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PENROSE TRANSFORM AND MONOGENIC SECTIONS

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Abstract. The Penrose transform gives an isomorphism between the kernel of the 2-Dirac operator over an affine subset and the third sheaf cohomology group on the twistor space. In the paper we give an integral formula which realizes the isomorphism and decompose the kernel as a module of the Levi factor of the parabolic subgroup. This gives a new insight into the structure of the kernel of the operator.

1. Introduction

Let us denote by $\mathrm{V}_{\mathbb{R}}(k, n+k)$ the Grassmanian of null k-planes in $\mathbb{R}^{n+k,k}$ with a quadratic form of signature $(n+k, k)$. This space is the homogeneous model for a parabolic geometry. For each $k \geq 2$ and $n \geq 2k$ there is a sequence of invariant differential operators starting with a first order operator called the *k-Dirac operator* (in the parabolic setting), see [8]. These sequences belong to singular character and are interesting from the point of the k-Dirac operator (in the Euclidean setting) studied in Clifford analysis, see [4].

The sequence starting with the 2-Dirac operator D_1 consists of three operators

$$\Gamma(\mathcal{V}_1) \xrightarrow{D_1} \Gamma(\mathcal{V}_2) \xrightarrow{D_2} \Gamma(\mathcal{V}_3) \xrightarrow{D_3} \Gamma(\mathcal{V}_4) \to 0 \tag{1}$$

where D_2 is a second order operator while D_1, D_3 are first order operators. The graph of the sequence depends only on k so we suppress the parameter n and talk about the k-Dirac operator. However, the size of the graph with increasing k grows rapidly. A recursive formula for constructing these graphs can be found in [5].

The sequences starting with the k-Dirac operator are coming from the Penrose transform. The Penrose transform is explained in [2] and [10]. The Penrose transform for the k-Dirac operator, i.e. the relative BGG sequences and the direct images, is described in [7]. Similar sequences were obtained by the Penrose transform also in [1] on quaternionic manifolds.

The Penrose transform lives in the holomorphic category. Thus we have to replace real groups and spaces by their complex analogues. Let $\mathrm{V}_{\mathbb{C}}(k, n)$ be the Grassmannian of complex k-dimensional subspaces in $\mathbb{C}^{k+n}$ that are totally null with respect to a fixed non-degenerate, symmetric, complex bilinear form. We

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may realize an affine subset of the real Grassmanian $\mathrm{V}_{\mathbb{R}}(k, n+k)$ as the set $\mathbb{R}^m \hookrightarrow \mathbb{C}^m$ with the canonical inclusion where $\mathbb{C}^m$ is an affine subset of the complex Grassmannian $\mathrm{V}_{\mathbb{C}}(k, n+k)$ and $m = nk + \binom{k}{2}$. Then a real analytic section defined over a subset $\mathcal{V}$ of $\mathbb{R}^m$ extends uniquely to a holomorphic section over a small open neighbourhood of the set $\mathcal{V}$ in $\mathbb{C}^m$. This induces a natural bijection between the set of germs of real analytic sections over $\mathbb{R}^m$ and the set of germs of holomorphic sections over $\mathbb{C}^m$. Thus we can naturally interpret results from the holomorphic category in the real analytic category.

A section $\psi \in \Gamma(\mathcal{V}_1)$ such that $D_1\psi = 0$ is called a *monogenic section*. In this paper we study the real analytic monogenic sections of the 2-Dirac operator D_1 in the case $n = 6$ over an affine subset of $\mathrm{V}_{\mathbb{R}}(2, 8)$. In this case we can write down explicitly the isomorphism (12) given by the Penrose transform between the third sheaf cohomology group $H^3(\mathcal{W}, \mathcal{O}_\lambda)$ and the kernel of D_1 over the affine subset. This is the integral formula (14) which to my knowledge has not been considered yet. This is a similar formula as the integral formula for the Maxwell equations given in [2] and [10]. We will use the Čech definition of $H^i(\mathcal{W}, \mathcal{O}_\lambda)$. The third cohomology group is discussed in some detail in Section 4.1. Moreover we give the decomposition of the space of homogeneous monogenic sections with respect to a maximal reductive subgroup of the parabolic subgroup, see Theorem 6.1.

1.1. **Notation.** We will denote by $M(k, n, \mathbb{C})$, resp. $M(k, \mathbb{C})$, resp. $A(k, \mathbb{C})$ the space of complex matrices of size $k \times n$, resp. $k \times k$, resp. the space of skew-symmetric matrices of size $k \times k$. The identity matrix in $M(k, \mathbb{C})$ is denoted by 1_k. We denote the span of vectors $v_1, \ldots, v_k$ by $\langle v_1, \ldots, v_k \rangle$. If $A \in M(n+k, k, \mathbb{C})$ is a matrix, then we associate to A the k-plane in $\mathbb{C}^{n+k}$ which is spanned by the columns of the matrix A.

2. The parabolic geometry

Let $\{e_1, e_2, e_3, e_4, e_5, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_1, \bar{e}_2\}$ be the standard basis of $\mathbb{C}^{10}$. Let h be the complex bilinear product uniquely determined by $h(e_i, \bar{e}_j) = \delta_{ij}, h(e_i, e_j) = h(\bar{e}_i, \bar{e}_j) = 0$ for all $1 \leq i, j \leq 5$. Let $\mathrm{G} := \{g \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{10}) \mid \forall u, v \in \mathbb{C}^{10} : h(gu, gv) = h(u, v), \det(g) = 1\}$. Let $x_0 := \langle e_1, e_2 \rangle$ and let $\mathrm{P} := \{g \in \mathrm{G} \mid g(x_0) = x_0\}$. The space G/P is naturally isomorphic to the Grassmannian $\mathrm{V}_{\mathbb{C}}(2, 8)$. Let G_0 be the stabilizer of $x_0^c := \langle e_3, e_4, e_5, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ in P. Then G_0 is a maximal reductive subgroup of P isomorphic to $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{SO}(6, \mathbb{C})$. Lie algebra $\mathfrak{g}$ of G consists of the matrices of the form

