpha(D_1(\theta))\big)\,,$$

where $\eta_t(s,\tau)=\eta(t,s,\tau)$ and $\sup\limits_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)d\tau ds=\eta^*<\infty$.

(H3) For each $t\in J$, $a(t,s)$ is measurable on $[0,t]$ and $a(t)=\operatorname{ess\,sup}\{|a(t,s)|, 0\le s\le t\}$ is bounded on J. The map $t\to a_t$ is continuous from J to $L^\infty(J,\mathbb{R})$, here, $a_t(s)=a(t,s)$.

Set $a=\sup\limits_{t\in J} a(t)$.

Theorem 3.4. *Suppose that the assumptions* (H1)–(H3) *hold with*

$$16a\eta^*<1\,, \tag{11}$$

then the problem (1) *has at least one mild solution on* $(-\infty,T]$.

Proof. We transform the problem (1) into a fixed-point problem. Define a mapping Φ from Ω into itself by

$$\Phi(x)(t)=\begin{cases}\phi(t)\,, & t\in\ (-\infty,0]\,;\\ -Q(t)\phi(0)+\displaystyle\int_0^t\int_0^s R(t-s)a(s,\tau)f\big(\tau,x_\tau,x(\tau)\big)\,d\tau\,ds, & t\in J\,.\end{cases}$$

Clearly, fixed points of the operator Φ are mild solutions of the problem (1).

For $\phi\in\mathcal{B}$, we will define the function $y(\cdot)\colon(-\infty,T]\to X$ by

$$y(t)=\begin{cases}\phi(t)\,, & \text{if}\quad t\in\ (-\infty,0]\,;\\ 0\,, & \text{if}\quad t\in J\,.\end{cases}$$

Then $y_0=\phi$. For each $z\in C(J,X)$ with $z(0)=0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t)=\begin{cases}0, & \text{if}\quad t\in\ (-\infty,0]\,;\\ z(t)\,, & \text{if}\quad t\in J\,.\end{cases}$$

If $x(\cdot)$ verifies (7), we can decompose it as $x(t) = y(t) + \overline{z}(t)$, for $t \in J$, which implies $x_t = y_t + \overline{z}_t$, for every $t \in J$ and the function $z(t)$ satisfies

$$z(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big)\, d\tau\, ds\,.$$

Let

$$Z_0 = \{z \in \Omega : z_0 = 0\}\,.$$

For any $z \in Z_0$, we have

$$\|z\|_{Z_0} = \sup_{t\in J}\|z(t)\| + \|z_0\|_{\mathcal{B}} = \sup_{t\in J}\|z(t)\|\,.$$

Thus $(Z_0, \|\cdot\|_{Z_0})$ is a Banach space. We define the operator $\widetilde{\Phi}\colon Z_0 \to Z_0$ by:

$$\widetilde{\Phi}(z)(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big)\, d\tau\, ds\,.$$

Obviously, the operator Φ has a fixed point is equivalent to $\widetilde{\Phi}$ has one, so it turns to prove that $\widetilde{\Phi}$ has a fixed point. Let $r > 0$ and consider the set

$$B_r = \{z \in Z_0 : \|z\|_{Z_0} \le r\}\,.$$

□

We need the following lemma.

Lemma 3.5. *Set*

$$C_1^* = \sup_{t\in J} C_1(t)\,; \qquad C_2^* = \sup_{\eta\in J} C_2(\eta)\,. \tag{12}$$

Then for any $z \in B_r$ we have

$$\|y_t + \overline{z}_t\|_{\mathcal{B}} \le C_2^*\|\phi\|_{\mathcal{B}} + C_1^* r := r^*\,,$$

and

$$\|f(t, y_t + \overline{z}_t, y(t) + \overline{z}(t)\| \le \mu_1(t)r^* + \mu_2(t)r\,. \tag{13}$$

