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NOETHER'S THEOREM FOR A FIXED REGION

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Abstract. We give an elementary proof of Noether's first Theorem while stressing the magical fact that the global quasi-symmetry only needs to hold for one fixed integration region. We provide sufficient conditions for gauging a global quasi-symmetry.

1. Introduction

We shall assume that the reader is familiar with Noether's Theorem in its most basic formulation. For a general introduction to the subject and for references, see *e.g.*, Goldstein's book [6] and the Wikipedia entry for Noether's Theorem [17]. The purpose of this paper is to state and prove Noether's Theorem in a powerful field-theoretic setting with a minimum of assumptions. At the same time, we aim at being self-contained and using as little mathematical machinery as practically possible.

Put into one sentence, the first Theorem of Noether states that a continuous, global, off-shell quasi-symmetry of an action S implies a local on-shell conservation law, *i.e.*, a continuity equation for a Noether current, which is valid in each world-volume point. Strictly speaking, Noether herself [11] and the majority of authors talk about *symmetry/invariance* rather than *quasi-symmetry/quasi-invariance*, but since quasi-symmetry is a very useful, natural and relatively mild generalization, we shall only use the notion of quasi-symmetry here, cf. Section 9. The term *global* is defined in Section 7.

The traditional treatment of Noether's first Theorem assumes that the global quasi-symmetry of the action S holds for *every* integration region, see *e.g.*, Noether [11], Hill [7], Goldstein [6], Bogoliubov and Shirkov [4], Trautman [16], Komorowski [10], Ibragimov [8], Sarlet and Cantrijn [15], Olver [12], and Ramond [14]. In the case of Olver [12], this assumption is hidden inside his definition of symmetry. Adding to the confusion, Goldstein [6] and Ramond [14] do never explicitly state that they require the quasi-symmetry of the action S to hold for *every* integration region, but this is the only interpretation that is consistent with their further conclusions, technically speaking, because their Noether identity contains only the bare (rather than the improved) Noether current.

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There is also a non-integral version of Noether's Theorem based on a quasi-symmetry of the Lagrangian density $\mathcal{L}(x)$ (or the Lagrangian form $\mathcal{L}(x)\mathrm{d}^d x$) rather than the action S, see *e.g.*, Arnold [1], or José and Saletan [9]. We shall here only discuss integral formulations.

TAB. 1: Flow-diagram of Noether's first Theorem. The $J^\mu(x)$ in Table 1 is an (improved) Noether current, cf. Section 9, and Y_0^α is a vertical generator of quasi-symmetry, see Section 5. The term *on-shell* and the wavy equality sign "$\approx$" means that the equations of motion $\delta\mathcal{L}(x)/\delta\phi^\alpha(x) \approx 0$ has been used.

$$\begin{array}{c}
\text{Continuous global off-shell quasi-symmetry of} \\
S_{\mathcal{V}} = \int_{\mathcal{V}} \mathrm{d}^d x \ \mathcal{L}(x) \text{ for a fixed region } \mathcal{V}. \\
\Downarrow \\
\text{Continuous global off-shell quasi-symmetry of} \\
S_{\mathcal{U}} = \int_{\mathcal{U}} \mathrm{d}^d x \ \mathcal{L}(x) \text{ for every region } \mathcal{U} \subseteq \mathcal{V}. \\
\Downarrow \\
\text{Local off-shell Noether identity:} \\
\forall \phi : \quad d_\mu J^\mu(x) \equiv -\frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} Y_0^\alpha(x). \\
\Downarrow \\
\text{Local on-shell conservation law:} \\
d_\mu J^\mu(x) \approx 0.
\end{array}$$

If the action S has quasi-symmetry for *every* integration region, it is, in retrospect, not surprising that one can derive a *local* conservation law for a Noether current via localization techniques, *i.e.*, by chopping the integral S into smaller and smaller neighborhoods around a single world-volume point. It would be much more amazing, if one could derive a *local* conservation law from only the knowledge that the action S has a quasi-symmetry for *one fixed* integration region. Our main goal with this paper is to communicate to a wider audience that this is possible! More precisely, the statement is, firstly, that the global quasi-symmetry of the action S only needs to hold for *one fixed* region of the world volume, namely the pertinent full world volume $\mathcal{V}$, and secondly, that this will, in turn, imply a global quasi-symmetry for *every* smaller region $\mathcal{U} \subseteq \mathcal{V}$. (We assume that the target space M is contractible, cf. Section 2, and that the quasi-symmetry is projectable, cf. Section 5.) It is for aesthetic and practical reasons nice to minimize the assumptions, and when formulated with a fixed region, the conclusions in Noether's first Theorem are mesmerizingly strong, cf. Table 1. The crucial input is the strong assumption that the quasi-symmetry of S should be valid *off-shell*, *i.e.*, for *every* possible configurations of the field ϕ; not just for configurations that satisfy equations of motion. To our knowledge, a proof of these facts has not been properly written down anywhere in the literature in elementary terms, although the key idea is outlined by, *e.g.*, Polchinski [13]. (See also de Wit and Smith [5].)

The paper is organized as follows. The main proof and definitions are given in Sections 2–9, while Section 10 and Appendix A provide some technical details. Sections 11–13 contain examples from classical mechanics of a global, off-shell, symmetry with respect to one fixed region that is *not* a symmetry for generic regions. Finally, Appendix B provides closed formulas and sufficient conditions for gauging a global quasi-symmetry.

2. World volume and target space

Consider a field $\phi : \mathcal{V} \to M$ from a fixed d-dimensional world volume $\mathcal{V}$ to a target space M. (We use the term *world volume* rather than the more conventional term *space-time*, because space-time in, *e.g.*, string theory is associated with the target space.) We will first consider the special case where $\mathcal{V} \subseteq \mathbb{R}^d$, and postpone the general case where $\mathcal{V}$ is a general manifold to Section 10. Here $\mathbb{R}$ denotes the set of real numbers. We will always assume for simplicity that the target space M has global coordinates y^α, so that one can describe the field ϕ with its coordinate functions $y^\alpha = \phi^\alpha(x)$, $x \in \mathcal{V}$. We furthermore assume that the y^α-coordinate region (which we identify with the target space M) is *star-shaped* around a point (which we take to be the origin $y=0$), *i.e.*,

$$\forall y \in M \forall \lambda \in [0,1] : \quad \lambda y \in M \ . \tag{2.1}$$

The world volume $\mathcal{V}$ and the target space M are also called the *horizontal* and the *vertical space*, respectively.

3. Action $S_\mathcal{V}$

The *action* $S_\mathcal{V}$ is given as a local functional

$$S_\mathcal{V}[\phi] \ := \ \int_\mathcal{V} \mathrm{d}^d x \ \mathcal{L}(x) \tag{3.1}$$

over the world volume $\mathcal{V}$, where the *Lagrangian density*

$$\mathcal{L}(x) \ = \ \mathcal{L}(\phi(x), \partial\phi(x), x) \tag{3.2}$$

depends smoothly on the fields $\phi^\alpha(x)$, their first derivatives $\partial_\mu \phi^\alpha(x)$, and explicitly on the point x. Phrased mathematically, the Lagrangian density $\mathcal{L} \in C^\infty(M \times M^d \times \mathcal{V})$ is assumed to be a smooth function on the 1-jet space. Please note that the ϕ and the $\partial\phi$ dependence will often not be written explicitly, cf., *e.g.*, the right–hand side of eq. (3.1). Since we do not want to impose boundary conditions on the field $\phi(x)$ (at least not at this stage), the notion of functional/variational derivative $\delta S_\mathcal{V}/\delta\phi(x)$ may be ill-defined, see *e.g.*, Ref. [3]. In contrast, the Euler-Lagrange derivative $\delta\mathcal{L}(x)/\delta\phi(x)$ is always well-defined, cf. eq. (6.5), even if the principle of stationary/least action has an incomplete formulation (at this stage). So when we speak of *equations of motion* and *on-shell*, we mean the equations $\delta\mathcal{L}(x)/\delta\phi(x) \approx 0$. (We should finally mention that Noether's Theorem also holds if the Lagrangian density $\mathcal{L}$ contains higher derivatives $\partial^2\phi$, $\partial^3\phi, \ldots, \partial^n\phi$, of the field ϕ, and/or if the world volume $\mathcal{V}$ and/or if the target space M are supermanifolds, but we shall for simplicity not consider this here.)

We will consider three cases of the fixed world volume $\mathcal{V}$.

(1) Case $\mathcal{V} = \mathbb{R}^d$: The reader who does not care about subtleties concerning boundary terms can assume $\mathcal{V}=\mathbb{R}^d$ from now on (and ignore hats "$\wedge$" on some symbols below).