$$\begin{pmatrix} A & Y_1 & Y_2 & Y_{12} \\ X_1 & B & D & -Y_2^T \\ X_2 & C & -B^T & -Y_1^T \\ X_{12} & -X_2^T & -X_1^T & -A^T \end{pmatrix} \tag{2}$$

where $A \in M(2, \mathbb{C})$, $B \in M(3, \mathbb{C})$, $C,\ D \in A(3, \mathbb{C})$, $Y_i,\ X_i^T \in M(2, 3, \mathbb{C})$, X_{12}, $Y_{12} \in A(2, \mathbb{C})$. There is a G_0-invariant grading

$$\mathfrak{g} \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

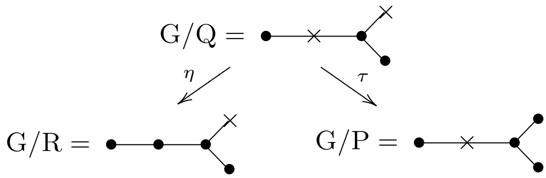
such that $\mathfrak{g}_0$ is Lie algebra of G_0 and Lie algebra $\mathfrak{p}$ of P is $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The subalgebra $\mathfrak{g}_0$ corresponds to the blocks A, B, C, D, $\mathfrak{g}_1$ to the blocks Y_1, Y_2, $\mathfrak{g}_2$ to the block Y_{12}, $\mathfrak{g}_{-2}$ to the block X_{12} and $\mathfrak{g}_{-1}$ to the blocks X_1, X_2. Let us denote by $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Let $\pi\colon \mathrm{G} \to \mathrm{G}/\mathrm{P}$ be the canonical projection. Let $\mathrm{G}_- := \exp(\mathfrak{g}_-)$ and let $\mathcal{U} := \pi(\mathrm{G}_-)$. The map exp is a biholomorphism between $\mathfrak{g}_-$ and G_- and the map π is a biholomorphism between G_- and $\mathcal{U}$. Thus $\mathcal{U}$ is an affine subset of $\mathrm{V}_{\mathbb{C}}(2,8)$ biholomorphic to $M(6,2,\mathbb{C}) \times A(2,\mathbb{C})$.

3. The double fibration

Let $z_0 := \langle e_1, \ldots, e_5 \rangle$ and set $\mathrm{R} := \{g \in \mathrm{G} \mid g(z_0) = z_0\}, \mathrm{Q} := \mathrm{P} \cap \mathrm{R}$. Then R and Q are parabolic subgroups of G. Then we have the diagram of double fibration

(3)



The twistor space G/R is the connected component of z_0 in the Grassmannian $\mathrm{V}_{\mathbb{C}}(5,5)$. This is the Grassmannian of self-dual 5-planes. Lie algebra of R is the subspace of $\mathfrak{g}$ where the matrices C, X_2, X_{12} from (2) are zero. Let $\bar{z}_0 = \langle \bar{e}_1, \ldots, \bar{e}_5 \rangle$ and let $\mathrm{R}_0 := \{g \in \mathrm{R} \mid g(\bar{z}_0) = \bar{z}_0\}$. Then R_0 is a maximal reductive subgroup of R isomorphic to $\mathrm{GL}(5,\mathbb{C})$ and its Lie algebra consists of the matrices where A, B, X_1, Y_1 are arbitrary and the other matrices are zero.

The correspondence space G/Q consists of the pairs (z, x) with $x \in \mathrm{G}/\mathrm{P}$, $z \in \mathrm{G}/\mathrm{R}$ such that $x \subset z$. Lie algebra $\mathfrak{q}$ of Q sits in the blocks A, B, D, Y_1, Y_2, Y_{12}. Let $\mathrm{Q}_0 := \mathrm{G}_0 \cap \mathrm{R}_0$. Then Q_0 is a maximal reductive subgroup of Q isomorphic to $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})$. Lie algebra $\mathfrak{q}_0$ of Q_0 corresponds to the blocks A, B.

Let us denote $\mathcal{V} := \tau^{-1}(\mathcal{U})$ and let $\mathcal{W} := \eta(\mathcal{V})$.

3.1. **Projection τ.** Let us recall that $x_0 = \langle e_1, e_2 \rangle, x_0^c = \langle e_3, \ldots, e_5, \bar{e}_3, \ldots, \bar{e}_5 \rangle$ and $z_0 = \langle e_1, \ldots, e_5 \rangle$. Let us notice that $x_0^\perp = x_0 \oplus x_0^c$ and that the invariant product h defined on $\mathbb{C}^{10}$ descends to a non-degenerate inner product on $x_0^\perp / x_0$. The projection $\tau|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$ sends $(z, x) \mapsto x$. The fibre $\tau^{-1}(x_0)$ is isomorphic to the set of all null 5-planes in G/R which contains the null 2-plane x_0. Any null 5-plane z with $(z, x_0) \in \tau^{-1}$ is contained in $x_0^\perp$ and the projection $z \mapsto z/x_0$ identifies the fibre $\tau^{-1}(x_0)$ with a set of null 3-planes in $x_0^\perp / x_0$. We know that $(z_0, x_0) \in \tau^{-1}(x_0)$ and $z_0/x_0 \cong y_0$ where $y_0 := \langle e_3, e_4, e_5 \rangle$. The fibre $\tau^{-1}(x)$ is isomorphic to P/Q and in particular is connected. We deduce that the fibre is biholomorphic to the connected component of y_0 in the space of all null 3-planes in $x_0^\perp / x_0 \cong \mathbb{C}^6$. This connected component is biholomorphic to the family of α-planes in $\mathbb{C}^6$.