Proof. Using (6), (12) and (10), we obtain

$$\begin{aligned}
\|y_t + \overline{z}_t\|_{\mathcal{B}} &\le \|y_t\|_{\mathcal{B}} + \|\overline{z}_t\|_{\mathcal{B}}\\
&\le C_1(t)\sup_{0\le\tau\le t}\|y(\tau)\| + C_2(t)\|y_0\|_{\mathcal{B}} + C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\| + C_2(t)\|z_0\|_{\mathcal{B}}\\
&\le C_2(t)\|\phi\|_{\mathcal{B}} + C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\|\\
&\le C_2^*\|\phi\|_{\mathcal{B}} + C_1^* r := r^*.
\end{aligned}$$

Also, we get

$$\begin{aligned}
\|f(t, y_t + \overline{z}_t, y(t) + \overline{z}(t)\| &\le \mu_1(t)\|y_t + \overline{z}_t\|_{\mathcal{B}} + \mu_2(t)\|y(t) + \overline{z}(t)\|\\
&\le \mu_1(t)r^* + \mu_2(t)r.
\end{aligned}$$

The lemma is proved. □

Now we prove that $\widetilde{\Phi}$ has a fixed point. The proof will be given in three steps.

Step 1: $\widetilde{\Phi}$ is continuous.
Let $\{z^k\}_{k\in\mathbb{N}}$ be a sequence such that $z^k \to z$ in B_r as $k\to\infty$. Then for each $t\in J$, we have

$$\begin{aligned}\|\widetilde{\Phi}(z^k)(t)-\widetilde{\Phi}(z)(t)\| &\le \int_0^t\int_0^s \|R(t-s)a(s,\tau)\big[f\big(\tau,y_\tau+\overline{z}^k_\tau,y(\tau)+\overline{z}^k(\tau)\big)\\ &\quad - f\big(\tau,y_\tau+\overline{z}_\tau,y(\tau)+\overline{z}(\tau)\big)\big]\|\,d\tau\,ds\\ &\le a\ C_{q,M}\int_0^t\int_0^s (t-s)^{q-1}\|f\big(\tau,y_\tau+\overline{z}^k_\tau,y(\tau)+\overline{z}^k(\tau)\big)\\ &\quad - f\big(\tau,y_\tau+\overline{z}_\tau,y(\tau)+\overline{z}(\tau)\big)\|\,d\tau\,ds.\end{aligned}$$

Since f is of Carathéodory type, we have by the Lebesgue Dominated Convergence Theorem that

$$\|\widetilde{\Phi}(z^k)(t)-\widetilde{\Phi}(z)(t)\|\to 0 \quad \text{when} \quad k\to\infty\,.$$

Consequently,

$$\lim_{k\to\infty}\|\widetilde{\Phi}(z^k)-\widetilde{\Phi}(z)\|_{Z_0}=0\,.$$

Thus $\widetilde{\Phi}$ is continuous.

Step 2: $\widetilde{\Phi}$ maps B_r into itself. Let

$$r\ge \frac{M\|\phi\|_{\mathcal{B}}+\frac{T^q aC_{q,M}r^*\|\mu_1\|_{L^1(J,\mathbb{R}^+)}}{q}}{1-\frac{T^q aC_{q,M}\|\mu_2\|_{L^1(J,\mathbb{R}^+)}}{q}}\,.$$

Then for each $z\in B_r$ and $t\in J$ we have

$$\begin{aligned}\|\widetilde{\Phi}(z)(t)\| &\le \|Q(t)\phi(0)\|\\ &\quad+\int_0^t\int_0^s \|R(t-s)a(s,\tau)f\big(\tau,y_\tau+\overline{z}_\tau,y(\tau)+\overline{z}(\tau)\big)\|\,d\tau\,ds\\ &\le M\|\phi\|_{\mathcal{B}}+a\ C_{q,M}\int_0^t\int_0^s (t-s)^{q-1}\left[\mu_1(\tau)r^*+\mu_2(\tau)r\right]\,d\tau\,ds\\ &\le M\|\phi\|_{\mathcal{B}}+a\ C_{q,M}r^*\int_0^t\int_0^s (t-s)^{q-1}\mu_1(\tau)\,d\tau\,ds\\ &\quad+a\ C_{q,M}r\int_0^t\int_0^s (t-s)^{q-1}\mu_2(\tau)\,d\tau\,ds\\ &\le M\|\phi\|_{\mathcal{B}}+\frac{T^q a\ C_{q,M}}{q}\left[r^*\|\mu_1\|_{L^1(J,\mathbb{R}^+)}+r\|\mu_2\|_{L^1(J,\mathbb{R}^+)}\right]\\ &\le r\,.\end{aligned}$$

Step 3: $\widetilde{\Phi}(B_r)$ is bounded and equicontinuous.