(2) Case $\mathcal{V} \subset \mathbb{R}^d$: For notational reasons it is convenient to assume that the original Lagrangian density $\mathcal{L} \in C^\infty(M \times M^d \times \mathcal{V})$ in eq. (3.1) and every admissible field configuration $\phi : \mathcal{V} \to M$ can be smoothly extended to some function $\mathcal{L} \in C^\infty(M \times M^d \times \mathbb{R}^d)$ and to functions $\phi : \mathbb{R}^d \to M$, which, with a slight abuse of notation, are called by the same names, respectively. The construction will actually not depend on which such smooth extensions are used, as will become evident shortly. Then it is possible to write the action (3.1) as an integral over the whole $\mathbb{R}^d$.

$$S_{\mathcal{V}}[\phi] \;=\; \int_{\mathbb{R}^d} \mathrm{d}^d x \; \hat{\mathcal{L}}(x) \;, \qquad \hat{\mathcal{L}}(x) \;:=\; 1_{\mathcal{V}}(x)\mathcal{L}(x) \;, \tag{3.3}$$

where

$$1_{\mathcal{V}}(x) \;:=\begin{cases} 1 & \text{for} \quad x \in \mathcal{V} \;, \\ 0 & \text{for} \quad x \in \mathbb{R}^d \backslash \mathcal{V} \;, \end{cases} \tag{3.4}$$

is the *characteristic function* for the region $\mathcal{V}$ in $\mathbb{R}^d$. Note that $1_{\mathcal{V}} : \mathbb{R}^d \to \mathbb{R}$ and $\hat{\mathcal{L}} : M \times M^d \times \mathbb{R}^d \to \mathbb{R}$ are *not* continuous functions. It is necessary to impose a regularity condition for the boundary $\partial\mathcal{V}$ of the region $\mathcal{V}$. Technically, the boundary $\partial\mathcal{V} \subset \mathbb{R}^d$ should have Lebesgue measure zero.

(3) Case $\mathcal{V}$ is a general manifold: See Section 10.

4. Total derivative d_μ

The *total derivative* d_μ is an explicit derivative ∂_μ plus implicit differentiation through ϕ, $\partial\phi^\alpha$, …, *i.e.*,

$$d_\mu \;=\; \partial_\mu + \phi^\alpha_\mu(x) \frac{\partial}{\partial \phi^\alpha(x)} + \phi^\alpha_{\mu\nu}(x) \frac{\partial}{\partial \phi^\alpha_\nu(x)} + \ldots \;, \tag{4.1}$$

where the following shorthand notation is used

$$\begin{aligned} d_\mu &:= \frac{d}{dx^\mu} \;, & \partial_\mu &:= \frac{\partial}{\partial x^\mu} \;, \\ \phi^\alpha_\mu(x) &:= \partial_\mu \phi^\alpha(x) \;, & \phi^\alpha_{\mu\nu}(x) &:= \partial_\mu \partial_\nu \phi^\alpha(x) \;, \qquad \ldots \;. \end{aligned} \tag{4.2}$$

5. Variation of x, ϕ and $\mathcal{V}$

We will assume that the reader is familiar with the notion of infinitesimal variations in a field-theoretic context. See *e.g.*, Goldstein [6], cf. Table 2. Consider an infinitesimal variation δ of the coordinates $x^\mu \to x'^\mu$, of the fields $\phi^\alpha(x) \to \phi'^\alpha(x')$,

TAB. 2: Conversion between notation used by various authors.

	Noether [11]	Hill [7]	Goldstein [6]	Bogoliubov & Shirkov [4]	Ramond [14]	This paper
Action	I	J	I	$\mathcal{A}$	S	S
Lagrangian density	f	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
Field	u_i	ψ^α	η_ρ	u_i	Φ	ϕ^α
Region		R	Ω			$\mathcal{V}$
Infinitesimal variation	Δ, δ	δ	δ	δ	δ	δ
Vertical variation	$\overline{\delta}$	δ_*	$\overline{\delta}$	$\overline{\delta}$	δ_0	δ_0
Generator		η^α	Ψ_ρ	Ψ_i		Y^α
Euler-Lagrange deriv.	ψ_i	$[\mathcal{L}]_\alpha$				$\frac{\delta \mathcal{L}(x)}{\delta \phi^\alpha(x)}$
Bare Noether current	$-B$			$-\theta^i$	$-\jmath^\mu$	$\jmath^\mu$

and of the region $\mathcal{V} \to \mathcal{V}' := \{x' \mid x \in \mathcal{V}\}$, *i.e.*,
(5.1)
$$\begin{array}{rclcl} x'^\mu - x^\mu &=:& \delta x^\mu &=& X^\mu(x)\varepsilon(x)\ , \\ \phi'^\alpha(x') - \phi^\alpha(x) &=:& \delta\phi^\alpha(x) &=& Y^\alpha(x)\varepsilon(x)\ , \\ \phi'^\alpha(x) - \phi^\alpha(x) &=:& \delta_0\phi^\alpha(x) &=& Y_0^\alpha(x)\varepsilon(x)\ , \\ d'_\mu\phi'^\alpha(x') - d_\mu\phi^\alpha(x) &=:& \delta d_\mu\phi^\alpha(x) &\neq& d_\mu\delta\phi^\alpha(x)\ , \\ d_\mu\phi'^\alpha(x) - d_\mu\phi^\alpha(x) &=:& \delta_0 d_\mu\phi^\alpha(x) &=& d_\mu\delta_0\phi^\alpha(x)\ , \end{array}$$
$$X^\mu(x) \text{ and } \varepsilon(x) \text{ are independent of } \phi \text{ (also known as \textit{projectable} [12])}\ ,$$
$$Y^\alpha(x) \;=\; Y^\alpha(\phi(x), \partial\phi(x), x)\ , \qquad Y_0^\alpha(x) \;=\; Y_0^\alpha(\phi(x), \partial\phi(x), x)\ ,$$

where $\varepsilon\colon \mathcal{V} \to \mathbb{R}$ is an arbitrary infinitesimal function, and where $X^\mu, Y^\alpha, Y_0^\alpha \in C^\infty(M \times M^d \times \mathcal{V})$ are *generators* of the variation, and in differential-geometric terms, they are *vector fields*.

(While working with infinitesimal quantities has intuitive advantages, it requires a comment to make them mathematically well-defined. The ε-function should more correctly be viewed as a product $\varepsilon(x) = \varepsilon_0 h(x)$, where ε_0 is the underlying 1-parameter of the variation, and $h(x)$ is a function. A 1-*parameter* means a 1-dimensional parameter. Then, for instance, the first equation in (5.1) should more properly be written $x'^\mu - x^\mu = \varepsilon(x) X^\mu(x) + \varepsilon_0 o(1)$, where the little-o notation $o(1)$ means any function of ε_0 that vanishes in the limit $\varepsilon_0 \to 0$. We shall not write such $o(1)$ terms explicitly to avoid clutter. The term $\varepsilon_0 o(1)$ is also written as $o(\varepsilon_0)$ in the little-o notation. An alternative method is to view ε_0 as an exterior 1-form, so that the square $\varepsilon_0 \wedge \varepsilon_0 = 0$ vanishes.)

In the case $\mathcal{V}\subset\mathbb{R}^d$, the above functions are for notational reasons assumed to be smoothly extended to $\varepsilon\colon\mathbb{R}^d\to\mathbb{R}$ and $X^\mu, Y^\alpha, Y_0^\alpha \in C^\infty(M\times M^d\times\mathbb{R}^d)$, which, with a slight abuse of notation, are called by the same names, respectively. (Again the choice of extensions will not matter.) The generator $Y^\alpha(x)$ can be decomposed in a vertical and a horizontal piece,

$$\delta \;=\; \delta_0 + \delta x^\mu d_\mu \;, \qquad Y^\alpha(x) \;=\; Y_0^\alpha(x) + \phi_\mu^\alpha(x) X^\mu(x) \;. \tag{5.2}$$

In other words, only the vertical and horizontal generators, Y_0^α and X^μ, respectively, are independent generators of the variation δ. The variation $\delta\mathcal{V}$ of the region $\mathcal{V}$ is by definition completely specified by the horizontal part X^μ. The main property of the vertical variation δ_0 that we need in the following, is that it commutes ($[\delta_0, d_\mu]=0$) with the total derivative d_μ. This should be compared with the fact that in general $[\delta, d_\mu]\neq 0$.

(In the case of Noether's second Theorem and local gauge symmetry, the generators $X^\mu, Y^\alpha, Y_0^\alpha$ in eq. (5.1) could in general be differential operators that act on $\varepsilon(x)$, but since we are here only interested in Noether's first theorem, and ultimately letting $\varepsilon(x)$ be an x-independent constant ε_0, cf. eq. (7.1), such differential operators will not contribute, so we will here for simplicity assume that the generators $X^\mu, Y^\alpha, Y_0^\alpha$ are just functions.)