3.2. **The family of α-planes in the Grassmanian $\mathbf{V}_{\mathbb{C}}(3,6)$.** It will be convenient to make the following identifications. We identify $x_0^\perp / x_0 \cong \Lambda^2 \mathbb{C}^4$ such that the basis $\{e_3, e_4, e_5, \bar{e}_3, \bar{e}_4, \bar{e}_5\}$ goes to the basis

$$\{f_0 \wedge f_1, f_0 \wedge f_2, f_0 \wedge f_3, f_2 \wedge f_3, -f_1 \wedge f_3, f_1 \wedge f_2\}\,, \tag{4}$$

of $\Lambda^2\mathbb{C}^4$ where $\{f_0, f_1, f_2, f_3\}$ is the standard basis of $\mathbb{C}^4$. Under this identification, the quadratic form on $\mathbb{C}^6$ goes to the quadratic form Q on $\Lambda^2\mathbb{C}^4$ which is determined by $\alpha \wedge \alpha = Q(\alpha) f_0 \wedge f_1 \wedge f_2 \wedge f_3$ for all $\alpha \in \Lambda^2\mathbb{C}^4$. The natural action of $\mathrm{SL}(4,\mathbb{C})$ on $\Lambda^2\mathbb{C}^4$ preserves this quadratic form and it turns out that this identifies $\mathrm{SL}(4,\mathbb{C}) \cong \mathrm{Spin}(6,\mathbb{C})$. The corresponding isomorphism $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$ is determined by

$$\begin{pmatrix} A_1 & E_{12} & 0 & 0 \\ E_{21} & A_2 & E_{23} & 0 \\ 0 & E_{32} & A_3 & E_{34} \\ 0 & 0 & E_{43} & A_4 \end{pmatrix} \to \tag{5}$$

$$\begin{pmatrix} A_1 + A_2 & E_{23} & 0 & 0 & 0 & 0 \\ E_{32} & A_1 + A_3 & E_{34} & 0 & 0 & E_{12} \\ 0 & E_{43} & A_1 + A_4 & 0 & -E_{12} & 0 \\ 0 & 0 & 0 & -A_1 - A_2 & -E_{32} & 0 \\ 0 & 0 & -E_{21} & -E_{23} & -A_1 - A_3 & -E_{43} \\ 0 & E_{21} & 0 & 0 & -E_{34} & -A_1 - A_4 \end{pmatrix}.$$

The Grassmannian $\mathrm{V}_{\mathbb{C}}(3,6)$ is the disjoint sum of two families. The first family, called the family of α-planes, can be identified with $\mathbb{CP}^3$ by the following mapping. Let $v \in \mathbb{C}^4$ be a representative of $\pi \in \mathbb{CP}^3$. Let $\{v, v_1, v_2, v_3\}$ be a basis of $\mathbb{C}^4$. We assign to π the 3-plane $\langle v \wedge v_1, v \wedge v_2, v \wedge v_3\rangle, i = 1, 2, 3$. It is easy to see that the 3-plane is null and that the map is well defined. The other family, called the family of β-planes, can be identified with $\mathbb{P}(\mathbb{C}^4)^*$ by the assignment $[\omega] \in \mathbb{P}(\mathbb{C}^4)^* \mapsto \langle v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3\rangle$ where $\{v_1, v_2, v_3\}$ is a basis of $Ker(\omega)$. One can easily check that this map is well defined.

3.3. **Affine coordinates on the family of α-planes.** Since the family of α-planes is biholomorphic to $\mathbb{CP}^3$ we know that there is an affine covering $\{\mathcal{U}_0, \dots, \mathcal{U}_3\}$ of the family of α-planes. Let us write down the affine charts on $\mathcal{U}_0$ and $\mathcal{U}_1$. Let $v = (v_0, v_1, v_2, v_3) \in \mathbb{C}^4$ be a non-zero vector and let us assume that $v_0 \neq 0$, resp. $v_1 \neq 0$. Let $w_0 := v_0^{-1} v = (1, \zeta_1, \zeta_2, \zeta_3)$, resp. $w_1 := v_1^{-1} v = (\rho_1, 1, \rho_2, \rho_3)$. Then $\{w_0 \wedge f_1, w_0 \wedge f_2, w_0 \wedge f_3\}$, resp. $\{-w_1 \wedge f_0, w_1 \wedge f_2, -w_1 \wedge f_3\}$ is a unique basis of the α-plane corresponding to $[v]$ such that the matrix whose columns are the coefficients of the vectors with respect to the preferred basis has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix}, \text{ resp. } \begin{pmatrix} 1 & 0 & 0 \\ \rho_2 & \rho_1 & 0 \\ \rho_3 & 0 & -\rho_1 \\ 0 & -\rho_3 & -\rho_2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6}$$

Thus we see that ζ_1, ζ_2, ζ_3, resp. ρ_1, ρ_2, ρ_3 give natural coordinates on the set $\mathcal{U}_0$, resp. $\mathcal{U}_1$. Similarly for $v_3 \neq 0$ and $v_4 \neq 0$. The change of coordinates between $\mathcal{U}_0$

and $\mathcal{U}_1$ is

$$\zeta_1^{-1} = \rho_1, \zeta_2\zeta_1^{-1} = \rho_2, \zeta_3\zeta_1^{-1} = \rho_3 \,. \tag{7}$$

3.4. **The set $\mathcal{W}$.** The set $\mathcal{V} := \tau^{-1}(\mathcal{U})$ is biholomorphic to $\tau^{-1}(x_0) \times \mathcal{U}$ such that $\tau : \mathcal{V} \to \mathcal{U}$ is the canonical projection on the second factor. Then $\{\mathcal{V}_i := \mathcal{U}_i \times \mathcal{U} \mid i = 0,1,2,3\}$ is an affine covering of $\mathcal{V}$. For $i = 0,1,2,3$ put $\mathcal{W}_i := \eta(\mathcal{V}_i)$. Then $\mathfrak{W} := \{\mathcal{W}_i \mid i = 0,1,2,3\}$ is an affine covering of $\mathcal{W} = \eta(\mathcal{V})$. A full discussion to this can be found in [9]. Then each self-dual 5-plane in $\mathcal{W}_0$, resp. $\mathcal{W}_1$ has a unique basis given by the columns of the matrix of the form