By Step 2, it is obvious that $\widetilde{\Phi}(B_r) \subset B_r$ is bounded. For the equicontinuity of $\widetilde{\Phi}(B_r)$. Let $\tau_1, \tau_2 \in J$ with $\tau_1 > \tau_2$, and let $z \in B_r$. Then

$$\begin{aligned}
\big\|\widetilde{\Phi}(z)(\tau_1) - \widetilde{\Phi}(z)(\tau_2)\big\| \le{}& \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
&+ \Big\| \int_0^{\tau_1}\int_0^{s} R(\tau_1 - s)a(s,\tau)f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big)\, d\tau\, ds \\
&- \int_0^{\tau_2}\int_0^{s} R(\tau_2 - s)a\big(s,\tau)f(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big)\, d\tau\, ds\Big\| .
\end{aligned}$$

Set

$$G\big(\cdot, y_\cdot + \overline{z}_\cdot, y(\cdot) + \overline{z}(\cdot)\big) = \int_0^{\cdot} a(\cdot,\tau)f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big)\, d\tau ,$$

then

$$\begin{aligned}
\big\|\widetilde{\Phi}(z)(\tau_1) - \widetilde{\Phi}(z)(\tau_2)\big\| \le{}& \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
&+ \Big\| \int_0^{\tau_1} R(\tau_1 - s)G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \\
&- \int_0^{\tau_2} R(\tau_2 - s)G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds\Big\| \\
\le{}& \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
&+ \Big\| \int_0^{\tau_2} R(\tau_1 - s)G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \\
&+ \int_{\tau_2}^{\tau_1} R(\tau_1 - s)G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \\
&- \int_0^{\tau_2} R(\tau_2 - s)G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds\Big\| \\
\le{}& \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
&+ \left\| \int_0^{\tau_2} [R(\tau_1 - s) - R(\tau_2 - s)]G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \right\| \\
&+ \int_{\tau_2}^{\tau_1} \|R(\tau_1 - s)\|\|G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\|\, ds \\
\le{}& I_1 + I_2 + I_3 ,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
I_2 &= \left\| \int_0^{\tau_2} [R(\tau_1 - s) - R(\tau_2 - s)]G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \right\| \\
I_3 &= \int_{\tau_2}^{\tau_1} \|R(\tau_1 - s)\|\|G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\|\, ds.
\end{aligned}$$

I_1 tends to zero as $\tau_2 \to \tau_1$, since $Q(t)$ is a strongly continuous operator.
For I_2, using (8) and (13), we have

$$\begin{aligned}
I_2 \le & \Big\| \int_0^{\tau_2} [q \int_0^{\infty} \sigma(\tau_1 - s)^{q-1} \xi_q(\sigma) S\big((\tau_1 - s)^q \sigma\big)\, d\sigma \\
& - q \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) S\big((\tau_2 - s)^q \sigma\big)\, d\sigma] G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big)\, ds \Big\| \\
\le & \; q \int_0^{\tau_2} \int_0^{\infty} \sigma \| [(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}] \xi_q(\sigma) S\big((\tau_1 - s)^q \sigma\big) \\
& \times G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big) \| \, d\sigma \, ds \\
& + q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S\big((\tau_1 - s)^q \sigma\big) - S\big((\tau_2 - s)^q \sigma\big) \| \\
& \times \| G(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)) \| \, d\sigma \, ds \\
\le & \; C_{q,M} \int_0^{\tau_2} \big| (\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1} \big| \, \| G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big) \| \, ds \\
& + q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S\big((\tau_1 - s)^q \sigma\big) - S\big((\tau_2 - s)^q \sigma\big) \| \\
& \times \| G\big(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)\big) \| \, d\sigma \, ds \\
\le & \; a \left[r^* \|\mu_1\|_{L^1(J,\mathbb{R}^+)} + r \|\mu_2\|_{L^1(J,\mathbb{R}^+)} \right] \\
& \times [C_{q,M} \int_0^{\tau_2} \big| (\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1} \big| \, ds \\
& + q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S\big((\tau_1 - s)^q \sigma\big) - S\big((\tau_2 - s)^q \sigma\big) \| \, d\sigma \, ds] \, .
\end{aligned}$$