6. Variation of $S_{\mathcal{V}}$

The infinitesimal variation $\delta S_{\mathcal{V}}$ of the action $S_{\mathcal{V}}$ comes in general from four types of contributions:

– Variation of the Lagrangian density $\mathcal{L}(x)$.

$$\delta\mathcal{L}(x) \;=\; \mathcal{L}(\phi'(x'), \partial'\phi'(x'), x') - \mathcal{L}(\phi(x), \partial\phi(x), x) \;. \tag{6.1}$$

– Variation of the measure $\mathrm{d}^d x$, which leads to a Jacobian factor.

$$\delta \mathrm{d}^d x \;=\; \mathrm{d}^d x' - \mathrm{d}^d x \;=\; \mathrm{d}^d x \; d_\mu \delta x^\mu \;. \tag{6.2}$$

– Boundary terms at $|x|=\infty$. In the way we have set up the action (3.3) on the whole $\mathbb{R}^d$ there are no boundary contributions at $|x|=\infty$ in both case 1 and 2.
– Variation of the characteristic function $1_{\mathcal{V}}(x)$. The characteristic function $1_{\mathcal{V}}(x)$ is invariant under the variation, due to a compensating variation $\delta\mathcal{V}$ of the region $\mathcal{V}$.

$$\delta 1_{\mathcal{V}}(x) \;=\; 1_{\mathcal{V}'}(x') - 1_{\mathcal{V}}(x) \;=\; 0 \;. \tag{6.3}$$

An arbitrary infinitesimal variation $\delta S_{\mathcal{V}}$ of the action $S_{\mathcal{V}}$ therefore consists of the two first contributions.

$$\begin{aligned}
\delta S_{\mathcal{V}} \;&=\; \int_{\mathcal{V}'} \mathrm{d}^d x' \; \mathcal{L}(\phi'(x'), \partial'\phi'(x'), x') - \int_{\mathcal{V}} \mathrm{d}^d x \; \mathcal{L}(\phi(x), \partial\phi(x), x) \\
&=\; \int_{\mathcal{V}} \mathrm{d}^d x \, [\delta\mathcal{L}(x) + \mathcal{L}(x) d_\mu \delta x^\mu] \;=\; \int_{\mathcal{V}} \mathrm{d}^d x \, [\delta_0 \mathcal{L}(x) + d_\mu(\mathcal{L}(x)\delta x^\mu)] \\
&=\; \int_{\mathcal{V}} \mathrm{d}^d x \left[\frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} \delta_0\phi^\alpha(x) + d_\mu \left(\frac{\partial\mathcal{L}(x)}{\partial\phi_\mu^\alpha(x)} \delta_0\phi^\alpha(x) + \mathcal{L}(x)\delta x^\mu \right) \right]
\end{aligned}$$

$$(6.4) \qquad = \int_{\mathcal{V}} \mathrm{d}^d x \, [f(x)\varepsilon(x) + \jmath^\mu(x) d_\mu \varepsilon(x)] \ .$$

Here $\delta\mathcal{L}(x)/\delta\phi^\alpha(x)$ is the Euler-Lagrange derivative

$$(6.5) \qquad \frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} \ := \ \frac{\partial\mathcal{L}(x)}{\partial\phi^\alpha(x)} - d_\mu \frac{\partial\mathcal{L}(x)}{\partial\phi^\alpha_\mu(x)} \ = \ \mathrm{function}(\phi(x), \partial\phi(x), \partial^2\phi(x), x) \ ,$$

i.e., the equations of motion are of at most second order. In equation (6.4) we have defined the *bare Noether current* as

$$(6.6) \qquad \jmath^\mu(x) \ := \ \frac{\partial\mathcal{L}(x)}{\partial\phi^\alpha_\mu(x)} Y^\alpha_0(x) + \mathcal{L}(x) X^\mu(x) \ = \ \jmath^\mu(\phi(x), \partial\phi(x), x) \ ,$$

and a function

$$(6.7) \qquad f(x) \ := \ \frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} Y^\alpha_0(x) + d_\mu \jmath^\mu(x) \ = \ f(\phi(x), \partial\phi(x), \partial^2\phi(x), x) \ .$$

In differential-geometric terms,

$$(6.8) \quad \jmath^\mu(x) \ \to \ \overline{\jmath}^\nu(\overline{x}) \ = \ \frac{\jmath^\mu(x)}{\det \frac{\partial\overline{x}}{\partial x}} \frac{\partial\overline{x}^\nu}{\partial x^\mu} \qquad \text{and} \qquad f(x) \ \to \ \overline{f}(\overline{x}) \ = \ \frac{f(x)}{\det \frac{\partial\overline{x}}{\partial x}}$$

behave as a density-valued vector-field and a density under passive coordinate transformations $x^\mu \to \overline{x}^\nu = \overline{x}^\nu(x)$, respectively.

7. Global variation

The variation (5.1) is by definition called *global* (or *rigid*) if

$$(7.1) \qquad \varepsilon(x) \ = \ \varepsilon_0$$

is an x-independent infinitesimal 1-parameter. Except for Appendix B, let us from now on specialize the variation (5.1) to the global type (7.1). Then eq. (6.4) becomes

$$(7.2) \qquad \delta S_{\mathcal{V}} \ = \ \varepsilon_0 F_{\mathcal{V}} \ , \qquad F_{\mathcal{V}}[\phi] \ := \ \int_{\mathcal{V}} \mathrm{d}^d x \ f(x) \ .$$

8. Smaller regions $\mathcal{U} \subseteq \mathcal{V}$

Note that $\jmath^\mu(x)$ and $f(x)$, from eqs. (6.6) and (6.7), respectively, are both independent of the region $\mathcal{V}$ in the sense that if one had built the action

$$(8.1) \qquad S_{\mathcal{U}}[\phi] \ := \ \int_{\mathcal{U}} \mathrm{d}^d x \ \mathcal{L}(x)$$

from a smaller region $\mathcal{U} \subseteq \mathcal{V}$, and smoothly extended the pertinent functions to $\mathbb{R}^d$ as in eq. (3.3), one would have arrived at another set of functions $\jmath^\mu(x)$ and $f(x)$, that would agree with the previous ones within the smaller region $x \in \mathcal{U}$. Similar to eq. (7.2), the corresponding global variation $\delta S_{\mathcal{U}}$ is just

$$(8.2) \qquad \delta S_{\mathcal{U}} \ = \ \varepsilon_0 F_{\mathcal{U}} \ , \qquad F_{\mathcal{U}}[\phi] \ = \ \int_{\mathcal{U}} \mathrm{d}^d x \ f(x) \ , \qquad \mathcal{U} \subseteq \mathcal{V} \ .$$

9. Quasi-symmetry

We will in the following use again and again the fact that an integral is a boundary integral if and only if its Euler-Lagrange derivative vanishes, cf. Appendix A. Assume that for a fixed region $\mathcal{V}$, the action $S_{\mathcal{V}}$ has an off-shell quasi-symmetry under a global variation (5.1, 7.1). By definition, a global off-shell *quasi-symmetry* means that the corresponding infinitesimal variation $\delta S_{\mathcal{V}}$ of the action is an integral over a smooth function $g(x) = g(\phi(x), \partial\phi(x), \partial^2\phi(x), \ldots, x)$, *i.e.*,

$$\forall\phi: \quad \delta S_{\mathcal{V}} \ \equiv \ \varepsilon_0 \int_{\mathcal{V}} \mathrm{d}^d x \ g(x) \ , \tag{9.1}$$

where

$$\begin{array}{c} g(x) \text{ is locally a divergence :} \\ \forall x_0 \in \mathcal{V} \exists \text{ local } x_0 \text{ neighborhood } \mathcal{W} \subseteq \mathcal{V} \ , \\ \exists g^{\mu}(x) = g^{\mu}(\phi(x), \partial\phi(x), \partial^2\phi(x), \ldots, x) \forall x \in \mathcal{W}: \quad g(x) \ = \ d_{\mu} g^{\mu}(x) \ . \end{array} \tag{9.2}$$

The integrand g is allowed to also depend on a finite number of higher derivatives $\partial^2\phi$, $\partial^3\phi$, …, of the field ϕ. As usual we assume that the function g can be extended smoothly to $\mathbb{R}^d$. In differential-geometric terms, the g function behaves as a density under passive coordinate transformations $x^{\mu} \to \overline{x}^{\nu} = \overline{x}^{\nu}(x)$. It follows that $\int_{\mathcal{V}} \mathrm{d}^d x \ g(x)$ is a boundary integral with identically vanishing Euler-Lagrange derivative

$$\frac{\delta g(x)}{\delta\phi^{\alpha}(x)} \ \equiv \ 0 \ . \tag{9.3}$$

(One of the aspects of Noether's Theorem, that we suppress in this note for simplicity, is the full Lie group G of quasi-symmetries. We only treat *one* infinitesimal quasi-symmetry at a time, cf. the 1-parameter ε_0. Thus we will also only derive *one* conservation law at a time. Technically speaking, the only remnant of G, that is treated here, is a $u(1)$ Lie subalgebra.)