(8)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ z_{11} & z_{21} & 0 & -\zeta_3 & \zeta_2 \\ z_{21} & z_{22} & \zeta_3 & 0 & -\zeta_1 \\ z_{31} & z_{32} & -\zeta_2 & \zeta_1 & 0 \\ 0 & z_0 & -z_{11} & -z_{21} & -z_{31} \\ -z_0 & 0 & -z_{21} & -z_{22} & -z_{32} \end{pmatrix}, \text{ resp. } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ w_{31} & w_{32} & \rho_2 & \rho_1 & 0 \\ w_{21} & w_{22} & \rho_3 & 0 & -\rho_1 \\ w_{11} & w_{21} & 0 & -\rho_3 & -\rho_2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & w_0 & * & * & * \\ -w_0 & 0 & * & * & * \end{pmatrix}$$

where $*$ are determined by the other entries. We write the left matrix from (8) in the block form

$$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_3 \\ B_1 & B_2 \\ B_0 & -B_1^T \end{pmatrix}, B_0 = \begin{pmatrix} 0 & z_0 \\ -z_0 & 0 \end{pmatrix}, \tag{9}$$
$$B_1 = (z_{ij}), B_2 = \begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix}.$$

The change of coordinates on $\mathcal{W}_0 \cap \mathcal{W}_1$ is

$$\begin{aligned} w_0 &= z_0 + z_{32}z_{21}\zeta_1^{-1} - z_{22}z_{31}\zeta_1^{-1} \,, \\ w_{11} &= z_{11} + z_{31}\zeta_3\zeta_1^{-1} + z_{21}\zeta_2\zeta_1^{-1} \,, \\ w_{12} &= z_{12} + z_{32}\zeta_3\zeta_1^{-1} + z_{22}\zeta_2\zeta_1^{-1} \,, \\ w_{ij} &= (-1)^i\zeta_1^{-1}z_{ij} \,, \quad i = 2,3, \; j = 1,2 \end{aligned} \tag{10}$$

and those in (7).

4. Sections of the bundle $\mathcal{O}_\lambda$ over the set $\mathcal{W}$

Let $\mathbb{C}_\lambda$ be an one-dimensional R-module with highest weight $\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$. Let $\mathcal{O}_\lambda$ be the sheaf of holomorphic sections of the line bundle $\mathcal{C}_\lambda := \mathrm{G} \times_{\mathrm{R}} \mathbb{C}_\lambda$. Let $\mathcal{M} \subset \mathrm{G}/\mathrm{R}$ be an open subset. Let $\mathcal{O}_\lambda(\mathcal{M})$ be the space of holomorphic sections of the bundle $\mathcal{C}_\lambda$ over $\mathcal{M}$ and let $\mathcal{O}(\mathcal{M})$ be the space of holomorphic functions on $\mathcal{M}$.

Let $z \in \mathcal{W}_0$ and let $\{v_1, \dots, v_5\}$ be the preferred basis of z from (8). Then $\{v_1, \dots, v_5, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_1, \bar{e}_2\}$ is null orthogonal basis of $\mathbb{C}^{10}$. Let $g \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{10})$ be the linear map such that for all $1 \le i \le 5 : g(e_i) = v_i,\ g(\bar{e}_i) = \bar{e}_i$. Then $g \in \mathrm{G}$ and the map $z \mapsto \rho_0(z) := g$ is a section of the principal R-bundle over $\mathcal{W}_0$. We define similarly for $i = 1, 2, 3$ sections ρ_i over $\mathcal{W}_i$. Let us write the transition function of the bundle $\mathcal{C}_\lambda$ on $\mathcal{W}_0 \cap \mathcal{W}_1$ in the preferred trivializations ρ_0 and ρ_1. Let $f_0 \in \mathcal{O}(\mathcal{W}_0)$, $f_1 \in \mathcal{O}(\mathcal{W}_1)$. Then f_0, f_1 defines an element of $\mathcal{O}_\lambda(\mathcal{W}_0 \cup \mathcal{W}_1)$ iff

$$f_0(z_0, z_{ij}, \zeta_i) = \zeta_1^{-5} f_1(w_0, w_{ij}, \rho_i) \tag{11}$$

on $\mathcal{W}_0 \cap \mathcal{W}_1$. All the transition functions are rational functions and thus we can also consider rational sections of $\mathcal{C}_\lambda$ over $\mathcal{W}$.

4.1. **Cohomology groups.** There is a spectral sequence which relates the sheaf cohomology $H^i(\mathcal{W}, \mathcal{O}_\lambda)$ to the cohomology of the sequence starting with the 2-Dirac operator over $\mathcal{U}$, for more see [2]. On the first page of the spectral sequence appears the direct images of the relative BGG sequence living on the correspondence space. These direct images can be found in [7]. The spectral sequence converges on the second page. In particular for $k = 2$, $n = 6$, we conclude that

$$\mathcal{P}\colon H^3(\mathcal{W}, \mathcal{O}_\lambda) \cong \mathrm{Ker}(D_1, \mathcal{U}) \tag{12}$$

where the bundles from (1) are $\mathcal{V}_1 = \mathrm{G} \times_{\mathrm{P}} (\mathbb{C}_\nu \otimes \mathbb{S}_+)$, $\mathcal{V}_2 = \mathrm{G} \times_{\mathrm{P}} (\mathbb{C}_\mu \otimes \mathbb{S}_+)$ where the $\mathrm{GL}(2, \mathbb{C})$-module $\mathbb{C}_\nu$, resp. $\mathbb{C}^2_\mu$ has highest weight $\left(\frac{5}{2}, \frac{5}{2}\right)$, resp. $\left(\frac{7}{2}, \frac{5}{2}\right)$ and $\mathbb{S}_+ \cong \mathbb{C}^4, \mathbb{S}_- \cong (\mathbb{C}^4)^*$ as $\mathrm{SL}(4, \mathbb{C})$-modules.