Clearly, the first term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. From the continuity of $S(t)$ in the uniform operator topology for $t > 0$, the second term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. In view of (13), we have

$$\begin{aligned}
I_3 \le & \; C_{q,M} \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} \| G(s, y_s + \overline{z}_s, y(s) + \overline{z}(s)) \| \, ds \\
\le & \; a \; C_{q,M} \left[r^* \|\mu_1\|_{L^1(J,\mathbb{R}^+)} + r \|\mu_2\|_{L^1(J,\mathbb{R}^+)} \right] \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} \, ds \, .
\end{aligned}$$

As $\tau_2 \to \tau_1$, I_3 tends to zero.
So $\widetilde{\Phi}(B_r)$ is equicontinuous.

Now let V be a subset of B_r such that $V \subset \overline{\mathrm{conv}}(\widetilde{\Phi}(V) \cup \{0\})$. Moreover, for any $\varepsilon > 0$ and bounded set D, we can take a sequence $\{v_n\}_{n=1}^{\infty} \subset D$ such that $\alpha(D) \le 2\alpha(\{v_n\}) + \varepsilon$ ([9, p.125]). Thus, for $\{v_n\}_{n=1}^{\infty} \subset V$, and using Lemmas

2.3–2.5 and (H2), we get

$$\begin{aligned}
\alpha\big(\widetilde{\Phi}V\big) &\le 2\alpha\big(\{\widetilde{\Phi}v_n\}\big)+\varepsilon = 2\sup_{t\in J}\alpha\big(\{\widetilde{\Phi}v_n(t)\}\big)+\varepsilon\\
&= 2\sup_{t\in J}\alpha\Big(\Big\{\int_0^t R(t-s)\int_0^s a(s,\tau)f\big(\tau,y_\tau+v_{n\tau},y(\tau)+v_n(\tau)\big)\,d\tau\,ds\Big\}\Big)+\varepsilon\\
&\le 4\sup_{t\in J}\int_0^t \alpha\Big(\Big\{R(t-s)\int_0^s a(s,\tau)f\big(\tau,y_\tau+v_{n\tau},y(\tau)+v_n(\tau)\big)\,d\tau\,ds\Big\}\Big)+\varepsilon\\
&\le 8\sup_{t\in J}\int_0^t\int_0^s \alpha(\{R(t-s)a(s,\tau)f\big(\tau,y_\tau+v_{n\tau},y(\tau)+v_n(\tau)\big)\,d\tau\,ds\})+\varepsilon\\
&\le 8\ a\sup_{t\in J}\int_0^t\int_0^s \alpha\big(\{R(t-s)f\big(\tau,y_\tau+v_{n\tau},y(\tau)+v_n(\tau)\big)\,d\tau\,ds\}\big)+\varepsilon\\
&\le 8\ a\ \sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)\big[\alpha(v_n(\tau))+\sup_{-\infty<\theta\le 0}\alpha(v_n(\theta+\tau))\big]\,d\tau\,ds+\varepsilon\\
&\le 8\ a\ \sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)\big[\alpha(v_n)+\sup_{0<\mu\le\tau}\alpha(v_n(\mu))\big]\,d\tau\,ds+\varepsilon\\
&\le 16\ a\ \alpha(v_n)\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)\,d\tau\,ds+\varepsilon\\
&\le 16\ a\ \eta^*\alpha(V)+\varepsilon\,.
\end{aligned}$$

Therefore, in view of Lemma 2.3, we have

$$\alpha(V)\le\alpha(\widetilde{\Phi}V)\le 16\ a\ \eta^*\alpha(V)+\varepsilon\,,$$

since ε is arbitrary we obtain that

$$\alpha(V)\le 16\ a\ \eta^*\alpha(V)\,.$$

This means that

$$\alpha(V)(1-16\ a\ \eta^*)\le 0\,.$$

By (11) it follows that $\alpha(V)=0$. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_r. Applying now Theorem 2.7, we conclude that $\widetilde{\Phi}$ has a fixed point which is a solution of the problem (1).