A quasi-symmetry is promoted to a *symmetry*, if $\delta S_{\mathcal{V}} \equiv 0$. (It is natural to ask if it is always possible to turn a quasi-symmetry into a symmetry by modifying the action $\delta S_{\mathcal{V}}$ with a boundary integral? The answer is in general no, see Section 13 for a counterexample. Thus the notion of quasi-symmetry is an essential generalization of the original notion of symmetry discussed by Noether [11].)

The variational formula (7.2) together with the definition (9.1) of a quasi-symmetry yield

$$\forall\phi: \quad \int_{\mathcal{V}} \mathrm{d}^d x \ f(x) \ \equiv \ F_{\mathcal{V}}[\phi] \ \equiv \ \int_{\mathcal{V}} \mathrm{d}^d x \ g(x) \ . \tag{9.4}$$

Now define the zero functional

$$\forall\phi: \quad Z_{\mathcal{V}}[\phi] \ \equiv \ \int_{\mathcal{V}} \mathrm{d}^d x \ (f-g)(x) \ \equiv \ 0 \ . \tag{9.5}$$

By performing an arbitrary variation $\delta\phi(x)$ with support in the interior $x \in \mathcal{V}^{\circ}$ of $\mathcal{V}$ away from any boundaries, one concludes that the Euler-Lagrange derivative

$\delta(f-g)(x)/\delta\phi^{\alpha}(x)$ must vanish identically in the bulk $x \in \mathcal{V}^{\circ}$ (=the interior of $\mathcal{V}$),

$$\forall\phi\forall x\in\mathcal{V}^{\circ}: \quad \frac{\delta f(x)}{\delta\phi^{\alpha}(x)} \stackrel{(9.3)}{=} \frac{\delta(f-g)(x)}{\delta\phi^{\alpha}(x)} \;=\; 0\;, \tag{9.6}$$

And by continuity, $\delta f(x)/\delta\phi^{\alpha}(x)$ must vanish for all $x \in \mathcal{V}$. Lemma A.1 in Appendix A now yields the following.

$$\begin{gathered}\text{The integrand } f(x) \text{ is locally a divergence}:\\ \forall x_0 \in \mathcal{V}\exists \text{ local } x_0 \text{ neighborhood } \mathcal{W}\subseteq\mathcal{V},\\ \exists f^{\mu}(x) = f^{\mu}(\phi(x),\partial\phi(x),\partial^2\phi(x),x)\forall x\in\mathcal{W}: \quad f(x) \;=\; d_{\mu}f^{\mu}(x)\;.\end{gathered} \tag{9.7}$$

Equations (8.2), (9.1) and (9.2) then imply that the global variation is an off-shell quasi-symmetry of the action $S_{\mathcal{U}}$ for all smaller regions $\mathcal{U}\subseteq\mathcal{V}$, which is one of the main conclusions. One can locally define an *improved Noether current* as

$$J^{\mu}(x) \;:=\; \jmath^{\mu}(x) - f^{\mu}(x) = J^{\mu}(\phi(x),\partial\phi(x),\partial^2\phi(x),x)\;. \tag{9.8}$$

Equation (6.7) then immediately yields the sought–for off–shell Noether identity (9.9).

Theorem 9.1 (Local Off–shell Noether identity)**.** *A continuous, global, off-shell quasi-symmetry* (5.1), (7.1), (9.1) *of an $S_{\mathcal{V}}$ action* (3.1) *implies a local off–shell Noether identity*

$$d_{\mu}J^{\mu}(x) \;=\; d_{\mu}\jmath^{\mu}(x) - f(x) \stackrel{(6.7)}{=} -\frac{\delta\mathcal{L}(x)}{\delta\phi^{\alpha}(x)}Y_0^{\alpha}(x)\;. \tag{9.9}$$

10. Case 3: General manifold $\mathcal{V}$

If the world volume $\mathcal{V}$ is a manifold, one decomposes $\mathcal{V}=\sqcup_a\mathcal{V}_a$ in a disjoint union of coordinate patches. (Disjoint modulo zero Lebesgue measure of pertinent boundaries.) Under an infinitesimal variation (5.1), the world volume transforms $\mathcal{V}\to\mathcal{V}'=\sqcup_a\mathcal{V}'_a$, where $\mathcal{V}'_a := \{x' \mid x\in\mathcal{V}_a\}$. Each coordinate patch $\mathcal{V}_a$ and its variation $\mathcal{V}'_a$ are identified with subsets $\subseteq\mathbb{R}^d$. The $S_{\mathcal{V}}$ action (3.1) decomposes

$$S_{\mathcal{V}} \;=\; \sum_a S_a\;, \qquad S_a[\phi] \;=\; \int_{\mathcal{V}_a} \mathrm{d}^d x\; \mathcal{L}_a(x)\;, \qquad \mathcal{L}_a(x) \;=\; \mathcal{L}_a(\phi(x),\partial\phi(x),x)\;, \tag{10.1}$$

The variational formula (6.4) becomes

$$\delta S_{\mathcal{V}} \;=\; \sum_a \int_{\mathcal{V}^a} \mathrm{d}^d x\,[f_a(x)\varepsilon(x) + \jmath_a^{\mu}(x)d_{\mu}\varepsilon(x)]\;, \tag{10.2}$$

the global variation formula (7.2) becomes

$$\delta S_{\mathcal{V}} \;=\; \varepsilon_0 F_{\mathcal{V}}\;, \qquad F_{\mathcal{V}} \;:=\; \sum_a F_a\;, \qquad F_a[\phi] \;:=\; \int_{\mathcal{V}_a}\mathrm{d}^d x\; f_a(x)\;, \tag{10.3}$$

the bare Noether current (6.6) becomes

$$\jmath_a^{\mu}(x) \;:=\; \frac{\partial\mathcal{L}_a(x)}{\partial\phi_{\mu}^{\alpha}(x)}Y_{0a}^{\alpha}(x) + \mathcal{L}_a(x)X_a^{\mu}(x)\;, \tag{10.4}$$

and the function (6.7) becomes

$$f_a(x) \ := \ \frac{\delta \mathcal{L}_a(x)}{\delta\phi^\alpha(x)} Y^\alpha_{0a}(x) + d_\mu \jmath^\mu_a(x) \ . \tag{10.5}$$

The only difference is that all quantities now carry a chart-subindex "a". The definition (10.6) of a global off-shell quasi-symmetry becomes

$$\forall\phi: \quad \delta S_{\mathcal{V}} \ \equiv \ \varepsilon_0 \sum_a G_a \ , \qquad G_a[\phi] \ := \ \int_{\mathcal{V}_a} \mathrm{d}^d x \ g_a(x) \ , \tag{10.6}$$

where the integrand g_a is locally a divergence, so that the integral $\sum_a G_a$ only receives contributions from external boundaries, *i.e.*, contributions from internal boundaries cancel pairwise. Then eq. (9.4) is replaced by

$$\forall\phi: \quad \sum_a F_a \ \equiv \ F_{\mathcal{V}} \ \equiv \ \sum_a G_a \ . \tag{10.7}$$

Now define the zero functional

$$Z_{\mathcal{V}}[\phi] \ := \ \sum_a (F_a[\phi] - G_a[\phi]) \ = \ \sum_a \int_{\mathcal{V}_a} \mathrm{d}^d x \ (f_a - g_a)(x) \ = \ 0 \ . \tag{10.8}$$

By performing an arbitrary variation $\delta\phi$ with support inside a single chart $\mathcal{V}_a$ away from any boundaries, one concludes that the Euler-Lagrange derivative vanishes identically in the interior $\mathcal{V}_a^\circ$ of $\mathcal{V}_a$,

$$\forall\phi\forall x \in \mathcal{V}_a^\circ : \quad \frac{\delta f_a(x)}{\delta\phi^\alpha(x)} \ = \ \frac{\delta(f_a - g_a)(x)}{\delta\phi^\alpha(x)} \ = \ 0 \ . \tag{10.9}$$

Hence one can proceed within a single coordinate patch $\mathcal{V}_a$, as already done in previous Sections, and prove the sought–for off–shell Noether identity (9.9) at the interior point $x \in \mathcal{V}_a^\circ$. All the constructions are geometrically covariant; they do not depend on the choice of coordinate patches $\mathcal{V}_a$, or the positions of patch boundaries, so the Noether identity (9.9) holds for all points $x \in \mathcal{V}$.