The Leray theorem states that $H^i(\mathcal{W}, \mathcal{O}_\lambda) \cong \check{H}^i(\mathfrak{W}, \mathcal{O}_\lambda)$ where $\check{H}^*(\mathfrak{W}, \mathcal{O}_\lambda)$ are the Čech cohomology groups computed with respect to the affine covering $\mathfrak{W}$, see for example [10]. The co-chains groups are $C^j(\mathfrak{W}, \mathcal{O}_\lambda) = 0,\ j \ge 0, C^3(\mathfrak{W}, \mathcal{O}_\lambda) = \big\{(\bigcap_{i=0}^3 \mathcal{W}_i, f) \mid f \in \mathcal{O}_\lambda(\bigcap_{i=0}^3 \mathcal{W}_i)\big\}, C^2(\mathfrak{W}, \mathcal{O}_\lambda) = \dots$. In particular notice that from this follows that for $k = 2, n = 6$ the sequence starting with the 2-Dirac operator is locally exact. By definition we have that $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda) := C^3(\mathfrak{W}, \mathcal{O}_\lambda)/\operatorname{Im}(\delta^2)$ where δ^2 is the Čech co-differential. We will denote the cohomology classes by [ ]. Let us make some simple observations about $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$. We will work with the affine chart on $\mathcal{W}_0$ from (8).

Let us first notice that $\bigcap_{i=0,1,2,3} \mathcal{W}_i = \{z \in \mathcal{W}_0 \mid \zeta_1 \ne 0, \zeta_2 \ne 0, \zeta_3 \ne 0\}$. Thus if $f \in \mathcal{O}_\lambda(\bigcap_{i=0,1,2,3} \mathcal{W}_i)$ is a holomorphic section then f is the converging sum of rational sections $f(s_0, s_{ij}, r_k) = z_0^{s_0} \prod_{ij} z_{ij}^{s_{ij}} \zeta_1^{-r_1} \zeta_2^{-r_2} \zeta_3^{-r_3}$ where s_0, $s_{ij} \ge 0$ and r_1, r_2, $r_3 \in \mathbb{Z}$. It is easy to see that $[f(s_0, s_{ij}, r_k)] = 0$ if $r_1 < 0$ or $r_2 < 0$ or $r_3 < 0$. From the formula (11) follows that if $5 + s_0 + \sum s_{ij} > r_1 + r_2 + r_3$ then $f(s_0, s_{ij}, r_k)$ extends to a rational section on $\mathcal{W}_1 \cap \mathcal{W}_2 \cap \mathcal{W}_3$ and thus $[f(s_0, s_{ij}, r_k)] = 0$. However notice that this does not characterize $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$. For example the cohomology class of the section $z_{31}^i \zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-3}$ is trivial for any $i \ge 0$ although the relation does not hold. The full characterization of the $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$ will be given in Theorem 6.1 where we give it as a direct sum of G_0-modules.

5. The correspondence $x \in \mathcal{U} \mapsto \eta \circ \tau^{-1}(x) \subset \mathcal{W}$

Let us write the correspondence on $\mathcal{W}_0$. Then

$$\begin{pmatrix} 1_2 \\ X_1 \\ X_2 \\ X_{12} - \frac{1}{2}(X_1^T X_2 + X_2^T X_1) \end{pmatrix} \in \mathcal{U} \mapsto \tag{13}$$

$$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_3 \\ X_2 - \zeta X_1 & \zeta \\ X_{12} + \frac{1}{2}(X_2^T X_1 - X_1^T X_2) + X_1^T \zeta X_1 & -X_2^T - X_1^T \zeta \end{pmatrix} \subset \mathcal{W}_0$$

where the blocks of the matrix in the second row are those from (9). In particular $X_1, X_2 \in M(3,2,\mathbb{C}), X_{12} \in A(2,\mathbb{C}), \zeta \in A(3,\mathbb{C})$. Notice that the set $\hat{x} := \eta \circ \tau^{-1}(x) \subset \mathcal{W}$ is biholomorphic to $\mathbb{CP}^3$ for each $x \in \mathcal{U}$.

5.1. **Integration.** The isomorphism (12) is given by integrating over the fibres of the correspondence (13). Let $f \in C^3(\mathcal{W}, \mathcal{O}_\lambda)$. We write $f \equiv f(B_0, B_1, B_2)$ where (B_0, B_1, B_2) are the matrices from (9). Then the integral formula expresses $\mathcal{P}(f)(X_{12}, X_1, X_2)$ as

$$\frac{1}{(2\pi i)^3} \int_{(S^1)^3} (1, \zeta_1, \zeta_2, \zeta_3) f(X_{12} + \tfrac{1}{2}(X_2^T X_1 - X_1^T X_2) \\ + X_1^T \zeta X_2, X_2 - \zeta X_1, \zeta)\, d\zeta_1\, d\zeta_2\, d\zeta_3 \tag{14}$$

where $d\zeta_1 d\zeta_2 d\zeta_3$ is the holomorphic top form on $\hat{x}$ which is homogeneous of degree 4 in the homogeneous coordinates. The section f is homogeneous of degree -5 in the homogeneous coordinates. Thus the integrand is homogeneous of degree zero and the integration does not depend on the choice of trivialization on $\hat{x}$. For example

$$\mathcal{P}(\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}) = \frac{1}{(2\pi i)^3} \int_{(S^1)^3} \frac{(1, \zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3}{\zeta_1 \zeta_2 \zeta_3} = (1, 0, 0, 0) \tag{15}$$

is a constant spinor on $\mathcal{U}$. Let us recall at this point that $\mathrm{Spin}(6,\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$ and that the spinor representation $\mathbb{S}_+$ of $\mathrm{Spin}(6,\mathbb{C})$ is isomorphic to the standard representation of $\mathrm{SL}(4,\mathbb{C})$. Thus we can view the right hand side of (15) as a section of $\Gamma(\mathrm{G} \times_{\mathrm{P}} (\mathbb{C}_\nu \otimes \mathbb{S}_+))$ over $\mathcal{U}$. Lemma 8.6.1 in [9] shows that the integral formula (14) really gives monogenic spinors.