4. An example

In this section we give an example to illustrate the above results. Consider the following fractional integrodifferential equations

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}v(t,\zeta) &= \frac{\partial^2}{\partial\zeta^2}v(t,\zeta)+\int_0^t(t-s)\int_{-\infty}^0\gamma_1(\theta)\sin(s|v(s+\theta,\zeta)|)\,d\theta\,ds\\
&\quad+\int_0^t(t-s)\frac{s^2}{2}\sin|v(s,\zeta)|\int_0^s\cos v(\iota,\zeta)\,d\iota\,ds \qquad (14)\\
v(\theta,\zeta) &= v_0(\theta,\zeta),\qquad -\infty<\theta\le 0\,,
\end{aligned}$$

where $t,\ \zeta \in [0,1]$, $\gamma_1\colon (-\infty,0] \to \mathbb{R}$, $v_0\colon (-\infty,0]\times[0,1] \to \mathbb{R}$ are continuous functions, and $\int_{-\infty}^{0} |\gamma_1(\theta)|\, d\theta < \infty$.

Set $X = L^2([0,1],\mathbb{R})$ and define A by

$$D(A) = H^2((0,1)) \cap H_0^1((0,1)),$$

$$Au = u''.$$

Then, A generates a compact, analytic semigroup $S(t)$ of uniformly bounded, linear operators such that $\|S(t)\| \le 1$.
Let the phase space $\mathcal{B} = C((-\infty,0],X)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\|_{\mathcal{B}} = \sup_{-\infty<\theta\le 0} |\varphi(\theta)|, \quad \forall \varphi \in \mathcal{B},$$

then we can see that $C_1(t) = 0$ in (6).

For $t \in [0,1]$, $\zeta \in [0,1]$ and $\varphi \in C((-\infty,0],X)$, we set

$$x(t)(\zeta) = v(t,\zeta),$$

$$\phi(\theta)(\zeta) = v_0(\theta,\zeta), \qquad \theta \in (-\infty,0],$$

$$a(t,s) = t - s,$$

$$f\big(t,\varphi,x(t)\big)(\zeta) = \int_{-\infty}^{0} \gamma_1(\theta) \sin\big(s|\varphi(\theta)(\zeta)|\big)\, d\theta + \frac{s^2}{2} \sin|x(t)(\zeta)| \int_0^s \cos x(\iota)(\zeta)\, d\iota.$$

Thus, problem (14) can be rewritten as the abstract problem (1).
Moreover, for $t \in [0,1]$, we can see

$$\begin{aligned}\|f(t,\varphi,x(t))(\zeta)\| &\le t\|\varphi\|_{\mathcal{B}} \int_{-\infty}^{0} |\gamma_1(\theta)| d\theta + \frac{t^3}{2}\|x(t)\| \\ &= \mu_1(t)\|\varphi\|_{\mathcal{B}} + \mu_2(t)\|x(t)\|,\end{aligned}$$

where

$$\mu_1(t) = t\int_{-\infty}^{0} |\gamma_1(\theta)|\, d\theta, \quad \mu_2(t) = t^3/2.$$

Then (14) has a mild solution by Theorem 3.4.
For example, if we put

$$\gamma_1(\theta) = e^{\theta}, \qquad q = \frac{1}{2},$$

then

$$C_{q,M} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} = 1/\sqrt{\pi}, \qquad \|\mu_2\|_{L^1(J,\mathbb{R}^+)} = 1/8.$$

Thus, we see

$$\frac{aT^q C_{q,M}\|\mu_2\|_{L^1(J,\mathbb{R}^+)}}{q} = \frac{1}{4\sqrt{\pi}} < 1.$$

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Corresponding author:
Mouffak Benchohra,
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie
E-mail: `benchohra@univ-sba.dz`