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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF e-KÄHLER MANIFOLDS

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of e-Kähler spaces (i.e. classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical, pseudo- and hyperbolic Kähler manifolds (shortly *e-Kähler*) in dependence on the smoothness class of the metric. We present well known facts, which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc., see [2, 3, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

2. Kähler manifolds

In the following definition we introduce generalizations of Kähler manifolds.

Definition 1. An n-dimensional (pseudo-)Riemannian manifold (M, g) is called an *e-Kähler manifold* K_n, if beside the metric tensor g, a tensor field F ($\neq \mathrm{Id}$) of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold:

$$F^2 = e\,\mathrm{Id}\,; \quad g(X, FX) = 0\,; \quad \nabla F = 0\,, \tag{1}$$

where $e = \pm 1$, X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in K_n.

If $e = -1$, K_n is a (*pseudo-*)*Kähler space* (also *elliptic Kähler space*) and F is a *complex structure*. As *A-spaces*, these spaces were first considered by P. A. Shirokov, see [14]. Independently they were studied by E. Kähler [5].

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If $e = +1$, K_n is a *hyperbolic Kähler space* (also *para Kähler space*, see [1]) and F is a *product structure*. The spaces K_n^+ were considered by P. K. Rashevskij [13].

The e-Kähler spaces introduced here are called shortly "Kähler" in the literature [10, 16]. By our definition we want to give a unified notation for all clases.

3. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^1

Assume the e-Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here K_n, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Likewise, as in [11] we introduce the following notations.

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$, $(\neq 0)$, $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \mathrm{Span}\,\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda\,,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

Definition 3. A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any analytically planar curve in K_n onto an analytically planar curve in $\bar{K}_n$.

Assume a holomorphically projective mapping $f\colon K_n \to \bar{K}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f\colon K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [10, 16]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + e\psi(FX)FY + e\psi(FY)FX \tag{2}$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective.* Beside these facts it was proved [10, 16] that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$. In local form:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i + e\psi_{\bar{i}}\delta^h_{\bar{j}} + e\psi_{\bar{j}}\delta^h_{\bar{i}}\,,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F^h_i are components of ψ, F and δ^h_i is the Kronecker delta, $\psi_{\bar{i}} = \psi_\alpha F^\alpha_i$, $\delta^h_{\bar{i}} = F^h_i$.

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots\,\bar{i}\,\dots} = A_{\dots\,k\,\dots}F^k_i\,.$$

Equations (2) are equivalent to the following equations

$$\text{(3)}\qquad \begin{aligned}\nabla_Z\bar g(X,Y) &= 2\psi(Z)\bar g(X,Y)+\psi(X)\bar g(Y,Z)+\psi(Y)\bar g(X,Z)\\ &\quad - e\psi(FX)\bar g(FY,Z)-e\psi(FY)\bar g(FX,Z)\,.\end{aligned}$$

In local form:

$$\bar g_{ij,k}=2\psi_k\bar g_{ij}+\psi_i\bar g_{jk}+\psi\bar g_{ik}-e\psi_{\bar i}\bar g_{\bar j k}-e\psi_{\bar j}\bar g_{\bar i k}\,,$$

where " , " denotes the covariant derivative on K_n. It is known that

$$\psi_i=\partial_i\Psi,\quad \Psi=\frac{1}{2(n+2)}\ln\left|\frac{\det\bar g}{\det g}\right|,\quad \partial_i=\partial/\partial x^i\,.$$

Domashev, Kurbatova and Mikeš [3, 6, 16] proved that equations (2) and (3) are equivalent to

$$\text{(4)}\qquad \begin{aligned}\nabla_Z a(X,Y) &= \lambda(X)g(Y,Z)+\lambda(Y)g(X,Z)\\ &\quad - e\lambda(FX)g(FY,Z)-e\lambda(FY)g(FX,Z)\,.\end{aligned}$$

In local form:

$$a_{ij,k}=\lambda_i g_{jk}+\lambda_j g_{ik}-e\lambda_{\bar i}g_{\bar j k}-e\lambda_{\bar j}g_{\bar i k}\,,$$

where

$$\text{(5)}\qquad \text{(a)}\ \ a_{ij}=\mathrm{e}^{2\Psi}\bar g^{\alpha\beta}g_{\alpha i}g_{\beta j};\qquad \text{(b)}\ \ \lambda_i=-\,\mathrm{e}^{2\Psi}\bar g^{\alpha\beta}g_{\beta i}\psi_\alpha\,.$$

From (4) follows $\lambda_i=\partial_i\lambda=\partial_i(\frac14\,a_{\alpha\beta}g^{\alpha\beta})$. On the other hand [10]:

$$\text{(6)}\qquad \bar g_{ij}=\mathrm{e}^{2\Psi}\tilde g_{ij},\quad \Psi=\frac12\ln\left|\frac{\det\tilde g}{\det g}\right|,\qquad \|\tilde g_{ij}\|=\|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}\,.$$

The above formulas are the criterion for holomorphically projective mappings $K_n\to\bar K_n$, globally as well as locally.