6. Decomposition of monogenic sections into irreducible G_0-modules

It is convenient to introduce a grading on the space of all polynomial spinors on $\mathcal{U}$. We trivialize the P-bundle over $\mathcal{U}$ by the map $\mathcal{U} \xrightarrow{\pi^{-1}} \mathrm{G}_- \hookrightarrow \mathrm{G}$. We write coordinates on $\mathfrak{g}_-$ and thus also on $\mathcal{U}$ as $X_1 = (x'_{ij})_{j=1,2}^{i=1,2,3}$, $X_2 = (x_{ij})_{j=1,2}^{i=1,2,3}$, $X_{12} = \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix}$. We will denote polynomials on the affine space $\mathfrak{g}_-$ with the same letters as the coordinates.

Let us first define the *weighted degree* of linear polynomials by setting $\deg_w(x_{12}) := 2$, $\deg_w(x_{ij}) = \deg_w(x'_{ij}) := 1$. Let us extend it to the set of monomials in $\mathbb{C}[x_{12}, x_{ij}, x'_{ij}]$ by requiring that deg_w is a homomorphism of $(\mathbb{C}[x_{12}, x_{ij}, x'_{ij}], \cdot)$ and $(\mathbb{Z}, +)$. Let N_k be the vector space generated by the monomials of the weighted degree k. Finally, let M_k be the space of monogenic spinors whose components belong to N_k. If we extend this gradation naturally also to $\Gamma(\mathcal{V}_2)$ over $\mathcal{U}$, then the operator D_1 is homogeneous of degree -1.

We denote by $\mathrm{W}_{(a,b)}$, resp. $\mathrm{V}_{(a,b,c,d)}$ an irreducible $\mathrm{GL}(2,\mathbb{C})$, resp. $\mathrm{SL}(4,\mathbb{C})$-module with highest weight (a, b), resp. (a, b, c, d). The space of linear monogenic spinors is a G_0-irreducible module isomorphic to $\mathrm{W}_{(\frac{7}{2},\frac{5}{2})} \otimes \mathrm{V}_{(2,1,0,0)}$. As a particular case of Theorem 6.1 we write down the decomposition of the space of quadratic monogenic spinors M_2 which is $M_2 = \mathrm{W}_{(\frac{9}{2},\frac{5}{2})} \otimes \mathrm{V}_{(3,2,0,0)} \oplus \mathrm{W}_{(\frac{7}{2},\frac{7}{2})} \otimes \mathrm{V}_{(3,1,1,0)} \oplus \mathrm{W}_{(\frac{7}{2},\frac{7}{2})} \otimes \mathrm{V}_{(1,0,0,0)}$. The proof of Theorem 6.1 gives a way how to find highest weight vectors of these modules viewed as elements of $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$ which we write as in Section 4.1 as rational functions of coordinates from the left hand side in (8). Let us write the dimension of the modules, the highest weight vectors and the corresponding monogenic spinors on $\mathcal{U}$ obtained by using the formula (14). We have the following table

$$
\begin{aligned}
&\mathrm{W}_{(\frac{9}{2},\frac{5}{2})} \otimes \mathrm{V}_{(3,2,0,0)} : 180, \frac{z_{11}^2}{\zeta_1\zeta_2\zeta_3}, (x_{11}^2, 0, 0, 0)\,,\\
&\mathrm{W}_{(\frac{7}{2},\frac{7}{2})} \otimes \mathrm{V}_{(3,1,1,0)} : 36, \frac{z_{11}z_{22} - z_{12}z_{21}}{\zeta_1\zeta_2\zeta_3}, (x_{11}x_{22} - x_{21}x_{12}, 0, 0, 0)\,,\\
&\mathrm{W}_{(\frac{7}{2},\frac{7}{2})} \otimes \mathrm{V}_{(1,0,0,0)} : 4, \frac{z_0}{\zeta_1\zeta_2\zeta_3} - \frac{z_{22}z_{31} - z_{21}z_{32}}{\zeta_1^2\zeta_2\zeta_3}\\
&\qquad\qquad - \frac{z_{11}z_{32} - z_{12}z_{31}}{\zeta_1\zeta_2^2\zeta_3} - \frac{z_{12}z_{21} - z_{11}z_{22}}{\zeta_1\zeta_2\zeta_3^2}\,,
\end{aligned}
$$

$$
(3x_{12} + \frac{1}{2}\sum_{i=1}^{3}(x'_{i1}x_{i2} - x_{i1}x'_{i2}), x_{21}x_{32} - x_{31}x_{22}, x_{31}x_{12} - x_{11}x_{32}, x_{11}x_{22} - x_{21}x_{12})\,.
$$

In general we have the following theorem.

Theorem 6.1. *Let us keep the notation as above. Then the space M_k of monogenic spinors of weighted degree k on $\mathcal{U}$ decomposes into irreducible* G_0*-modules*

$$
M_k \cong \bigoplus_{a,b,l\geq 0, 2a+b+2l=k} \mathrm{W}_{(\frac{5}{2}+l+a+b, \frac{5}{2}+l+a)} \otimes \mathrm{V}_{(2a+b+1, a+b, a, 0)}\,. \tag{16}
$$

In particular, the decomposition of algebraic monogenic spinors into irreducible G_0*-modules is multiplicity free.*

Proof. Let us recall that the sections are in bijective correspondence with equivariant functions on the total space and that the action of G_0 on $\check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$ is induced by the left action on the total space of the parabolic geometry. Thus Lie algebra $\mathfrak{g}_0$ of G_0 acts by the right invariant vector fields, i.e. let $X \in \mathfrak{g}_0$ and let R_X be the corresponding right invariant vector field then for any equivariant function f

the action is $(Xf)(g) = \frac{d}{dt}\big|_0 f(e^{-tX}g) = -R_X f(g)$. Let us now compute the weight of

$$f = \frac{z_0^{s_0}\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1}\zeta_2^{r_2}\zeta_3^{r_3}} \in C^3(\mathcal{W}, \mathcal{O}_\lambda)\,. \tag{17}$$