11. Example: Particle with external force

Consider the action for a non-relativistic point particle of mass m moving in one dimension,

$$S_{\mathcal{V}}[q] \ := \ \int_{t_i}^{t_f} \mathrm{d}t \ L(t) \ , \qquad L(t) \ := \ \frac{1}{2} m \left(\dot{q}(t)\right)^2 + q(t)F(t) \ , \qquad x^0 \equiv t \ . \tag{11.1}$$

Assume that the particle experiences a given background external force $F(t)$ that is independent of q and happens to satisfy that the total momentum transfer ΔP for the whole time period $[t_i, t_f]$ is zero

$$\Delta P \ = \ \int_{t_i}^{t_f} \mathrm{d}t \ F(t) \ = \ 0 \ . \tag{11.2}$$

The fixed region is in this case $\mathcal{V} = [t_i, t_f]$. One can write

$$S_{\mathcal{V}}[q] \ = \ \int_{\mathbb{R}} \mathrm{d}t \ \hat{L}(t) \ , \qquad \hat{L}(t) \ := \ 1_{\mathcal{V}}(t) L(t) \ , \tag{11.3}$$

The Euler-Lagrange derivative is

$$\begin{aligned} \frac{\delta \hat{L}(t)}{\delta q(t)} &= 1_{\mathcal{V}}(t)\frac{\delta L(t)}{\delta q(t)} - \frac{\partial L(t)}{\partial \dot{q}(t)}\partial_0 1_{\mathcal{V}}(t) \\ &= 1_{\mathcal{V}}(t)\left[F(t) - m\ddot{q}(t)\right] + m\dot{q}(t)\left[\delta(t-t_f) - \delta(t-t_i)\right] \ . \end{aligned} \tag{11.4}$$

The principle of stationary/least action in classical mechanics says that $\delta\hat{L}(t)/\delta q(t) \approx 0$ is the equations of motion for the system. This yields Newton's second law in the bulk,

$$\forall t \in \mathcal{V}^\circ : \quad \frac{\delta L(t)}{\delta q(t)} \ = \ F(t) - m\ddot{q}(t) \ \approx 0 \ . \tag{11.5}$$

and Neumann conditions at the boundary,

$$\dot{q}(t_i) \ \approx \ 0 \ , \qquad \dot{q}(t_f) \ \approx \ 0 \ . \tag{11.6}$$

Note that we here take painstaking care of representing the model (11.1) as it was mathematically given to us. The delta functions at the boundary in eq. (11.4) may or may not reflect the physical reality. For instance, if the variational problem has additional conditions, say, a Dirichlet boundary condition $q(t_i)=q_i$ at $t=t_i$, then any variation of q must obey $\delta q(t_i)=0$, and one will be unable to deduce the corresponding equation of motion for $t=t_i$, and therefore one cannot conclude the Neumann boundary condition (11.6) at $t=t_i$. If the system is unconstrained at $t=t_i$, it will probably make more physical sense to *impose* Neumann boundary condition (11.6) at $t=t_i$ from the very beginning, rather than to derive it as an equation of motion. Similarly for the other boundary $t=t_f$.

Consider now a global variation

$$\delta t \ = \ 0 \ , \qquad \delta q(t) \ = \ \delta_0 q(t) \ = \ \varepsilon_0 \ , \tag{11.7}$$

where ε_0 is a global, t-independent infinitesimal 1-parameter, *i.e.*, the horizontal and vertical generators are $X^0(t)=0$ and $Y(t)=Y_0(t)=1$, respectively. This vertical variation $\delta=\delta_0$ is *not* necessarily a symmetry of the Lagrangian

$$\delta L(t) \ = \ \varepsilon_0 F(t) \ , \tag{11.8}$$

but it is a symmetry of the action

$$\delta S_{\mathcal{V}} \ = \ \varepsilon_0 \Delta P \ = \ 0 \ , \tag{11.9}$$

due to the condition (11.2). We stress that the global variation (11.7) is *not* necessarily a symmetry of the action for other regions $\mathcal{U}$. The bare Noether current is the momentum of the particle

$$\jmath^0(t) \ = \ \frac{\partial L(t)}{\partial \dot{q}(t)} Y_0(t) \ = \ m\dot{q}(t) \ . \tag{11.10}$$

The function

$$f(t) \ := \ \frac{\delta L(t)}{\delta q(t)} Y_0(t) + d_0 \jmath^0(t) \ = \ F(t) \ . \tag{11.11}$$

from eq. (6.7) can be written as a total time derivative

$$f(t) \ = \ d_0 f^0(t) \ , \tag{11.12}$$

if one defines the accumulated momentum transfer

$$f^0(t) \ := \ \int^t \mathrm{d}t' \ F(t') \ . \tag{11.13}$$

The improved Noether current is then

$$J^0(t) \ := \ \jmath^0(t) - f^0(t) \ = \ m\dot{q}(t) - f^0(t) \ . \tag{11.14}$$

The off-shell Noether identity reads

$$d_0 J^0(t) \ = \ m\ddot{q}(t) - F(t) \ = \ -\frac{\delta L(t)}{\delta q(t)} Y_0(t) \ . \tag{11.15}$$

12. Example: Particle with fluctuating zero-point energy

Consider the action for a non-relativistic point particle of mass m moving in one dimension,

$$S_{\mathcal{V}}[q] := \int_{t_i}^{t_f} \mathrm{d}t \ L(t) \ , \quad L(t) := T(t) - V(t) \ , \quad T(t) := \frac{1}{2} m \left(\dot{q}(t)\right)^2 \ . \tag{12.1}$$

Assume that the background fluctuating zero-point energy $V(t)$ is independent of q and happens to satisfy that

$$V(t_i) \ = \ V(t_f) \ . \tag{12.2}$$

The fixed region is in this case $\mathcal{V} \equiv [t_i, t_f]$. The Euler-Lagrange derivative is

$$0 \ \approx \ \frac{\delta L(t)}{\delta q(t)} \ = \ -m\ddot{q}(t) \ . \tag{12.3}$$

Consider now a global variation

$$\delta t \ = \ -\varepsilon_0 \ , \qquad \delta q(t) \ = \ 0 \ , \qquad \delta_0 q(t) \ = \ \varepsilon_0 \dot{q}(t) \ , \tag{12.4}$$

where ε_0 is a global, t-independent infinitesimal 1-parameter, *i.e.*, the generators are $X^0(t) = -1$, $Y(t) = 0$ and $Y_0(t) = \dot{q}(t)$. This variation (12.4) is *not* necessarily a symmetry of the Lagrangian

$$\delta L(t) \ = \ \varepsilon_0 \dot{V}(t) \ , \tag{12.5}$$

but it is a symmetry of the action

$$\begin{aligned} \delta S_{\mathcal{V}} &= \int_{t_i}^{t_f} \mathrm{d}t \left(\delta L(t) + L(t) d_0 \delta t\right) \\ &= \varepsilon_0 \int_{t_i}^{t_f} \mathrm{d}t \ \dot{V}(t) \ = \ \varepsilon_0 \left[V(t_f) - V(t_i)\right] \ = \ 0 \ , \end{aligned} \tag{12.6}$$

due to the condition (12.2). We stress that the variation (12.4) is *not* necessarily a symmetry of the action for other regions $\mathcal{U}$. The bare Noether current is the total energy of the particle

$$\jmath^0(t) \ := \ \frac{\partial L(t)}{\partial \dot{q}(t)} Y_0(t) + L(t) X^0(t) \ = \ T(t) + V(t) \ . \tag{12.7}$$

The function $f(t)$ from eq. (6.7) is a total time derivative of the zero-point energy

$$f(t) \ :=\ \frac{\delta L(t)}{\delta q(t)}Y_0(t) + d_0 \jmath^0(t) \ =\ \dot{V}(t) \ =\ d_0 f^0(t) \tag{12.8}$$

if one defines $f^0(t) = V(t)$. The improved Noether current is the kinetic energy

$$J^0(t) \ :=\ \jmath^0(t) - f^0(t) \ =\ T(t) \ . \tag{12.9}$$

The off-shell Noether identity reads

$$d_0 J^0(t) \ =\ \dot{T}(t) \ =\ m\dot{q}(t)\ddot{q}(t) \ =\ -\frac{\delta L(t)}{\delta q(t)}Y_0(t) \ . \tag{12.10}$$

Notice that one may need to improve the bare Noether current $\jmath^0(t) \to J^0(t)$ even in cases of an exact symmetry (12.6) of the action.