4. Holomorphically projective mapping theory for $K_n\to\bar K_n$ of class C^2

Let K_n and $\bar K_n\in C^2$ be e-Kähler manifolds, then for holomorphically projective mappings $K_n\to\bar K_n$ the Riemann and the Ricci tensors transform in this way

$$\text{(7)}\qquad \begin{aligned}&\text{(a)}\quad \bar R^h_{ijk}=R^h_{ijk}+\delta^h_k\psi_{ij}-\delta^h_j\psi_{ik}-e\delta^h_{\bar k}\psi_{i\bar j}+e\delta^h_{\bar j}\psi_{i\bar k}+2e\delta^h_{\bar i}\psi_{j\bar k}\,;\\ &\text{(b)}\quad \bar R_{ij}=R_{ij}-(n+2)\psi_{ij}\,,\end{aligned}$$

where $\psi_{ij}=\psi_{i,j}-\psi_i\psi_j+\psi_{\bar i}\psi_{\bar j}$ $(\psi_{ij}=\psi_{ji}=-e\psi_{\bar i\bar j})$.

The tensor of holomorphically projective curvature, which is defined in the following form

$$\text{(8)}\qquad P^h_{ijk}=R^h_{ijk}+\frac{1}{n+2}\big(\delta^h_kR_{ij}-\delta^h_jR_{ik}-e\delta^h_{\bar k}R_{i\bar j}+e\delta^h_{\bar j}R_{i\bar k}+2e\delta^h_{\bar i}R_{j\bar k}\big)\,,$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar P^h_{ijk}=P^h_{ijk}$.

The integrability conditions of equations (4) have the following form

$$a_{i\alpha}R^{\alpha}_{jkl} + a_{j\alpha}R^{\alpha}_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k} \tag{9}$$
$$- eg_{\bar{i}k}\lambda_{\bar{j},l} - eg_{\bar{j}k}\lambda_{\bar{i},l} + eg_{\bar{i}l}\lambda_{\bar{j},k} + eg_{\bar{j}l}\lambda_{\bar{i},k}\,.$$

We make the remark that the formulas introduced above, (7), (8) and (9), are not valid when $K_n \notin C^2$ or $\bar{K}_n \notin C^2$.

After contraction with g^{jk} we get:

$$a_{i\alpha}R^{\alpha}_{k} + a_{\alpha\beta}R^{\alpha\ \beta}_{ik} = e\lambda_{\bar{i},\bar{k}} - (n-1)\lambda_{i,k},$$

where $R^{\alpha}{}_{il}{}^{\beta} = g^{\beta k}R^{\alpha}{}_{ilk}$; $R^{\alpha}_{l} = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.

We contract this formula with $F^{i}_{i'}F^{k}_{k'}$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

$$\lambda_{\bar{i},\bar{k}} = -e\lambda_{i,k}\,, \tag{10}$$

and ([3, 9, 10, 15])

$$n\lambda_{i,k} = \mu g_{ik} + a_{i\alpha}R^{\alpha}_{k} + a_{\alpha\beta}R^{\alpha}{}_{ik}{}^{\beta}\,. \tag{11}$$

Because λ_i is a gradient-like covector, from equation (11) follows $a_{i\alpha}R^{\alpha}_{j} = a_{j\alpha}R^{\alpha}_{i}$.

From (10) follows that the vector field $\lambda_{\bar{i}}$ ($\equiv \lambda_{\alpha}F^{\alpha}_{i}$) is a Killing vector field, i.e. $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$.

5. Holomorphically projective mappings between $K_n \in C^r$ $(r > 2)$ and $\bar{K}_n \in C^2$

We proof the following theorem

Theorem 1. *If $K_n \in C^r$ $(r > 2)$ admits holomorphically projective mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

The proof of this theorem follows from the following lemmas.

Lemma 1 (see [4])**.** *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If*

$$\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1 \tag{12}$$

then $\lambda^h \in C^2$ and $\varrho \in C^1$.

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i\lambda^h - \varrho\delta^h_i \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 2. *If $K_n \in C^3$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^3$.*

Proof. In this case equations (4) and (11) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^h_{ijk}, R^h{}_{ij}{}^k, R_{ij}, R^h_i \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (11) in the form (12), where

$$\lambda^h = g^{h\alpha}\lambda_{\alpha} \in C^1,\ \varrho = \mu/n \text{ and } f^h_i = \frac{1}{n}\,(-\lambda^{\alpha}\Gamma^h_{\alpha i} - g^{h\gamma}a_{\alpha\gamma}R^{\alpha}_{i} + g^{h\gamma}a_{\alpha\beta}R^{\alpha}{}_{i\gamma}{}^{\beta}) \in C^1\,.$$

From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (6) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for holomorphically projective mappings between e-Kähler manifolds K_n and $\bar{K}_n$ of class C^3 holds the following third set of equations [6, 8, 9, 15, 10, 16]:

$$\mu_{,k} = 2\lambda_\alpha R^\alpha_k \,. \tag{13}$$

If $K_n \in C^r$ and $\bar{K}_n \in C^2$, then by Lemma 2, $\bar{K}_n \in C^3$ and (13) holds. Because the system (4), (11) and (13) is closed, we can differentiate equations (4) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ ($\equiv \bar{K}_n \in C^r$).

Remark. Moreover, in this case from equation (13) follows that the function $\mu \in C^{r-1}$.

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