Let us denote for $j = 1,2 : c_j := s_{1j} + s_{2j} + s_{3j}$, for $i = 1,2,3 : s_i := s_{i1} + s_{i2}$ and let $r := r_1 + r_2 + r_3$, $s := s_1 + s_2 + s_3$. We write weights as $\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C})$-weights. Then the formulas (2), (5), (8) give that the weight of f is

$$(c_1 + s_0 + \frac{5}{2}, c_2 + s_0 + \frac{5}{2}) \oplus (5 + s - r, r_1 + s_1, r_2 + s_2, r_3 + s_3). \tag{18}$$

Let us denote the R-module structure on $\mathbb{C}_\lambda$ by σ. Let A_{12} be a standard positive root in $\mathfrak{sl}(2,\mathbb{C})$. The action of $\mathfrak{g}_0$ on the space of polynomials on $\mathcal{W}_0$ is determined by

$$\begin{aligned} E_{12}z_0 &= z_{22}z_{31} - z_{21}z_{32},\ E_{12}z_{1j} = -\zeta_2 z_{2j} - \zeta_3 z_{3j},\ E_{12}z_{kj} = z_{kj}\zeta_1\,,\\ E_{12}\zeta_i &= \zeta_i\zeta_1,\ A_{12}z_{i2} = z_{i1},\ E_{23}\zeta_1 = -\zeta_2\,,\\ E_{23}z_{2j} &= z_{1j},\ E_{34}\zeta_2 = -\zeta_3,\ E_{34}z_{3j} = z_{2j},\ E_{21}\zeta_1 = 1\,,\\ E_{32}z_{1j} &= z_{2j},\ E_{32}\zeta_2 = -\zeta_1,\ E_{43}\zeta_3 = -\zeta_2,\ E_{43}z_{2j} = z_{3j}\,, \end{aligned} \tag{19}$$

where $i = 1,2,3$, $j = 1,2$, $k = 2,3$ and all other terms are zero. Since $f \in \mathcal{O}_\lambda(\mathcal{W}_0)$ is a section, we have take into account also the vertical part of the right invariant vector fields corresponding to $E_* \in \mathfrak{g}_0$. We denote the action of the vertical part of R_{E_*} by $\dot\sigma(E_*)$. We find that

$$\dot\sigma(E_{12})f = 5\zeta_1 f,\ \dot\sigma(A_{12})f = \dot\sigma(E_{23})f = \dot\sigma(E_{34})f = 0\,. \tag{20}$$

This and the Leibniz rule allows us to compute the action of $\mathfrak{g}_0$ on $C^3(\mathfrak{W}, \mathcal{O}_\lambda)$. In particular, with respect to A_{12}, E_{23}, E_{34}, f behaves as a usual function while when differentiating with respect to E_{12} there is the additional term $5\zeta_1 f$ appearing. For example

$$E_{12}(z_{11}/\zeta_1) = 5z_{11} - (\zeta_2 z_{21} + \zeta_3 z_{31})/\zeta_1 - z_{11} = 4z_{11} - (\zeta_2 z_{21} + \zeta_3 z_{31})/\zeta_1\,.$$

$\square$

Lemma 6.1. *Let f be a $\mathfrak{q}_0$-highest weight vector in the space of polynomials on the block B_1 in (9), i.e. $f \in \mathbb{C}[z_{ij}]$. Then f is $A(z_{11}z_{22} - z_{21}z_{12})^a z_{11}^b$ for some $A \in \mathbb{C}, a, b = 0,1,2,\dots$*

Proof. See Theorem 5.2.7. on $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$-duality from [6]. $\square$

Lemma 6.2. *Let f be the rational section from* (17) *such that the weight of f is dominant and $s_0 = 0, r_i \geq 1$. Then the class $[E_{12}^{r_1+r_2+r_3-3}E_{23}^{r_2+r_3-2}E_{34}^{r_3-1}f] \in \check{H}^3(\mathfrak{W}, \mathcal{O}_\lambda)$ is non-zero.*

Proof. We have that

$$E_{34}^{r_3-1}f = A\frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1}\zeta_2^{r_2+r_3-1}\zeta_3} + \zeta_3^{-2}(\dots)\,,$$

$$E_{23}^{r_2+r_3-2}E_{34}^{r_3-1}f = B\frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1+r_2+r_3-2}\zeta_2\zeta_3} + \zeta_2^{-2}(\dots) + \zeta_3^{-2}(\dots)\,,$$

$$E_{12}^{r_1+r_2+r_3-3}E_{23}^{r_2+r_3-2}E_{34}^{r_3-1}f = C\frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1\zeta_2\zeta_3} + \zeta_1^{-2}(\dots) + \zeta_2^{-2}(\dots) + \zeta_3^{-2}(\dots)\,,$$

where $\dots$ denotes sections where ζ_i appear only in denominators and

$$A = (-1)^{r_3-1}r_2(r_2+1)\dots(r_2+r_3-2)\,,$$
$$B = (-1)^{r_2+r_3-2}r_1(r_1+1)\dots(r_1+r_2+r_3-3)A\,,$$
$$C = AB(s_2+s_3+5-r)(s_2+s_3+5-(r-1))\dots(s_2+s_3+1)\,.$$

Since the weight of f is by assumption dominant, then $5+s-r \geq r_1+s_1 > 0$ and thus $5+s_2+s_3-r > 0$. It follows that $C \neq 0$ and thus also

$$\mathcal{P}(E_{12}^{r_1+r_2+r_3-3}E_{23}^{r_2+r_3-2}E_{34}^{r_3-1}f) = C\prod_{ij}(x_{ij}^{s_{ij}},0,0,0) + \dots \tag{21}$$

where $\dots$ denotes some spinors whose first components are different from $\prod_{ij} x_{ij}^{s_{ij}}$. In particular we get that the cohomology class is non-zero. $\square$