13. Example: Quasi-symmetry vs. symmetry

Here we will consider a quasi-symmetry δ of a Lagrangian $L(t)$ that can *not* be turned into a symmetry by modifying the Lagrangian $L(t) \to \widetilde{L}(t) := L(t) + dF(t)/dt$ with a total derivative.

Let $L(t) = L(q(t), \dot{q}(t))$ be a Lagrangian that depends on position $q(t)$ and velocity $\dot{q}(t)$, but that does *not* depend explicitly on time t. Consider now a global variation

$$\delta t \ =\ 0, , \qquad \delta q(t) \ =\ \delta_0 q(t) \ =\ \varepsilon_0 \dot{q}(t) \ , \tag{13.1}$$

where ε_0 is a global, t-independent infinitesimal 1-parameter, *i.e.*, the generators are $X^0(t) = 0$ and $Y(t) = Y_0(t) = \dot{q}(t)$. This vertical variation $\delta = \delta_0$ is a quasi-symmetry of the Lagrangian

$$\delta L(t) \ =\ \varepsilon_0 \left(\frac{\partial L(t)}{\partial q(t)}\dot{q}(t) + \frac{\partial L(t)}{\partial \dot{q}(t)}\ddot{q}(t) \right) \ =\ \varepsilon_0 \dot{L}(t) \ , \tag{13.2}$$

but it is only a symmetry of the Lagrangian $\delta L(t) = 0$, if $L(t)$ does also not depend on position $q(t)$ and velocity $\dot{q}(t)$, *i.e.*, if the Lagrangian is only a constant. Thus, in order to modify the Lagrangian $L(t) \to \widetilde{L}(t) := L(t) + dF(t)/dt$, so that the new Lagrangian $\delta\widetilde{L}(t) = 0$ has a symmetry, the old Lagrangian $L(t)$ must be a total derivative to begin with.

The bare Noether current $\jmath^0(t)$ is

$$\jmath^0(t) \ :=\ \frac{\partial L(t)}{\partial \dot{q}(t)}Y_0(t) + L(t)X^0(t) \ =\ p(t)\dot{q}(t) \ . \tag{13.3}$$

The function $f(t)$ from eq. (6.7) is a total time derivative of the Lagrangian

$$f(t) \ :=\ \frac{\delta L(t)}{\delta q(t)}Y_0(t) + d_0 \jmath^0(t) \ =\ \dot{L}(t) \ =\ d_0 f^0(t) \tag{13.4}$$

if one defines $f^0(t) = L(t)$. The improved Noether current is the energy

$$J^0(t) \ :=\ \jmath^0(t) - f^0(t) \ =\ p(t)\dot{q}(t) - L(t) \ =\ h(t) \ . \tag{13.5}$$

The off-shell Noether identity reads

$$d_0 J^0(t) \;=\; \dot{h}(t) \;=\; -\frac{\delta L(t)}{\delta q(t)} Y_0(t) \;, \tag{13.6}$$

reflecting the well-known fact that the energy $h(t)$ is conserved when the Lagrangian does not depend explicitly on time t.

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A. Identically vanishing Euler-Lagrange derivative

We will prove in this Appendix A that an integral is a boundary integral if its Euler-Lagrange derivative vanishes. Consider a function

$$\mathcal{L} \;\in\; \mathcal{F}(M \times M^d \times M^{d(d+1)/2} \times \mathcal{V}) \;, \qquad \mathcal{L}(x) \;=\; \mathcal{L}(\phi(x), \partial\phi(x), \partial^2\phi(x), x) \;, \tag{A.1}$$

on the 2-jet space. The function $\mathcal{L}$ is assumed to be smooth in both vertical and horizontal directions.

Lemma A.1.

$$\begin{array}{c} \text{Identically vanishing Euler Lagrange derivatives of} \\ \mathcal{L}(x) = \mathcal{L}(\phi(x), \partial\phi(x), \partial^2\phi(x), x) : \\ \forall\phi\forall x \in \mathcal{V} : \quad \frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} \;\equiv\; 0 \;. \\ \\ \Downarrow \\ \\ \mathcal{L}(x) \text{ is locally a divergence :} \\ \forall x_0 \in \mathcal{V} \exists \text{ local } x_0 \text{ neighborhood } \mathcal{W} \subseteq \mathcal{V}, \\ \exists \Lambda^\mu(x) = \Lambda^\mu(\phi(x), \partial\phi(x), \partial^2\phi(x), x) \forall x \in \mathcal{W} : \quad \mathcal{L}(x) \;=\; d_\mu \Lambda^\mu(x) \;. \end{array} \tag{A.2}$$

Proof of Lemma A.1. Define a region with one more dimension

$$\widetilde{\mathcal{V}} \;:=\; \mathcal{V} \times [0,1] \;, \tag{A.3}$$

which locally has coordinates $\widetilde{x} := (x, \lambda)$. Define the field $\widetilde{\phi} : \widetilde{\mathcal{V}} \to M$ as

$$\widetilde{\phi}(\widetilde{x}) \;:=\; \lambda\phi(x) \;. \tag{A.4}$$

This makes sense, because the target space M is star-shaped around 0, cf. eq. (2.1). Define

$$\widetilde{\mathcal{L}}(\widetilde{x}) \;:=\; \mathcal{L}(\widetilde{\phi}(\widetilde{x}), \partial\widetilde{\phi}(\widetilde{x}), \partial^2\widetilde{\phi}(\widetilde{x}), x) \;=\; \mathcal{L}(x)\big|_{\phi(x) \to \widetilde{\phi}(\widetilde{x})} \;. \tag{A.5}$$

Note that $\widetilde{\mathcal{L}}$ does not depend on λ-derivatives of the $\widetilde{\phi}$-fields, nor explicitly on λ. Thus the total derivative with respect to λ reads

$$\frac{d\widetilde{\mathcal{L}}(\widetilde{x})}{d\lambda} = \frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^\alpha(\widetilde{x})} \frac{\partial\widetilde{\phi}^\alpha(\widetilde{x})}{\partial\lambda} + \frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^\alpha_\mu(\widetilde{x})} \frac{\partial\widetilde{\phi}^\alpha_\mu(\widetilde{x})}{\partial\lambda} + \sum_{\nu \le \mu} \frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^\alpha_{\mu\nu}(\widetilde{x})} \frac{\partial\widetilde{\phi}^\alpha_{\mu\nu}(\widetilde{x})}{\partial\lambda}$$

$$\text{(A.6)} \qquad \stackrel{(A.7)+(A.8)}{=} \frac{\delta\widetilde{\mathcal{L}}(\widetilde{x})}{\delta\widetilde{\phi}^{\alpha}(\widetilde{x})}\frac{\partial\widetilde{\phi}^{\alpha}(\widetilde{x})}{\partial\lambda} + d_{\mu}\widetilde{\Lambda}^{\mu}(\widetilde{x}) \stackrel{(A.7)}{=} d_{\mu}\widetilde{\Lambda}^{\mu}(\widetilde{x}) \ ,$$

where the Euler-Lagrange derivatives vanish by assumption

$$\frac{\delta\widetilde{\mathcal{L}}(\widetilde{x})}{\delta\widetilde{\phi}^{\alpha}(\widetilde{x})} := \frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}(\widetilde{x})} - d_{\mu}\frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}_{\mu}(\widetilde{x})} + \sum_{\nu\leq\mu} d_{\mu}d_{\nu}\frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}_{\mu\nu}(\widetilde{x})}$$

$$\text{(A.7)} \qquad = \left.\frac{\delta\mathcal{L}(x)}{\delta\phi^{\alpha}(x)}\right|_{\phi(x)\to\widetilde{\phi}(\widetilde{x})} = 0 \ ,$$

and we have defined some functions

$$\widetilde{\Lambda}^{\mu}(\widetilde{x}) := \left(\frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}_{\mu}(\widetilde{x})} - 2\sum_{\nu\leq\mu} d_{\nu}\frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}_{\mu\nu}(\widetilde{x})}\right)\frac{\partial\widetilde{\phi}^{\alpha}(\widetilde{x})}{\partial\lambda}$$

$$\text{(A.8)} \qquad + \sum_{\nu\leq\mu} d_{\nu}\left(\frac{\partial\widetilde{\mathcal{L}}(\widetilde{x})}{\partial\widetilde{\phi}^{\alpha}_{\mu\nu}(\widetilde{x})}\frac{\partial\widetilde{\phi}^{\alpha}(\widetilde{x})}{\partial\lambda}\right) \ .$$

Hence

$$\mathcal{L}(x) - \mathcal{L}(x)|_{\phi=0} = \left.\widetilde{\mathcal{L}}(\widetilde{x})\right|_{\lambda=1} - \left.\widetilde{\mathcal{L}}(\widetilde{x})\right|_{\lambda=0}$$

$$\text{(A.9)} \qquad = \int_0^1 \mathrm{d}\lambda \frac{d\widetilde{\mathcal{L}}(\widetilde{x})}{d\lambda} \stackrel{(A.6)}{=} d_{\mu}\int_0^1 \mathrm{d}\lambda\ \widetilde{\Lambda}^{\mu}(\widetilde{x}) \ .$$

On the other hand, the lower boundary

$$\text{(A.10)} \qquad h(x) := \mathcal{L}(x)|_{\phi=0}$$

in eq. (A.9) does not depend on ϕ, so one can, *e.g.*, locally pick a coordinate $t\equiv x^0$, so that $x^{\mu}=(t,\vec{x})$, and define

$$\text{(A.11)} \qquad H^0(x) := \int^t \mathrm{d}t'\ h(t',\vec{x}) \ , \qquad 0 = H^1 = H^2 = \ldots = H^{d-1} \ .$$

Then $h(x)=\partial_{\mu}H^{\mu}(x)$ is locally a divergence. Altogether, this implies that $\mathcal{L}(x)$ is locally a divergence. □

Remark. It is easy to check that the opposite arrow "⇑" in Lemma A.1 is also true. The Lemma A.1 can be generalized to n-jets, for any $n = 1, 2, 3, \ldots$, using essentially the same proof technique. We have focused on the $n=2$ case, since this is the case that is needed in the proof of Noether's first Theorem, cf. eq. (9.7). The fact that the $n=2$ case is actually needed for the physically relevant case, where the Lagrangian density depends on up to first order derivatives of the fields, is often glossed over in standard textbooks on classical mechanics. By (a dualized version of) the Poincaré Lemma, it follows that the local functions $\Lambda^{\mu} \to \Lambda^{\mu} + d_{\nu}\Lambda^{\nu\mu}$ are unique up to antisymmetric improvement terms $\Lambda^{\nu\mu}=-\Lambda^{\mu\nu}$, see *e.g.*, Ref. [2].

B. Gauging a global $u(1)$ quasi-symmetry

A global quasi-symmetry δ from eq. (5.1) is by definition promoted to a *gauge quasi-symmetry* if the variation $\delta S_{\mathcal{V}}$ of the action in eq. (6.4) is a boundary integral for arbitrary x-dependent $\varepsilon(x)$. Noether's second Theorem [11] states that a gauge quasi-symmetry δ implies an off-shell conservation law and an off-shell Noether identity, *i.e.*,

$$0 \;\equiv\; d_\mu J^\mu(x) \;\equiv\; -\frac{\delta\mathcal{L}(x)}{\delta\phi^\alpha(x)} Y_0^\alpha(x) \; . \tag{B.1}$$

As we shall see in eq. (B.19), it is often possible to gauge a global $u(1)$ quasi-symmetry δ by introducing an Abelian *gauge potential* $A_\mu = A_\mu(x)$ with infinitesimal Abelian *gauge transformation*

$$\delta A_\mu \;=\; \partial_\mu \varepsilon \; , \tag{B.2}$$

and adding certain terms to the Lagrangian density $\mathcal{L}$ that vanish for $A \to 0$. The Abelian field strength

$$F_{\mu\nu} \;:=\; \partial_\mu A_\nu - (\mu \leftrightarrow \nu) \tag{B.3}$$

is gauge invariant $\delta F_{\mu\nu} = 0$. In this appendix, we specialize to the case where the horizontal generator X^μ vanishes, and where the vertical generator Y_0^α does not depend on derivatives $\partial\phi$,

$$X^\mu(x) \;=\; 0 \; , \qquad Y_0^\alpha(x) \;=\; Y_0^\alpha(\phi(x), x) \; . \tag{B.4}$$

Assumption (B.4) is made in order for the sought-for gauged Lagrangian density $\mathcal{L}^{\rm gauged}$ to be minimally coupled, cf. eq. (B.18). It is useful to first introduce a bit of notation. The *jet-prolongated vector field* $\hat{Y}_0$ is defined as

$$\hat{Y}_0 \;:=\; J^\bullet Y_0 \;=\; Y_0^\alpha \frac{\partial}{\partial\phi^\alpha} + d_\mu Y_0^\alpha \, \frac{\partial}{\partial\phi_\mu^\alpha} + \sum_{\mu\leq\nu} d_\mu d_\nu Y_0^\alpha \, \frac{\partial}{\partial\phi_{\mu\nu}^\alpha} + \ldots \; . \tag{B.5}$$

The jet-prolongated vector field $\hat{Y}_0$ and the total derivative d_μ commute $[d_\mu, \hat{Y}_0] = 0$. The *covariant derivative* D_μ is defined as

$$D_\mu \;:=\; d_\mu - A_\mu Y_0^\alpha \frac{\partial}{\partial\phi^\alpha} \; . \tag{B.6}$$

The characteristic feature of the covariant derivative $D_\mu\phi^\alpha = d_\mu\phi^\alpha - A_\mu Y_0^\alpha$ is that it behaves covariantly under the gauge transformation δ,

$$\delta D_\mu\phi^\alpha \;=\; d_\mu\delta\phi^\alpha - Y_0^\alpha\delta A_\mu - A_\mu\delta Y_0^\alpha \;=\; d_\mu(\varepsilon Y_0^\alpha) - Y_0^\alpha\partial_\mu\varepsilon - A_\mu\frac{\partial Y_0^\alpha}{\partial\phi^\beta} Y_0^\beta\varepsilon \;=\; \varepsilon D_\mu Y_0^\alpha \; . \tag{B.7}$$

The *minimal* extension $\widetilde{h}(x)$ (which in this Appendix B is notationally denoted with a tilde "$\sim$") of a function

$$h(x) \;=\; h(\phi(x), \partial\phi(x), A(x), F(x), x) \; , \tag{B.8}$$

is defined by replacing partial derivatives ∂_μ with covariant derivatives D_μ, *i.e.*,

$$\widetilde{h}(x) \;:=\; h(\phi(x), D\phi(x), A(x), F(x), x) \; . \tag{B.9}$$

Here it is important that the h function in eq. (B.8) does *not* depend on higher x-derivatives of ϕ. (A minimal extension $\widetilde{h}$ of a function h, that depend on higher x-derivatives of ϕ, is only well-defined if the field strength $F_{\mu\nu}$ vanishes, so that the covariant derivatives D_μ commute.) Assumption (B.4) implies that

$$\begin{aligned}\left(d_\mu\hat{Y}_0\phi^\alpha\right)^\sim &= (d_\mu Y_0^\alpha)^\sim = \left(\partial_\mu Y_0^\alpha + \frac{\partial Y_0^\alpha}{\partial\phi^\beta}\partial_\mu\phi^\beta\right)^\sim = \partial_\mu Y_0^\alpha + \frac{\partial Y_0^\alpha}{\partial\phi^\beta}D_\mu\phi^\beta \\ &= D_\mu Y_0^\alpha = d_\mu Y_0^\alpha - A_\mu Y_0^\beta\frac{\partial Y_0^\alpha}{\partial\phi^\beta} = d_\mu\hat{Y}_0\phi^\alpha - A_\mu\hat{Y}_0 Y_0^\alpha \\ &= \hat{Y}_0 D_\mu\phi^\alpha \ . \end{aligned} \tag{B.10}$$

More generally, assumption (B.4) implies that the jet-prolongated vector field $\hat{Y}_0$ and the minimal extension "$\sim$" commute in the sense that if h is a function of type (B.8), then $\hat{Y}_0 h$ is also a function of type (B.8), and its minimal extension is

$$\begin{aligned}\left(\hat{Y}_0 h\right)^\sim &= \left(\frac{\partial h}{\partial\phi^\alpha}Y_0^\alpha + \frac{\partial h}{\partial\phi^\alpha_\mu}d_\mu Y_0^\alpha\right)^\sim \\ &\stackrel{(B.10)}{=} \frac{\partial\widetilde{h}}{\partial\phi^\alpha}Y_0^\alpha + \frac{\partial\widetilde{h}}{\partial D_\mu\phi^\alpha}D_\mu Y_0^\alpha \stackrel{(B.10)}{=} \hat{Y}_0\widetilde{h} \ .\end{aligned} \tag{B.11}$$