Lemma 6.3. *Let*

$$f = \sum_{k=1}^{K} g_k,\ \textit{where}\ g_k = \frac{f_k}{\prod_i \zeta_i^{r_i^k}}\,, \tag{22}$$

be a highest weight vector in $\check{H}^3(\mathfrak{W},\mathcal{O}_\lambda)$ *such that all* $f_k \in \mathbb{C}[z_{ij}]$*. Then* $K = r_1^1 = r_2^1 = r_3^1 = 1$ *and* f_1 *is a* $\mathfrak{q}_0$*-maximal polynomial given in Lemma 6.1.*

Proof. Each summand in (22) satisfy the assumptions of Lemma (6.2). Let us notice that from (18) follows that for all $1 \leq j,\, k \leq K$: $r_1^j + r_2^j + r_3^j = r_1^k + r_2^k + r_3^k$ and that $\deg(f_j) = \deg(f_k)$. We can order the summands in (22) in such a way that for all $k > 1$ the following holds: $r_1^1 > r_1^k$ or $r_1^1 = r_1^k$ and $r_2^1 > r_2^k$ or $r_1^1 = r_1^k$ and $r_2^1 = r_2^k$ and $r_3^1 > r_3^k$.

Let us assume that $r_1^1 r_2^1 r_3^3 \geq 2$. Let $E := E_{12}^{r_1^1+r_2^1+r_3^1-3}E_{23}^{r_2^1+r_3^1-2}E_{34}^{r_3^1-1}$. The formula (21) reveals that $\mathcal{P}(E(g_1)) \neq 0$. Similar manipulations give that $\mathcal{P}(E(g_1)) \neq -\mathcal{P}(E(f-g_1)$ and thus $\mathcal{P}(Ef) \neq 0$ and thus f is not a highest weight vector. Thus the only possibility is that $K = r_1^1 = r_2^1 = r_3^1 = 1$ and $f = \frac{f_1}{\zeta_1\zeta_2\zeta_3}$ with f_1 a $\mathfrak{q}_0$-highest weight vector from Lemma 6.2. $\square$

Lemma 6.4. *Let*

$$f = \sum_{i=0}^{s_0} z_0^{s_0-i}f_i = z_0^{s_0}f_0 + z_0^{s_0-1}f_1 + \dots \tag{23}$$

be a maximal highest weight vector such that $f_i \in C^3(\mathfrak{W},\mathcal{O}_\lambda)$ *are rational sections which do not depend on* z_0 *and* $[f_0] \neq 0$*. Then* f_0 *is also a highest weight vector* *and f is uniquely determined by s_0 and f_0. Conversely given a non-zero highest weight vector $f_0 \in C^3(\mathfrak{W}, \mathcal{O}_\lambda)$ that does not depend on the variable z_0 and $s_0 \geq 0$, then there exists a unique highest weight vector f of the form as in (23) for some f_i, $i = 1, \ldots, s_0$.*

Proof. We easily check that if f is highest weight vector then also f_0 is highest weight vector. Thus f_0 is a multiple non-zero of $z_{11}^a(z_{11}z_{22} - z_{12}z_{21})^b\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}$ for some $a, b \geq 0$. Let us check uniqueness of f given f_0 and s_0. Let $\hat{f}, \tilde{f}$ be two highest weight vectors of the same weight such that $\hat{f} = z_0^{s_0}f_0 + z_0^{s_0-1}(\ldots)$ and $\tilde{f} = z_0^{s_0}f_0 + z_0^{s_0-1}(\ldots)$. Then $\check{f} := \hat{f} - \tilde{f} = z_0^{t_0}\check{f}_0 + z_0^{t_0-1}\check{f}_1 + \ldots$ with $t_0 < s_0$ has to be a highest weight vector with the same weight as f and f'. Thus $\check{f}_0$ is a multiple of $z_{11}^c(z_{11}z_{22} - z_{12}z_{21})^d\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1}$ for some $c, d \geq 0$. But the formula (18) shows that then $a = c$, $b = d$ and thus also $t_0 = s_0$. Contradiction.

Let us consider a filtration $\{F_i \mid i \geq 0\}$ of $C^3(\mathfrak{W}, \mathcal{O}_\lambda)$ given by the degree of z_0, i.e. $F_i := \{g \in C^3(\mathfrak{W}, \mathcal{O}_\lambda) \mid \partial_{12}^{i+1}g = 0\}$ where ∂_{12} is the coordinate vector field corresponding to the variable z_0. From table (19), we conclude that the action of G_0 preserves this filtration. Let $f_{\mathrm{top}} = z_0^{s_0}f_0$ be the highest part of f. Let $V = \mathrm{G}_0 \cdot f_{\mathrm{top}} = \{\sum_{j=1}^M g_j \cdot f_{\mathrm{top}} \mid g_j \in \mathrm{G}_0, M < \infty\}$. Then clearly V is the smallest G_0-module which contains the vector f_{top}. Moreover $V \subset F_{s_0}$ and from the table (19) follows that V/F_{s_0-1} is spanned by f_{top}. We have that $V = \oplus_i V_i$ for some irreducible G_0-modules V_i. Let h_i be a maximal vector of V_i. Since the filtration F_i is G_0-equivariant, there exists i such that $h_i = f_{top} + l.o.t.$ where $l.o.t.$ means lower order terms in z_0-variables. From the uniqueness we have that h_i is up to a multiple the unique highest weight vector with the leading term $z_0^{s_0}f_0$. $\square$

Thus we have that any highest weight vector is uniquely determined by its leading term f_{top} with respect to the variable z_0. If $f_{\mathrm{top}} = z_0^l(z_{11}z_{22} - z_{12}z_{21})^a z_{11}^b$, then f is highest weight vector of the module $W_{(\frac{5}{2}+a+b+s_0, \frac{5}{2}+s_0+b)} \otimes V_{(2a+b+1, a+b, a, 0)}$.

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