Furthermore, the gauge transformation $\delta\widetilde{h}$ of the minimal extension $\widetilde{h}$ can be calculated with the help of the jet-prolongated vector field $\hat{Y}_0$ as

$$\begin{aligned}\delta\widetilde{h} &= \frac{\partial\widetilde{h}}{\partial\phi^\alpha}\delta\phi^\alpha + \left(\frac{\partial h}{\partial\phi^\alpha_\mu}\right)^\sim\delta D_\mu\phi^\alpha + \left(\frac{\partial h}{\partial A_\mu}\right)^\sim\delta A_\mu \\ &= \left(\varepsilon\hat{Y}_0 h + \frac{\partial h}{\partial A_\mu}\partial_\mu\varepsilon\right)^\sim \ .\end{aligned} \tag{B.12}$$

In particular, it follows from assumption (B.4) that the function $f=\hat{Y}_0\mathcal{L}$ from eq. (6.7) is a function of type (B.8), *i.e.*, f can not depend on higher x-derivatives of the field ϕ,

$$f(x) \ = \ f(\phi(x), \partial\phi(x), x) \ . \tag{B.13}$$

Equation (B.13) and Appendix A imply, in turn, that the local function $f^\mu(x) = f^\mu(\phi(x), \partial\phi(x), x)$ from eq. (9.7) must also be of type (B.8), and have derivatives

$$\frac{\partial f^\mu}{\partial\phi^\alpha_\nu} \ = \ -(\mu\leftrightarrow\nu) \tag{B.14}$$

that are $\mu\leftrightarrow\nu$ antisymmetric. The local function $f^\mu \to f^\mu + d_\nu f^{\nu\mu}$ is unique up to antisymmetric improvement terms $f^{\nu\mu} = -f^{\mu\nu}$. We will furthermore assume that

$$f^\mu \text{ is globally defined,} \tag{B.15}$$

and that f^μ has been chosen so that

$$\frac{\partial f^\mu}{\partial\phi^\alpha_\nu}Y_0^\alpha \ = \ (\mu\leftrightarrow\nu) \ . \tag{B.16}$$

The latter assumption (B.16) together with eq. (B.14) imply that

$$\frac{\partial f^\mu}{\partial \phi^\alpha_\nu} Y_0^\alpha \;=\; 0\ . \tag{B.17}$$

The (minimally coupled) *gauged Lagrangian density* $\mathcal{L}^{\rm gauged}$ is now defined as

$$\mathcal{L}^{\rm gauged} \;:=\; \left(\mathcal{L} + A_\mu f^\mu\right)^\sim \;=\; \widetilde{\mathcal{L}} + A_\mu \widetilde{f}^\mu\ . \tag{B.18}$$

The gauge transformation $\delta\mathcal{L}^{\rm gauged}$ of $\mathcal{L}^{\rm gauged}$ can be written as a divergence

$$\begin{aligned}
\delta\mathcal{L}^{\rm gauged} &\stackrel{(B.12)}{=} \left(\varepsilon \hat{Y}_0 \mathcal{L} + f^\mu \partial_\mu \varepsilon + A_\mu \varepsilon \hat{Y}_0 f^\mu\right)^\sim \\
&= \left(d_\mu(\varepsilon f^\mu) + \varepsilon A_\mu \big(\frac{\partial f^\mu}{\partial \phi^\alpha} Y_0^\alpha + \frac{\partial f^\mu}{\partial \phi^\alpha_\nu} d_\nu Y_0^\alpha\big)\right)^\sim \\
&= \partial_\mu(\varepsilon \widetilde{f}^\mu) + \varepsilon \frac{\partial \widetilde{f}^\mu}{\partial \phi^\alpha} D_\mu \phi^\alpha + \varepsilon A_\mu \left(\frac{\partial \widetilde{f}^\mu}{\partial \phi^\alpha} Y_0^\alpha + \frac{\partial \widetilde{f}^\mu}{\partial D_\nu \phi^\alpha} D_\nu Y_0^\alpha\right) \\
&\stackrel{(B.14)}{=} \partial_\mu(\varepsilon \widetilde{f}^\mu) + \varepsilon \frac{\partial \widetilde{f}^\mu}{\partial \phi^\alpha} \partial_\mu \phi^\alpha - \varepsilon A_\mu \frac{\partial \widetilde{f}^\nu}{\partial D_\mu \phi^\alpha} d_\nu Y_0^\alpha \\
&\stackrel{(B.20)}{=} d_\mu(\varepsilon \widetilde{f}^\mu)\ ,
\end{aligned} \tag{B.19}$$

because

$$\begin{aligned}
d_\mu \widetilde{f}^\mu - \partial_\mu \widetilde{f}^\mu - \frac{\partial \widetilde{f}^\mu}{\partial \phi^\alpha} \partial_\mu \phi^\alpha &= \frac{\partial \widetilde{f}^\mu}{\partial D_\nu \phi^\alpha} d_\mu D_\nu \phi^\alpha \stackrel{(B.14)}{=} -\frac{\partial \widetilde{f}^\mu}{\partial D_\nu \phi^\alpha} d_\mu \left(A_\nu Y_0^\alpha\right) \\
&\stackrel{(B.14)}{=} -\frac{\partial \widetilde{f}^\mu}{\partial D_\nu \phi^\alpha} \left(\frac{1}{2} F_{\mu\nu} Y_0^\alpha + A_\nu d_\mu Y_0^\alpha\right) \\
&\stackrel{(B.16)}{=} -\frac{\partial \widetilde{f}^\mu}{\partial D_\nu \phi^\alpha} A_\nu d_\mu Y_0^\alpha\ .
\end{aligned} \tag{B.20}$$

Equation (B.19) shows that the gauge transformation δ is a gauge quasi-symmetry of the (minimally coupled) gauged action

$$S_{\mathcal{V}}^{\rm gauged}[\phi, A] \;:=\; \int_{\mathcal{V}} \mathrm{d}^d x\ \mathcal{L}^{\rm gauged}(x)\ , \tag{B.21}$$

which was the goal of Appendix B.

Theorem B.1. *If an $S_{\mathcal{V}}$ action* (3.1) *has a global quasi-symmetry* (5.1), (7.1), (9.1) *of the form* (B.4), (B.15), (B.16), *then the (minimally coupled) gauged action* (B.21) *has a corresponding gauge quasi-symmetry.*

Finally, let us try to justify assumption (B.16), which was only used in the last equality of eq. (B.20). Notice that the function $f = \hat{Y}_0\mathcal{L}$ depends linearly on $\mathcal{L}$, so we may argue term by term in $\mathcal{L}$. Firstly, in the special case where the Lagrangian density $\mathcal{L} = d_\mu \Lambda^\mu$ is locally a divergence, the equations of motion $\delta\mathcal{L}/\delta\phi^\alpha$ vanish identically, cf. Appendix A, and we may pick $f^\mu = \jmath^\mu$ globally as the bare Noether current $\jmath^\mu := Y_0^\alpha \partial\mathcal{L}/\partial\phi^\alpha_\mu$, which clearly satisfies condition (B.16). Secondly, in the general case with general local f^μ, and under the additional assumption of a homotopy inverse to the jet prolongation $\hat{Y}_0$, there exists a local Λ^μ such that $f^\mu =$

$\hat{Y}_0\Lambda^\mu$. Since $[d_\mu, \hat{Y}_0]=0$, we have $\hat{Y}_0 d_\mu\Lambda^\mu = d_\mu\hat{Y}_0\Lambda^\mu = d_\mu f^\mu = f$. Because we have already discussed the special case of a local divergence, we may subtract the local divergence $d_\mu\Lambda^\mu$ from $\mathcal{L}$, so that the remaining Lagrangian density $\mathcal{L}' = \mathcal{L} - d_\mu\Lambda^\mu$ has vanishing f-function $f' = d_\mu f'^\mu$, because $f' = \hat{Y}_0\mathcal{L}' = \hat{Y}_0\mathcal{L} - \hat{Y}_0 d_\mu\Lambda^\mu = f - f = 0$. Thus we may pick the remaining f^μ-function globally as $f'^\mu = 0$, which clearly also satisfies condition (B.16).

If we consider a point x_0, where the vertical vector field $Y_0(x_0) \neq 0$ does not vanish, it is possible to locally stratify Y_0, *i.e.*, by changing target space coordinates ϕ^α, so that vertical vector field $Y_0 = \partial/\partial\phi^1$ (and hence the whole jet prolongation $\hat{Y}_0 = Y_0 = \partial/\partial\phi^1$) is just a differentiation with respect to a single coordinate ϕ^1, the homotopy inverse exists and is just an integration with respect to ϕ^1.

In fact, these arguments show under the assumption (B.4), that locally (away from singular points x_0 with $Y_0(x_0)=0$), it is possible to enhance a global quasi-symmetry into a genuine global symmetry with vanishing function $f \equiv 0$ by adding a local divergence term $d_\mu\Lambda^\mu$ to the Lagrangian density $\mathcal{L} \to \mathcal{L} + d_\mu\Lambda^\mu$.

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